Optimal Control and $H_{\infty}$ Output Feedback Design Options for Active Magnetic Bearing Spindle Position Regulation

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Abstract—For the demand of high speed and high accuracy, the use of active magnetic bearing (AMB) plays a key role in various industries such as clean rooms, compressors and satellites due to their contactless nature. In this research, two other control options for high speed machine were designed based on the optimal output feedback and $H_{\infty}$ output feedback control methods to improve the radial and axial position regulation of AMB. The output feedback control gain matrix with the minimum performance index is obtained by solving the Riccati equation and fed back to the system in order to achieve the system’s performance. The above designed controllers can efficiently regulate the radial and axial directions position deviation of for AMB systems. Simulations for the two control methods were carried out using Matlab and Simulink for AMB system models. Results show that the $H_{\infty}$ output feedback controller has a better position deviation control performance over the optimal output feedback under condition of decreasing the disturbance of reaction. Finally, simulations results demonstrate that the $H_{\infty}$ Output Feedback is effective.

Index Terms—Active Magnetic Bearing, Disturbance Attenuation, $H_{\infty}$ Output Feedback.
feedback controller design, are introduced in Section 3 and Section 4. The simulations are discussed in Section 5. Finally, conclusions are drawn in Section 6.

II. DYNAMICS OF AMB SYSTEM

The studied magnetic bearing-rotor system for a high speed machine is shown in Fig. 1. Each electromagnet in the left- and right-side stators generates its electromagnetic force \( F_{ax}, F_{ay}, F_{bx}, F_{by}, \) and \( F_{z} \) with radial and axial components to regulate the radial and axial deviation of the rotor. To consider the effects of load disturbance, five representative external disturbance forces: \( f_{dax}, f_{day}, f_{dax}, f_{day}, f_{dax} \), shown in Fig. 2, are also included in the derivation of the dynamics model.

The platform is representative of realistic magnetic bearing-rotor system for a high speed machine shown in Fig. 2. O is defined as the mass center and also the origin of the coordinate \((x, y, z)\). \( \Omega \) is the spinning speed of the rotor. \( a \) and \( b \) is the angular displacement from the equilibrium point, about axis-\( x \) and axis-\( y \), respectively. \( l_b \) and \( l_a \) are the distances of AMB A, AMB B and (radial) gap sensor from the mass center of the rotor, respectively.

The rotor is assumed as a rigid body. By using Newton’s second law, the rotor has five degrees of freedom and its equations of motions can be described as follows:

\[
\begin{align*}
\dot{m}\ddot{x} &= F_{ax} + F_{bx} + f_{dax} \\
\dot{m}\ddot{y} &= F_{ay} + F_{by} + f_{day} \\
\dot{m}\ddot{z} &= F_{z} + mg \\
J_x\ddot{\theta_x} &= -l_yF_{ax} + l_yF_{bx} - J_x\Omega \dot{\theta_x} \\
J_y\ddot{\theta_y} &= l_xF_{ax} - l_xF_{bx} + J_y\Omega \dot{\theta_y}
\end{align*}
\]

where \( \theta_x \) and \( \theta_y \) are the angular displacements of the rod. \( m, J_x, J_y \) are the mass and transverse moment of inertia of the rotor about \( x \) (or \( y \)). \( F_{pq} \) \((p=a \text{ and } b, \text{ q=x and y})\) is the radial magnetic force induced by a pair of AMB in the lateral direction. \( F_{z} \) is the axial magnetic force. \( q=x \) and \( y \) is referred to the AMB set at (A) and \( q=y \) to the AMB set at (B) in Fig. 2, respectively. That is, control forces produced by AMB on both sides of the rotor can be give by

\[
\begin{align*}
F_{ax} &= k_{ax} \cdot x + k_{ax} \cdot i_{ax} + k_{ax} \cdot i_{by} \\
F_{ay} &= k_{ay} \cdot y - k_{ay} \cdot i_{by} + k_{ay} \cdot i_{ax} + G_a
\end{align*}
\]

Owing to small lateral position deviation of the rotor, the deviation of the rotor can be represented by the air gap with respect to AMB, that is

\[
\begin{align*}
x &= \frac{l_b}{l_a + l_b} x_a + \frac{l_a}{l_a + l_b} x_b \\
y &= \frac{l_a}{l_a + l_b} y_a + \frac{l_b}{l_a + l_b} y_b \\
\tan \theta_x \equiv \theta_x &= \frac{y_a - y_b}{l_a + l_b} \\
\tan \theta_y \equiv \theta_y &= \frac{x_a - x_b}{l_a + l_b}
\end{align*}
\]
The control output vector can be written as 
\[ U = \begin{bmatrix} i_a & i_b \end{bmatrix}^T \] where, for the AMB A, 2 phase control currents \( i_a \) and \( i_b \) that are transformed from 3 phase control currents \( i_{ax}, i_{bx}, i_{cy} \) and \( i_{cz} \). For the AMB B, the calculating method of control currents is the same as that of the AMB A in the \( x \)- and \( y \)-direction. \( i_c \) is control currents in the \( z \)-direction. Finally, define the entire state vector shown in Fig. 2 contains ten states and can be expressed as

\[ X = (x_a, x_y, y_y, z_y, \dot{x}_a, \dot{x}_y, \dot{y}_y, \dot{z}_y, \dot{x}_z, \dot{z}_z)^T \] (5)

The control output vector can be written as

\[ Y = (y_1, y_2, y_3, y_4, y_5)^T = (x_a, x_y, y_y, z_y, z_z)^T \] (6)

It can be seen from (4)-(6) that the state equation of the 5-degrees-of-freedom AMB Spindle is a 10-input and 5-output strong-coupled MIMO system. A step disturbances \( d(t) \) is introduced in simulations to verify disturbance rejection properties of the design. The state model for the magnetic bearing system can be obtained:

\[ \dot{x} = A(x) + Bu + Bd \]
\[ y = Cx \] (7)

where \( y \) is the measurement, \( A, B, C \) and \( D \) are system matrix, input matrix, output matrix and disturbance matrix, respectively, and defined as follows:

\[ A = \begin{bmatrix} 0_{5 \times 5} & I_{5 \times 5} \\ M & 0_{2 \times 2} & 0_{2 \times 2} & -N & 0_{2 \times 1} \\ 0_{2 \times 2} & 0_{2 \times 2} & 1/k_i & 0_{2 \times 2} & 0 \end{bmatrix} \]

\[ B = \begin{bmatrix} 0_{5 \times 5} \\ P & 0_{2 \times 1} \\ Q & 0_{2 \times 1} \\ 0_{2 \times 2} & 1/k_i \end{bmatrix} \]

In this research, we use three air gap sensors to measure the radial an axial deviations of rotor and results are then fed back to an on-board microprocessor. That is, one is to measure the rotor position deviation in \( x \)-axis while the other two is to measure the rotor position deviation in \( y \)-axis and \( z \)-axis, respectively. The 5-degrees-of-freedom AMB is a state-dependent system with nonnegligible external disturbance. Fig. 2 shows the state trajectories when the system is at rest and experiencing a disturbance which \( f_{ax} \) is affecting the \( x \)-direction and \( f_{by} \) is affecting the \( y \)-direction. Therefore, a control strategy to account for disturbances is absolutely required.
III. OPTIMAL OUTPUT FEEDBACK CONTROL ALGORITHM

We assume the plant equation is given by the linear time-invariant state variable model:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]

The optimal output feedback method works to minimize the following quadratic performance index (PI)

\[
J = \frac{1}{2} \mathbf{E} \left\{ \int_0^\infty \mathbf{x}'(t)Q\mathbf{x}(t) + \mathbf{u}'(t)R\mathbf{u}(t)dt \right\}
\]

where \(Q \succeq 0\) and \(R \succ 0\). This performance index is evaluated subject to the initial condition \(x_0\), Levine [9] has assumed \(x_0\) to be a random variable with known covariance \(E\{x_0x_0^T\} = X_0\), \(E\{\cdot\}\) represents the expected value. When the optimal control problem has a solution, the feedback gain is determined by solving the following three coupled nonlinear matrix design equations of the form [9]

\[
\begin{align*}
0 &= \frac{\partial H}{\partial S} = A_S' P + PA_S + C'TK^T RKC + Q \\
0 &= \frac{\partial H}{\partial P} = A_S + S A_S^T + X \\
0 &= \frac{1}{2} \frac{\partial H}{\partial K} = RKCSCT - B'TPSC^T
\end{align*}
\]

where \(A_S = A - BK\) is assumed to be stable closed loop system matrix and \(X = x(0)x'(0)\). The first two of these are Lyapunov equation and the third is an equation for the feedback gain \(K\). We can assume that \(R\) is positive definite and \(CSCT\) is nonsingular, so the optimal output feedback gain \(K\) can be obtained by

\[
K = R^{-1}B'TPSC^T(CSCT)^{-1}
\]

IV. OPTIMAL H∞ OUTPUT FEEDBACK CONTROL ALGORITHM

Optimal output feedback gain is determined by solving three coupled matrix design equations in [15-17] is a harder question than \(H_\infty\) output feedback control which does not require an initial stabilizing static output feedback gain. Consider the following controlled linear time-invariant system with disturbance,

\[
\begin{align*}
\dot{x} &= Ax + Bu + Bd \\
y &= Cx
\end{align*}
\]

where \(x \in \mathbb{R}^n\), \(u \in \mathbb{R}^m\), \(y \in \mathbb{R}^r\) are the state, input and output vectors, respectively, and \(A, B, C\) are matrices with constant real coefficients and appropriate dimensions. The dynamics system (14) is depicted in Figure 3. The output feedback stabilization problem for system (14) consists in finding, if possible, a control law described by

\[
u = -K\phi = -KCy
\]

where, \(K\) is an \(m \times p\) matrix of constant feedback coefficients to be chosen to minimize the performance index (PI) of the system. Where \(d\) is a disturbance input and \(z\) is the performance output that satisfies

\[
\int_0^\infty \mathbf{z}'(t)\mathbf{z}(t)dt - \int_0^\infty x'(t)Qx + u'du'dt
\]

System (14) is said to be output feedback stabilizable if there exists a real matrix \(F\) such that \(A + BK\) is stable. (A2) The pair \((A, C)\) is said to be detectable if there exists a real matrix \(L\) such that \(A + LC\) is stable.

Definition 1-L₂ Gain Performance: For a causal linear time-invariant system, designing the control system to minimize the \(L_2\)-gain from \(d\) to \(z\) is bounded by \(\gamma\) as shown in figure 1. The problem is to find a static output-feedback controller, so that

\[
\left\| T_{zd}(j\omega) \right\|_2 = \sup_{d \leq \gamma} \frac{\left\| z(j\omega) \right\|_2}{\left\| d(j\omega) \right\|_2} \leq \gamma
\]

For all \(T \geq 0\) and all \(d \in L_2(0, T)\), with \(x(0) = 0\). The \(L_2\)-gain problem is equivalent to the solvability of the zero-sum game [18].

\[
V(x_0) = \min_{d \in L_2(0, T)} \max_{u \in L_2(0, T)} \int_0^T (x'Qx + u'ru + 2u'dd' \gamma^2 \|d\|^2)dt
\]

This is known as the \(H_\infty\) output feedback control problem.

Theorem 1: if the matrix gain \(k \in \mathbb{R}^m \times \mathbb{R}^n\) stabilizes the closed loop system \(Ac \equiv A - BKC\) and satisfies \(KC = R^{-1}(B'TP + L)\) where \(L \in \mathbb{R}^m \times \mathbb{R}^n\) is an arbitrary matrix and \(P \in \mathbb{R}^n \times \mathbb{R}^n\) is the positive definite solution of the Riccati equation

\[
PA_A + A'_TP + 1/\gamma^2 PDD'TP + Q + C'TK^TRKC = 0
\]

then the matrix \((A - BKC)\) is asymptotically stable.

Proof: Since \(P\) is symmetric and positive definite, we introduce the Lyapunov function

\[
0 = \frac{\partial H}{\partial S} = A_S' P + PA_S + C'TK^T RKC + Q
\]
\[ V(x) = x^T P x \]

Its time derivative along any trajectory of the closed-loop system is given by
\[ \dot{V}(x) = x^T \left\{ (A - BK) P + P(A - BK) \right\} x - x^T \left\{ Q + C^T K^T R K \right\} x \]

Obviously,
\[ -x^T \left\{ \frac{1}{2} P D^2 \right\} P x \leq 0, \text{ and} \]
\[ -x^T \left\{ Q + C^T K^T R K \right\} x \leq -x^T \{ Q \} x \leq 0 \]

So, \( \dot{V}(x) \leq 0 \ \forall x \in \mathbb{R}^n \)

This proves the closed-loop system stability. To prove its asymptotic stability it suffices to note that there is no \( x \in \mathbb{R}^n \) such that \( \dot{V}(x(t)) = 0 \ \forall t \geq 0 \) since, by assumption, the pair \( (A, Q^{1/2}) \) is observable. In this paper, it has been proved (Gadewadikar et al. [10]) that an optimal \( H_\infty \) output feedback can be solved if there exists a matrix \( P \geq 0 \) that satisfies the following Riccati equation:
\[ K = R^{-1} (B^T P_a + L_s) C^T (C C^T)^{-1} \]
\[ L = R K C - B^T P \]

where the matrix \( P > 0 \) satisfy the following Riccati equation
\[ PA + A^T P + Q + \frac{1}{2} P D^2 P - P B R^{-1} B^T P + L^T R^{-1} L = 0 \]

The influence of the weighting matrices \( Q \) and \( R \) are the investigated by setting the arbitrary coefficients of the matrices. This method avoids the shortcomings and great difficulties of Hamilton-Jacobi-Bellman (HJB) equation.

V. ANALYSIS OF THE FIVE-AXIS VERTICAL AMB SYSTEMS

To test the performance of the AMB systems control for the designed optimal output feedback and \( H_\infty \) output feedback control methods. Simulations have been carried out for each of radial direction independently. In this research, the nominal operation range of spinning speed is set at 10 \( K \) rpm. The nominal values of the parameters used in the simulations are chosen as Table 1.

Take axial direction and one of radial directions of spindle, for example. Define \( A_s, B_s, C_s, D_s \) as the corresponding system matrix, input matrix, output matrix and disturbance matrix at spinning speed 10 \( K \) rpm.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Current value/Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current stiffness of radial AMB A ( (k_{ax}) )</td>
<td>( k_{ax} )</td>
<td>1.36x10^7 N/m</td>
</tr>
<tr>
<td>Displacement stiffness of radial AMB A ( (k_{ar}) )</td>
<td>( k_{ar} )</td>
<td>-22.69 N/A</td>
</tr>
<tr>
<td>Current stiffness of radial AMB B ( (k_{bx}) )</td>
<td>( k_{bx} )</td>
<td>1.96x10^7 N/m</td>
</tr>
<tr>
<td>Displacement stiffness of radial AMB B ( (k_{br}) )</td>
<td>( k_{br} )</td>
<td>-32.68 N/A</td>
</tr>
<tr>
<td>Current stiffness of axial AMB ( (k_z) )</td>
<td>( k_z )</td>
<td>4.42x10^7 N/m</td>
</tr>
<tr>
<td>Distance between radial AMB A to the mass center ( (l_a) )</td>
<td>( l_a )</td>
<td>63.25 mm</td>
</tr>
<tr>
<td>Distance between radial AMB B to the mass center ( (l_b) )</td>
<td>( l_b )</td>
<td>60.75 mm</td>
</tr>
<tr>
<td>Distance between axial AMB to the mass center ( (l) )</td>
<td>( l )</td>
<td>119 mm</td>
</tr>
<tr>
<td>Transverse moment of inertia of rotor about ( x ) (or ( y )) ( (J_x) )</td>
<td>( J_x )</td>
<td>4.31x10^4 kg·m^2</td>
</tr>
<tr>
<td>Polar moment of inertia of rotor about ( z ) ( (J_z) )</td>
<td>( J_z )</td>
<td>5.29x10^5 kg·m^2</td>
</tr>
<tr>
<td>Mass of the rotor ( (m) )</td>
<td>( m )</td>
<td>1.7 kg</td>
</tr>
</tbody>
</table>

The eigenvalues of the open-loop system can be obtained as follows:
\[ 1.6249 \pm 0.6228i, \ 1.3454 \pm 0.0350i, \ -1.6249 \pm 0.6228i, \ -1.3454 \pm 0.0350i \text{ and } \pm 5.0990. \]

It is obvious that the open-loop five-axis AMB system is unstable. Here, a more effective controller is needed. It is required to select the output-feedback gains to yield stability with stabilized closed-loop response. We can select the control gain \( K \) using MATLAB in a few seconds using the above mentioned algorithms described in the earlier section for the given \( \gamma \), \( Q \), and \( R \). After repeating the design several times, the values of \( Q \) and \( R \) are chosen as follows:

\[ A_s = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1000 & 0 & 0 & 0 & 0 \\
20.62 & -5.94 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5.94 & 28.31 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 20.62 & -5.94 & 0 & 65.78 & -63.18 & 63.18 & -63.18 & 0 \\
0 & 0 & -5.94 & 28.31 & 0 & -63.18 & 65.78 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 260 & 0 & 0 & 0 & 0 
\end{bmatrix} \]
Then the $H_{\infty}$ output feedback gain matrix $K_1$ is obtained

$$B_N = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-34.41 & -34.41 & 9.91 & 9.91 & 0 \\
7.71 & 7.71 & -48.36 & -48.36 & 0 \\
-34.41 & 34.41 & 9.91 & -9.91 & 0 \\
6.88 & -6.88 & -47.21 & 47.21 & 0 \\
0 & 0 & 0 & 0 & 260
\end{bmatrix}$$

$$C_N = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$D_N = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
588.24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
588.24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 588.24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 588.24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

and resulting closed loop poles $S = -0.1196 \pm 0.9560i$, $-0.6317 \pm 0.4122i$, $-0.0371 \pm 0.3582i$, $-1.2011$, $-0.0000 \pm 1.6165i$.

The optimal output feedback gain for the same $Q$ and $R$ matrix $K_2$ leads to

$$K_2 = \begin{bmatrix}
-0.7577 & -0.2856 & -69.9378 & -63.4847 & 0.0000 \\
-0.8629 & -0.3459 & -67.5944 & -73.3349 & 0.0000 \\
1.0344 & 0.3912 & -159.5803 & -169.1656 & 0.0000 \\
1.1798 & 0.4746 & -162.8238 & -155.5394 & 0.0000 \\
0.0000 & -0.0000 & 0.0000 & 0.0000 & 11.0499
\end{bmatrix}$$

When the initialization of $x$ was 0.1 mm, $y$ was 0.03 mm and $z$ was 0 mm, a step disturbances $d(t)$ was introduced in simulations to verify the test for static levitation characteristic which had a nicer result. We plotted the response with $x$-, $y$- and $z$- direction according to H-Infinity output gain matrix given in equation (24) and the optimal output-feedback gain matrix given in equation (25), time response for a step command is shown in Figure 4, Figure 5 and Figure 6, respectively. From the above performance comparison plots, we can see that the adjusting time of position deviation accuracies of the $H_{\infty}$ output feedback controller are less than that of optimal output feedback controller in the $x$-, $y$- and $z$-direction. It is worthwhile to point out that the simulation results for H-Infinity output feedback controller with the prescribed loops have shown a better
levitation performance compared to the performance of the optimal output feedback controller when disturbance was introduced. At the same time, the simulation results have also shown that the steady-state error of system approaches to 0 and the overshoot of system is very small.

H-Infinity optimal output feedback gives superior results over the optimal output-feedback controller, simulation results show that this algorithm is superior to H-\infty algorithm for calculating optimal output feedback gains. IEEE Transactions on Automatic Control, vol.9, pp. 900–903. 1985.

VI. CONCLUSIONS

In this paper, the dynamics of the AMB is established and found that the open-loop system is unstable. In order to improve the time response of the system, a more effective controller is needed. Static levitation and high-speed rotation of five degree of freedom AMB is successfully performed which use this control system with H-Infinity output feedback and optimal output feedback algorithm. Compare with optimal output-feedback controller, simulation results show that H-Infinity optimal output feedback gives superior results when disturbance is present.

REFERENCES

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