Multi-criteria group decision making method based on intuitionistic linguistic aggregation operators

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Abstract. For multi-criteria group decision making problems with intuitionistic linguistic information, we define a new score function and a new accuracy function of intuitionistic linguistic numbers, and propose a simple approach for the comparison between two intuitionistic linguistic numbers. Based on the intuitionistic linguistic weighted arithmetic averaging (ILWAA) operator, we define two new intuitionistic linguistic aggregation operators, such as the intuitionistic linguistic ordered weighted averaging (ILOWA) operator and the intuitionistic linguistic hybrid aggregation (ILHA) operator, and establish various properties of these operators. The ILOWA operator weights the ordered positions of the intuitionistic linguistic numbers instead of weighting the arguments themselves. The ILHA operator generalizes both the ILWAA operator and the ILOWA operator at the same time, and reflects the importance degrees of both the given intuitionistic linguistic numbers and the ordered positions of these arguments. Furthermore, based on the ILHA operator and the ILWAA operator, we develop a multi-criteria group decision making approach, in which the criteria values are intuitionistic linguistic numbers and the criteria weight information is known completely. Finally, an example is given to illustrate the feasibility and effectiveness of the developed method.

Keywords: Multi-criteria group decision making, intuitionistic linguistic number, intuitionistic linguistic ordered weighted averaging (ILOWA) operator, intuitionistic linguistic hybrid aggregation (ILHA) operator

1. Introduction

In the socio-economic activities, there are a lot of multi-criteria decision making problems. A multi-criteria decision making problem is to find a desirable solution from a finite number of feasible alternatives assessed on multiple criteria, both quantitative and qualitative. Depending on quantitative aspects presented by each decision making problem we can handle different types of precise numerical values, but in other cases, the problems present qualitative aspects that are complex to assess by means of exact values. In the latter case, the use of the fuzzy linguistic approaches [1–14] has provided very good results. It handles qualitative aspects that are represented in qualitative terms by means of linguistic variables, that is, variables whose values are not numbers but linguistic terms, such as “poor”, “slightly poor”, “fair”, “slightly good”, “good”, etc.

In the last decades, a number of linguistic multi-criteria decision making problems were studied and many linguistic aggregation operators were presented [2, 5, 15–24]. Herrera et al. [2] proposed a consensus model for group decision making under linguistic assessments information. Herrera et al. [15] developed
generalized IFHA (GIFHA) operator. Xia and Xu [43] operator, generalized IFOW A (GIFOW A) operator and operators referred to as the generalized IFW A (GIFW A) (IFHA) operator. Zhao et al. [42] extended the IFW A, operator and intuitionistic fuzzy hybrid aggregation (IFOW A) operator. Xu and Xia [44] proposed some induced generalized intuitionistic fuzzy aggregation operators, including the induced generalized intuitionistic fuzzy Choquet integral operators and induced generalized intuitionistic fuzzy Dempster-Shafer operators. In addition, they established various properties of these operators and applied them in multi-criteria decision making fields.

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several group decision making processes using linguistic ordered weighted averaging (LOWA) operator. Herrera and Herrera-Viedma [5] presented three aggregation operators for weighted linguistic information, such as the linguistic weighted disjunction (LWD) operator, linguistic weighted conjunction (LWC) operator and linguistic weighted averaging (LWA) operator. Herrera and Herrera-Viedma [16] proposed three steps to follow in the linguistic decision analysis of group decision making problems with linguistic information. Xu [17] presented the linguistic hybrid aggregation (LHA) operator and applied it to group decision making. Tang and Zheng [18] developed a new linguistic modeling technique based on semantic similarity relation among linguistic labels. Wei [19] proposed some 2-tuple linguistic aggregation operators, such as the 2-tuple linguistic weighted harmonic averaging (TWHA) operator, 2-tuple linguistic ordered weighted harmonic averaging (TOWHA) operator and 2-tuple linguistic combined weighted harmonic averaging (TCWHA) operator, and developed a multi-criteria group decision making method based on the TWHA and TCWHA operators. Wei [20] proposed a grey relational analysis method for 2-tuple linguistic multi-criteria group decision making with incomplete weight information.

Atanassov [25] extended Zadeh’ fuzzy set [26] and introduced the notion of intuitionistic fuzzy set (IFS), which is characterized by a membership function and a non-membership function. Gau and Buehrer [27] presented the concept of vague set, but Basentine and Burillo [28] pointed out that vague sets are intuitionistic fuzzy sets. The IFS is more useful and effective in dealing with vagueness and uncertainty than the traditional fuzzy set [29–32], and has already been applied in many fields, such as decision analysis [33, 34], cluster analysis [35–37], pattern recognition [38, 39], forecasting [40], manufacturing grid [41] and so on.

Aggregation operators of intuitionistic fuzzy information and their applications to multi-criteria decision making is a new branch of IFS theory, which has attracted significant interest from researchers in recent years. Xu [34] proposed some intuitionistic fuzzy arithmetic aggregation operators, such as the intuitionistic fuzzy weighted averaging (IFWA) operator, intuitionistic fuzzy ordered weighted averaging (IFOWA) operator and intuitionistic fuzzy hybrid aggregation (IFHA) operator. Zhao et al. [42] extended the IFWA, IFOWA and IFHA operators to provide a new class of operators referred to as the generalized IFWA (GIFWA) operator, generalized IFOWA (GIFOWA) operator and generalized IFHA (GIFHA) operator. Xia and Xu [43] developed various generalized intuitionistic fuzzy point aggregation operators, such as the generalized intuitionistic fuzzy point weighted averaging (GIFPWA) operators, generalized intuitionistic fuzzy point ordered weighted averaging (GIFPOW A) operators and generalized intuitionistic fuzzy point hybrid averaging (GIFPHA) operators. Xu and Xia [44] proposed some induced generalized intuitionistic fuzzy aggregation operators, including the induced generalized intuitionistic fuzzy Choquet integral operators and induced generalized intuitionistic fuzzy Dempster-Shafer operators. In addition, they established various properties of these operators and applied them in multi-criteria decision making fields.

Shu, Wang, et al. extended the IFS and defined the concepts of intuitionistic triangular fuzzy number [45], intuitionistic trapezoidal fuzzy number [46] and intuitionistic linguistic number [47], respectively. Shu et al. [45] defined four operations of the intuitionistic triangular fuzzy numbers and used them in fault tree analysis. Wang and Li [47] proposed the intuitionistic linguistic weighted arithmetic averaging (ILWA) operator and developed a multi-criteria decision making approach in which the criteria values are intuitionistic linguistic numbers. Wang and Zhang [48], Wan and Dong [49] introduced some intuitionistic trapezoidal arithmetic aggregation operators, such as the intuitionistic trapezoidal weighted averaging (ITWA) operator, intuitionistic trapezoidal ordered weighted averaging (ITOWA) operator and intuitionistic trapezoidal hybrid aggregation (ITHA) operator, and developed several multi-criteria decision making approaches in which the criteria values are intuitionistic trapezoidal fuzzy numbers.

From the [47], we know that an intuitionistic linguistic number, characterized by a linguistic term, a membership function and a non-membership function, is a generalization of linguistic term and intuitionistic fuzzy number. In processes of cognition of things, people may not possess a precise or sufficient level of knowledge of the problem domain, due to the increasing complexity of the socio-economic environments. In such a case, they usually have some hesitation and indeterminacy in providing their linguistic evaluation values over the objects considered, which makes the results of cognitive performance reveal the characteristics of affirmation, negation and hesitation. As the linguistic term and intuitionistic fuzzy number cannot be used to completely express all the information in a situation as such, their applications are limited. The intuitionistic linguistic number can describe the fuzzy
characters of things more detailedly and comprehensively, therefore, it is more suitable and reasonable to express decision making information taking the form of intuitionistic linguistic numbers rather than linguistic terms and intuitionistic fuzzy numbers. In addition, the ILWAA operator presented in this reference weights only the intuitionistic linguistic numbers. To solve the drawback, we shall propose an intuitionistic linguistic ordered weighted averaging (ILOWA) operator. The ILOWA operator first reorders all the given intuitionistic linguistic numbers in descending order and then weights these reordered arguments, and finally aggregates all these ordered weighted arguments into a collective one. The fundamental characteristic of the ILOWA operator is to weight the ordered positions of the intuitionistic linguistic numbers instead of weighing the arguments themselves. Furthermore, weights represent different aspects in both the ILWAA operator and the ILOWA operator, and both the operators consider only one of them. Thus, in the following we shall propose another new aggregation operator called the intuitionistic linguistic hybrid aggregation (ILHA) operator. The ILHA operator first weights the given intuitionistic linguistic numbers, and then reorders the weighted arguments in descending order and weights these reordered arguments by the ILHA weights, and finally aggregates all the weighted arguments into a collective one. Obviously, the ILHA operator generalizes both the ILWAA operator and the ILOWA operator at the same time, and reflects the importance degrees of both the given intuitionistic linguistic numbers and the ordered positions of these arguments. In order to do so, we organize the paper as follows. In Section 2, we define a new score function and a new accuracy function of intuitionistic linguistic numbers, and based on these two functions, propose a simple approach for the comparison between two intuitionistic linguistic numbers. In Section 3, we propose two new aggregation operators called the intuitionistic linguistic ordered weighted averaging (ILOWA) operator and the intuitionistic linguistic hybrid aggregation (ILHA) operator, and study some desirable properties of these two operators. In Section 4, based on the ILHA operator and the ILWAA operator, we develop a multi-criteria group decision making approach, in which the criteria values are intuitionistic linguistic numbers and the criteria weight information is known completely. In Section 5, an illustrative example is given to verify the developed approach and to demonstrate its feasibility and effectiveness. Finally, conclusions of this paper are presented in Section 6.

2. Preliminaries

Definition 1 [2, 14]. Let $S = \{x_\theta | \theta = 0, 1, \cdots, 2l\}$, in which $l$ is a positive integer, $s_\theta$ represents a possible value for a linguistic variable, and it should satisfy the following characteristics:

1) The set is ordered: $s_a > s_b$ if $a > b$;
2) There is the negation operator: $\text{neg}(s_\theta) = s_\theta$ such that $a + b = 2l$.

then we call $S$ a discrete linguistic term set.

For example, let $l = 4$, then $S$ can be defined as:

$S = \{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{slightly poor}, s_4 = \text{fair}, s_5 = \text{slightly good}, s_6 = \text{good}, s_7 = \text{very good}, s_8 = \text{extremely good}\}$.

To preserve all the given information, the discrete linguistic term set $S$ should be extended to a continuous linguistic term set $\tilde{S} = \{\tilde{x}_\theta | \theta \in [0, q]\}$, in which $\tilde{s}_a > \tilde{s}_b$ if $a > b$, and $q (q > 2l)$ is a sufficiently large positive integer. If $\tilde{s}_\theta \in \tilde{S}$, then we call $\tilde{s}_\theta$ the original linguistic term, otherwise, we call $\tilde{s}_\theta$ the virtual linguistic term.

Let $s_a, s_b, s_c \in \tilde{S}$, then the following operational laws are valid [14]:

1) $s_a + s_b = s_c$;
2) $\lambda s_a = s_c, \lambda \in [0, 1]$.

Definition 2 [25]. Let $X$ be a universe of discourse, then an IFS $V$ in $X$ is given by:

$$V = \{(x, (\mu_V(x)) (\nu_V(x))) | x \in X\}$$

where the functions $\mu_V : X \rightarrow \{0, 1\}$ and $\nu_V : X \rightarrow [0, 1]$ determine the degree of membership and the degree of non-membership of the element $x$ to $V$, respectively, and for every $x \in X$:

$$0 \leq \mu_V(x) + \nu_V(x) = 1$$

Let $\pi_V(x) = 1 - \mu_V(x) - \nu_V(x)$ for all $x \in X$, then $\pi_V(x)$ is called the degree of hesitancy or indeterminacy of $x$ to $V$.

Generally, for convenience, $\alpha = (\mu_A, \nu_A)$ is called an intuitionistic fuzzy number, where $\mu_A \in [0, 1], \nu_A \in [0, 1]$ and $\mu_A + \nu_A \leq 1$.

Based on the linguistic term set and the IFS, Wang and Li [47] defined the intuitionistic linguistic number set as follows.

Definition 3 [47]. Let $X$ be a universe of discourse, then $\tilde{S}$, an intuitionistic linguistic number set $A$ in $X$ is an object having the following form:

$$A = \{(x, (\tilde{\mu}_A(x), \tilde{\nu}_A(x))) | x \in X\}$$
which is characterized by a linguistic term \(s_{\theta x_1}\), a membership function \(\mu_A\) and a non-membership function \(\nu_A\) of the element \(x\) to \(s_{\theta x_1}\), where

\[
\mu_A : X \rightarrow [0, 1], \quad x \rightarrow s_{\theta x_1} \rightarrow \mu_A(x) \\
\nu_A : X \rightarrow [0, 1], \quad x \rightarrow s_{\theta x_1} \rightarrow \nu_A(x)
\]

with the condition

\[
0 \leq \mu_A(x) + \nu_A(x) \leq 1, \quad x \in X
\]

Let \(\pi(x) = 1 - \mu_A(x) - \nu_A(x)\) for all \(x \in X\), then \(\pi(x)\) is called the degree of hesitancy of \(x\) to \(s_{\theta x_1}\).

When \(\mu_A(x) = 1\), \(\nu_A(x) = 0\), the intuitionistic linguistic number set is reduced to the linguistic term set.

For convenience, \(\beta = (\langle s_{\theta x_1}, \mu(\beta)\rangle, \nu(\beta))\) is called an intuitionistic linguistic number, where \(\tilde{s}_{\theta x_1}\) is a linguistic term, \(\mu(\beta) \in [0, 1], \nu(\beta) \in [0, 1], \mu(\beta) + \nu(\beta) \leq 1\), and let \(\Omega\) be the set of all intuitionistic linguistic numbers.

For example, \(\beta = (s_{0.5}, 0.6, 0.3)\) is an intuitionistic linguistic number, and from it, we know that the degree that the evaluation object belongs to \(s_{0.5}\) is 0.6, the degree that the evaluation object doesn’t belong to \(s_{0.5}\) is 0.3 and the degree of decision maker’s hesitancy is 0.1.

Obviously, the intuitionistic linguistic number generalizes both the linguistic term and the intuitionistic fuzzy number at the same time, and it is more practical and reasonable to express decision making information taking the form of intuitionistic linguistic numbers rather than linguistic terms and intuitionistic fuzzy numbers.

In the following, we introduce some operational laws of the intuitionistic linguistic numbers.

**Definition 4** [47]. Let \(\beta_1 = (\langle s_{\theta x_1}, \mu(\beta_1)\rangle, \nu(\beta_1))\) and \(\beta_2 = (\langle s_{\theta x_1}, \mu(\beta_2)\rangle, \nu(\beta_2))\) be two intuitionistic linguistic numbers, then

1. \(\beta_1 \oplus \beta_2 = (\langle s_{\theta x_1}, \mu(\beta_1) + \nu(\beta_1)\rangle, \nu(\beta_2))\) and \(\beta_2 \oplus \beta_1 = (\langle s_{\theta x_1}, \mu(\beta_2) + \nu(\beta_2)\rangle, \nu(\beta_1))\),
2. \(\lambda \beta_1 = (\langle s_{\theta x_1}, \mu(\beta_1)\rangle, \nu(\beta_1))\), \(\lambda \in [0, 1]\).

It can be easily proven that all the above results are also intuitionistic linguistic numbers. Based on Definition 4, we can further obtain the following relation.

1. \(\beta_1 \oplus \beta_2 = \beta_2 \oplus \beta_1\);
2. \((\beta_1 \oplus \beta_2) \oplus \beta_3 = \beta_1 \oplus (\beta_2 \oplus \beta_3)\);
3. \(\lambda(\beta_1 \oplus \beta_2) = \lambda \beta_1 \oplus \lambda \beta_2, \lambda \in [0, 1]\);
4. \(\lambda_1 \beta_1 + \lambda_2 \beta_2 = (\lambda_1 + \lambda_2)\beta_1 + \lambda_2 \beta_2, \lambda_1, \lambda_2 \in [0, 1]\).

The basis of the concepts of maximum expected values, minimum expected values and compromise expected values, Wang and Li [47] defined a score function and an accuracy function for the comparison between two intuitionistic linguistic numbers, but these two functions are cumbersome. In this paper, we define a new score function and a new accuracy function of the intuitionistic linguistic numbers.

**Definition 5.** Let \(\beta = (\langle s_{\theta x_1}, \mu(\beta)\rangle, \nu(\beta))\) be an intuitionistic linguistic number, the score of \(\beta\) can be evaluated by a new score function \(h(\beta)\) shown as

\[
h(\beta) = \theta(\beta) - \nu(\beta)
\]

where \(h(\beta) \in [-q, q]\).

**Definition 6.** Let \(\beta = (\langle s_{\theta x_1}, \mu(\beta)\rangle, \nu(\beta))\) be an intuitionistic linguistic number, the degree of accuracy of \(\beta\) can be evaluated by a new accuracy function \(H(\beta)\) shown as

\[
H(\beta) = \theta(\beta) + \nu(\beta)
\]

where \(H(\beta) \in [0, q]\).

Hong and Choi [33] showed that the relation between the score function and the accuracy function is similar to the relation between mean and variance in statistics. So, by using expectation-variance principle, we propose a simple method for the comparison between two intuitionistic linguistic numbers, which is based on the score function \(h(\beta)\) and the accuracy function \(H(\beta)\) and defined as follows.

**Definition 7.** Let \(\beta_1 = (\langle s_{\theta x_1}, \mu(\beta_1)\rangle, \nu(\beta_1))\) and \(\beta_2 = (\langle s_{\theta x_1}, \mu(\beta_2)\rangle, \nu(\beta_2))\) be two intuitionistic linguistic numbers, then

1. If \(h(\beta_1) < h(\beta_2)\), then \(\beta_1\) is smaller than \(\beta_2\), denoted by \(\beta_1 < \beta_2\);
2. If \(h(\beta_1) = h(\beta_2)\), then \(\beta_1\) is equal to \(\beta_2\), denoted by \(\beta_1 = \beta_2\);
3. If \(H(\beta_1) < H(\beta_2)\), then \(\beta_1\) is smaller than \(\beta_2\), denoted by \(\beta_1 < \beta_2\);
4. If \(H(\beta_1) > H(\beta_2)\), then \(\beta_1\) is bigger than \(\beta_2\), denoted by \(\beta_1 > \beta_2\).

3. Intuitionistic linguistic aggregation operators

Based on Definition 4, Wang and Li [47] presented the ILWAA operator, but the ILWAA operator weights only the intuitionistic linguistic numbers. In the following, we shall propose two new intuitionistic linguistic aggregation operators, such as the ILOWA operator and
the ILHA operator, and investigate various properties of these operators.

**Definition 8** [47]. Let \( \beta_j = (\mu(\beta_j), \nu(\beta_j), s(\beta_j)) \) \( (j = 1, 2, \ldots, n) \) be a collection of intuitionistic linguistic numbers and let ILWA: \( \Omega^n \rightarrow \Omega \), if

\[
\text{ILWA}_n(\beta_1, \beta_2, \ldots, \beta_n) = w_1\beta_1 \oplus w_2\beta_2 \oplus \cdots \oplus w_n\beta_n
\]

then ILWA is called an intuitionistic linguistic weighted arithmetic averaging operator of dimension \( n \), where \( w = (w_1, w_2, \ldots, w_n)^T \) is the weight vector of \( \beta_j \) \( (j = 1, 2, \ldots, n) \), with \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \).

1. Especially, if \( w = (1/n, 1/n, \ldots, 1/n)^T \), then the ILWA operator is reduced to the intuitionistic linguistic arithmetic averaging (ILAA) operator of dimension \( n \), which is defined as follows:

\[
\text{ILAA}_n(\beta_1, \beta_2, \ldots, \beta_n) = \frac{1}{n}(\beta_1 \oplus \beta_2 \oplus \cdots \oplus \beta_n)
\]

**Theorem 1.** Let \( \beta_j = (\mu(\beta_j), \nu(\beta_j), s(\beta_j)) \) \( (j = 1, 2, \ldots, n) \) be a collection of intuitionistic linguistic numbers, and \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector of \( \beta_j \) \( (j = 1, 2, \ldots, n) \), with \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \), then their aggregated value by using the ILAA operator is also an intuitionistic linguistic number.

\[
\text{ILAA}_n(\beta_1, \beta_2, \ldots, \beta_n) = \left\{ \begin{array}{l}
\sum_{j=1}^{n} w_j(\beta_j) \\
\sum_{j=1}^{n} w_j(\beta_j) + \sum_{j=1}^{n} w_j(\beta_j) + \sum_{j=1}^{n} w_j(\beta_j)
\end{array} \right.
\]

**Proof.** Obviously, from Definition 4, the aggregated value by using the ILAA operator is also an intuitionistic linguistic number. In the following, we prove Equation (5) by using mathematical induction on \( n \).

1) For \( n = 2 \), since

\[
\begin{align*}
\text{ILAA}_2(\beta_1, \beta_2) &= \sum_{j=1}^{2} w_j(\beta_j) \\
\text{ILAA}_2(\beta_1, \beta_2) &= \sum_{j=1}^{2} w_j(\beta_j) + \sum_{j=1}^{2} w_j(\beta_j) + \sum_{j=1}^{2} w_j(\beta_j)
\end{align*}
\]

then

\[
\text{ILAA}_2(\beta_1, \beta_2) = w_1\beta_1 \oplus w_2\beta_2
\]

2) If equation (5) holds for \( n = k \), that is

\[
\text{ILAA}_k(\beta_1, \beta_2, \ldots, \beta_k) = \left\{ \begin{array}{l}
\sum_{j=1}^{k} w_j(\beta_j) \\
\sum_{j=1}^{k} w_j(\beta_j) + \sum_{j=1}^{k} w_j(\beta_j) + \sum_{j=1}^{k} w_j(\beta_j)
\end{array} \right.
\]

Then, when \( n = k + 1 \), by Definition 4, we have

\[
\begin{align*}
\text{ILAA}_{k+1}(\beta_1, \beta_2, \ldots, \beta_k, \beta_{k+1}) &= \left\{ \begin{array}{l}
\sum_{j=1}^{k+1} w_j(\beta_j) \\
\sum_{j=1}^{k+1} w_j(\beta_j) + \sum_{j=1}^{k+1} w_j(\beta_j) + \sum_{j=1}^{k+1} w_j(\beta_j)
\end{array} \right.
\end{align*}
\]
Let \( \omega \) be a collection of intuitionistic linguistic numbers. An intuitionistic linguistic ordered weighted averaging (ILOWA) operator of dimension \( n \) is a mapping \( \text{ILOWA} : \Omega^n \rightarrow \Omega \), that has an associated weight vector \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) such that \( \omega_j \in [0, 1] \) and \( \sum \omega_j = 1 \). Furthermore, \( \text{ILOWA}_\omega(\beta_1, \beta_2, \ldots, \beta_n) = \sum_{j=1}^n \omega_j \beta_j \) is a permutation of \( (\beta_1, \beta_2, \ldots, \beta_n) \) that such that \( \beta_j \geq \beta_{j+1} \), for all \( j \). Especially, if \( \omega = (1/n, 1/n, \ldots, 1/n)^T \), then the ILOWA operator is reduced to the ILAA operator.

Similar to Theorem 1, we have the following.

**Theorem 2.** Let \( \beta_j = (\omega_{0j}, \mu(\beta_j), \nu(\beta_j)) \) (\( j = 1, 2, \ldots, n \)) be a collection of intuitionistic linguistic numbers, then their aggregated value by using the ILOWA operator is also an intuitionistic linguistic number, and

\[
\text{ILOWA}_\omega(\beta_1, \beta_2, \ldots, \beta_n) = \sum_{j=1}^n \omega_j \beta_j
\]

where \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) is the weight vector related to the ILOWA operator, with \( \omega_j \in [0, 1] \) and \( \sum \omega_j = 1 \), which can be determined similar to the OWA weights (for example, we can use the normal distribution based method [50]).

The ILOWA operator has the following properties.

**Theorem 3.** Let \( \beta_j = (\omega_{0j}, \mu(\beta_j), \nu(\beta_j)) \) (\( j = 1, 2, \ldots, n \)) be a collection of intuitionistic linguistic numbers and \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) is the weight vector related to the ILOWA operator, with \( \omega_j \in [0, 1] \) and \( \sum \omega_j = 1 \), then

1) (Idempotency) If all \( \beta_j \) (\( j = 1, 2, \ldots, n \)) are equal, i.e. \( \beta_j = \beta \) for all \( j \), then

\[
\text{ILOWA}_\omega(\beta, \beta, \ldots, \beta) = \beta.
\]

2) (Boundary)

\[
\beta^- \leq \text{ILOWA}_\omega(\beta_1, \beta_2, \ldots, \beta_n) \leq \beta^+,
\]

where \( \beta^- = (\min_{j} \omega_{0j}), \min_{j} \mu(\beta_j), \max_{j} \nu(\beta_j)) \), \( \beta^+ = (\max_{j} \omega_{0j}), \max_{j} \mu(\beta_j), \min_{j} \nu(\beta_j)) \).

3) (Monotonicity) Let \( \beta^*_j = (\omega_{0j}, \mu(\beta^*_j), \nu(\beta^*_j)) \) (\( j = 1, 2, \ldots, n \)) be a collection of intuitionistic linguistic numbers, if \( \omega_{0j} \leq \omega_{0j}', \mu(\beta_j) \leq \mu(\beta'_j), \nu(\beta_j) \geq \nu(\beta'_j) \), for all \( j \), then

\[
\text{ILOWA}_\omega(\beta_1, \beta_2, \ldots, \beta_n) \leq \text{ILOWA}_\omega(\beta'_1, \beta'_2, \ldots, \beta'_n)
\]

4) (Commutativity) Let \( \beta_j = (\omega_{0j}, \mu(\beta_j), \nu(\beta_j)) \) (\( j = 1, 2, \ldots, n \)) be a collection of intuitionistic
linguistic numbers, then
\[
\text{ILW}_{\omega}(\beta_1, \beta_2, \ldots, \beta_n) = \text{ILW}_{\omega}(\beta_1, \beta_2, \ldots, \beta_n),
\]
where \((\beta_1, \beta_2, \ldots, \beta_n)\) is any permutation of \((\beta_1, \beta_2, \ldots, \beta_n)\).

**Theorem 4.** Let \(\beta_j = (w_{\beta_j}, \mu(\beta_j), \nu(\beta_j))\) \((j = 1, 2, \ldots, n)\) be a collection of intuitionistic linguistic numbers and \(\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T\) is the weight vector related to the ILOWA operator, with \(\omega_j \in [0, 1]\) and \(\sum_{j=1}^n \omega_j = 1\), then
1) If \(\omega = (1, 0, \ldots, 0)^T\), then
\[
\text{ILOWA}_{\omega}(\beta_1, \beta_2, \ldots, \beta_n) = \max(\beta_j).
\]
2) If \(\omega = (0, 0, \ldots, 1)^T\), then
\[
\text{ILOWA}_{\omega}(\beta_1, \beta_2, \ldots, \beta_n) = \min(\beta_j).
\]
3) If \(\omega_j = 1, \omega_k = 0\), and \(j \neq k\), then
\[
\text{ILOWA}_{\omega}(\beta_1, \beta_2, \ldots, \beta_n) = \beta_{\tau(j)}
\]
where \(\beta_{\tau(j)}\) is the largest of \(\beta_j\) \((j = 1, 2, \ldots, n)\).

From Definition 8 and 9, we know that the ILWAA operator weights only the intuitionistic linguistic numbers, whereas the ILOWA operator weights only the ordered positions of the intuitionistic linguistic numbers instead of weighting the arguments themselves. To overcome this limitation, in what follows, we develop an ILHA operator, which weights both the given intuitionistic linguistic numbers and their ordered positions.

**Definition 10.** Let \(\beta_j = (w_{\beta_j}, \mu(\beta_j), \nu(\beta_j))\) \((j = 1, 2, \ldots, n)\) be a collection of intuitionistic linguistic numbers. An intuitionistic linguistic hybrid aggregation (ILHA) operator of dimension \(n\) is a mapping ILHA: \(\Omega^n \rightarrow \Omega\), which has an associated vector \(\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T\) with \(\omega_j \in [0, 1]\) and \(\sum_{j=1}^n \omega_j = 1\), such that
\[
\text{ILHA}_{\omega}(\beta_1, \beta_2, \ldots, \beta_n) = \omega \beta_1^1(1) \oplus \omega \beta_1^2(2) \oplus \cdots \oplus \omega \beta_n(\omega) \quad (8)
\]
where \(\beta_{\tau(j)}(j)\) is the \(j\)th largest of weighted intuitionistic linguistic numbers \((w_{\beta_1}, w_{\beta_2}, w_{\beta_3}, \ldots, w_{\beta_n})\), \(w = (w_1, w_2, \ldots, w_n)^T\) is the weight vector of \(\beta_j\), with \(w_j \in [0, 1]\) and \(\sum_{j=1}^n w_j = 1\), and \(n\) is the balancing coefficient, which plays a role of balance.

**Theorem 5.** Let \(\beta_j = (w_{\beta_j}, \mu(\beta_j), \nu(\beta_j))\) \((j = 1, 2, \ldots, n)\) be a collection of intuitionistic linguistic numbers and \(\beta_{\tau(j)}(j)\) is the \(j\)th largest of the intuitionistic linguistic numbers \((w_{\beta_1}, w_{\beta_2}, w_{\beta_3}, \ldots, w_{\beta_n})\), with \(w_j \in [0, 1]\) and \(\sum_{j=1}^n w_j = 1\), then
\[
\text{ILHA}_{\omega}(\beta_1, \beta_2, \ldots, \beta_n) = \left(\frac{\sum_{j=1}^n \omega_j(\beta_{\tau(j)}(1)^T)}{\sum_{j=1}^n \omega_j(\beta_{\tau(j)}(1)^T)}, \frac{\sum_{j=1}^n \omega_j(\beta_{\tau(j)}(2)^T)}{\sum_{j=1}^n \omega_j(\beta_{\tau(j)}(1)^T)}, \ldots, \frac{\sum_{j=1}^n \omega_j(\beta_{\tau(j)}(\omega)^T)}{\sum_{j=1}^n \omega_j(\beta_{\tau(j)}(1)^T)}\right) \quad (9)
\]
and the aggregated value derived by using the ILHA operator is also an intuitionistic linguistic number.

**Theorem 6.** If \(\omega = (1/n, 1/n, \ldots, 1/n)^T\), the ILHA operator is reduced to the ILOWA operator.

**Theorem 7.** If \(\omega = (1/n, 1/n, \ldots, 1/n)^T\), the ILHA operator is reduced to the ILWAA operator.

Obviously, the ILHA operator generalizes both the ILWAA operator and the ILOWA operator at the same time, and reflects the importance degrees of both the given intuitionistic linguistic numbers and the ordered positions of these arguments.

**Example.** Suppose \(\beta_1 = (x_{14}, 0.8, 0.1), \beta_2 = (x_{40}, 0.7, 0.2), \beta_3 = (x_{31}, 0.9, 0.1), \beta_4 = (x_{35}, 0.7, 0.1)\) and \(\beta_5 = (x_{38}, 0.8, 0.2)\) are five intuitionistic linguistic numbers, and \(w = (0.22, 0.26, 0.15, 0.17, 0.20)^T\) is the weight vector of \(\beta_j\) \((j = 1, 2, \ldots, 5)\), then by the operational law in Definition 4, we get the weighted intuitionistic linguistic numbers as
\[
\beta'_1 = (x_{40}, 0.8, 0.1), \quad \beta'_2 = (x_{70}, 0.7, 0.2),
\]
\[
\beta'_3 = (x_{31}, 0.9, 0.1), \quad \beta'_4 = (x_{35}, 0.7, 0.1), \quad \beta'_5 = (x_{38}, 0.8, 0.2).
\]

By Equation (1) in Definition 5, we calculate the scores of \(\beta'_j\) \((j = 1, 2, \ldots, 5)\):
Then, by (9), it follows that $h(p_1^k) > h(p_2^k) > h(p_3^k) > h(p_4^k)$ then
\[ \beta_{ij}^{(1)} = (7.70, 0.7, 0.2), \quad \beta_{ij}^{(2)} = (4.40, 0.8, 0.1), \quad \beta_{ij}^{(3)} = (3.15, 0.9, 0.1), \quad \beta_{ij}^{(4)} = (3.30, 0.8, 0.2), \]
\[ \beta_{ij}^{(5)} = (5.25, 0.7, 0.1). \]
Suppose that $s_0 = (0.1117, 0.2365, 0.3206, 0.2365, 0.1117)^T$ is the weight vector related to the ILHA operator (derived by the normal distribution based method [50]), then, by (9), it follows that
\[ \text{ILHA}_{w_d}(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) = (4.0447, 0.7996, 0.1391). \]

4. A group decision making method based on the ILHA and ILWAA operator

In the following, based on the ILHA and ILWAA operator, we develop a multi-criteria group decision making method, in which the criteria values are intuitionistic linguistic numbers and the criteria weight information is known completely.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives, and $B = \{B_1, B_2, \ldots, B_n\}$ be a set of criteria, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $B_j$ ($j = 1, 2, \ldots, n$), where $w_j \in [0, 1]$. Let $D = \{d_1, d_2, \ldots, d_t\}$ be a set of decision makers, and $e = (e_1, e_2, \ldots, e_t)^T$ be the weight vector of $d_k$ ($k = 1, 2, \ldots, t$) with $e_k \in [0, 1]$ and $\sum e_k = 1$. Assume that decision makers $D = \{d_1, d_2, \ldots, d_t\}$ represent the characteristics of the alternatives $A_i$ ($i = 1, 2, \ldots, m$) by the intuitionistic linguistic numbers $\beta_i^{(1)}, \beta_i^{(2)}, \ldots, \beta_i^{(n)}$ and derive the decision matrix $B_k = (\beta_i^{(j)})_{m \times n}$, where $\beta_i^{(j)}$ are the linguistic evaluation values of $A_i$ with respect to $B_j$, and $w_{ik}$ indicate, respectively, the degree of membership and the degree of non-membership that $A_i$ attaches to $s_{\beta_i^{(j)}}$ with respect to $B_j$ ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; k = 1, 2, \ldots, t$).

To get the best alternative(s), the group decision making procedure is given as follows.

**Step 1.** Normalize the decision making information in the matrix $R_k$ ($k = 1, 2, \ldots, t$). For the benefit-type criteria, we do nothing; for the cost-type criteria, we utilize the linguistic negation operator $\lambda_{\theta}(\beta_i^{(j)}) = \text{neg} \{s_{\beta_i^{(j)}}\} = s_{2-\beta_i^{(j)}}$ to make linguistic evaluation values be normalized.

For convenience, the normalized criteria values of the alternatives $A_i$ ($i = 1, 2, \ldots, m$) with respect to the criteria $B_j$ ($j = 1, 2, \ldots, n$) are also denoted by $\beta_i^{(j)} = (s_{\beta_i^{(j)}}, v_{\beta_i^{(j)}}) = (s_{\beta_i^{(j)}}, v_{\beta_i^{(j)}})$.\[ \text{ILWAA}_{v} (\beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_m^{(1)}) \]

**Step 2.** Utilize the ILWAA operator
\[ \beta_i^{(1)} = \text{ILWAA}_{v} (\beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_m^{(1)}) \]

**Step 3.** Utilize the ILHA operator
\[ \beta_i^{(1)} = \text{ILHA}_{w} (\beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_m^{(1)}) \]

**Step 4.** Utilize equation (1) to calculate the scores $h(\beta_i)$ of the collective overall values $\beta_i$ of the alternatives $A_i$ ($i = 1, 2, \ldots, m$).

**Step 5.** By Definition 7, utilize the scores $h(\beta_i)$ ($i = 1, 2, \ldots, m$) to rank the alternatives $A_i$ ($i = 1, 2, \ldots, m$), and then select the best one(s) (if there is no difference between two scores $h(\beta_i)$ and $h(\beta_j)$, then we need to calculate the accuracy degrees $H(h(\beta_i))$ and $H(h(\beta_j))$ of the collective overall values $\beta_i$ and $\beta_j$ by equation (2), respectively, and then rank the alternatives $A_i$ and $A_j$ in accordance with the accuracy degrees $H(h(\beta_i))$ and $H(h(\beta_j))$).
5. Illustrative example

In this section, a multi-criteria group decision making problem involving a company’s making a project investment is used to illustrate the developed method using the ILHA and ILWAA operator.

Consider that a risk investment company wants to make a project investment, which will be chosen from four alternative enterprises \( A_i \) \((i = 1, 2, 3, 4)\) by three decision makers whose weight vector is \( \omega = (0.3, 0.4, 0.3)^T \). In assessing the potential contribution of each enterprise, three factors are considered, \( B_1\)–profitability, \( B_2\)–competitiveness, and \( B_3\)–risk affordability. Suppose that the weight vector of \( B_j \) \((j = 1, 2, 3)\) is \( \omega_{B_j} = (0.3727, 0.3500, 0.2773)^T \), and the characteristic information of the alternatives \( A_i \) \((i = 1, 2, 3, 4)\) with respect to the criteria \( B_j \) \((j = 1, 2, 3)\) given by decision makers \( d_k \) \((k = 1, 2, 3)\) are represented by the intuitionistic linguistic numbers \( \beta_{ij}^{(k)} = (\lambda^{(k)}_{ij}, \mu^{(k)}_{ij}, \nu^{(k)}_{ij}) \) listed in Tables 1, 2 and 3.

Because all criteria \( B_j \) \((j = 1, 2, 3)\) are benefit-type, their values need not to be normalized. In the following, we utilize the developed approach to get the best investment enterprise.

Step 1. Utilize Equation (10) to aggregate the criteria values of the \( i \)th row of the decision matrix \( R_1 \) and derive the individual overall values \( \beta_i^{(k)} \) of the alternatives \( A_i \) by the decision makers \( d_k \) \((i = 1, 2, 3, 4; k = 1, 2, 3)\).

\[
\beta_i^{(1)} = (a_{4,2773}, 0.7349, 0.1976),
\beta_i^{(2)} = (a_{4,7000}, 0.7870, 0.1683),
\beta_i^{(3)} = (a_{4,9773}, 0.7557, 0.2443),
\beta_i^{(4)} = (a_{4,3272}, 0.8000, 0.1680),
\beta_i^{(2)} = (a_{4,2546}, 0.7915, 0.1263),
\beta_i^{(3)} = (a_{4,5139}, 0.7753, 0.1614),
\beta_i^{(4)} = (a_{4,2773}, 0.7631, 0.1667),
\beta_i^{(2)} = (a_{5,3272}, 0.7460, 0.1720),
\beta_i^{(3)} = (a_{5,7000}, 0.8187, 0.1570),
\beta_i^{(4)} = (a_{4,2773}, 0.8327, 0.1000),
\beta_i^{(2)} = (a_{4,3272}, 0.8371, 0.1395),
\beta_i^{(3)} = (a_{4,5139}, 0.8228, 0.1386).
\]

Step 2. Utilize Equation (11) to aggregate all the individual overall values \( \beta_i^{(1)}, \beta_i^{(2)}, \beta_i^{(3)} \) and derive the collective overall values \( \beta_i \) of the alternatives \( A_i \) \((i = 1, 2, 3, 4)\), where the weight vector of ILHA operator is \( \omega = (0.2429, 0.5142, 0.2429)^T \) which is determined by the weighting method based on normal distribution [50].

\[
\beta_1 = (a_{4,0684}, 0.7608, 0.2122),
\beta_2 = (a_{4,2933}, 0.7855, 0.1527),
\beta_3 = (a_{5,3924}, 0.7854, 0.1634),
\beta_4 = (a_{4,3467}, 0.8318, 0.1213).
\]

Step 3. By Equation (3), we calculate the scores \( h(\beta_i) \) of alternatives \( A_i \) \((i = 1, 2, 3, 4)\).

\[
h(\beta_1) = 2.5280, h(\beta_2) = 2.7146, h(\beta_3) = 3.3540, h(\beta_4) = 3.0885.
\]

Step 4. By Definition 7, we obtain that

\[ h(\beta_1) > h(\beta_2) > h(\beta_3) > h(\beta_4) \]

then

\[ A_1 > A_4 > A_2 > A_3. \]

So the best investment enterprise is \( A_3 \).
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