New Proofs of Identities of Lebesgue and Göllnitz via Tilings

David P. Little
Joint work with James Sellers

November 8, 2007

www.math.psu.edu/dlittle

Lebesgue Identites

In 1840, V. A. Lebesgue proved the following identities:

$$\sum_{n\geq 0} \frac{(-1;q)_n}{(q;q)_n} q^{\binom{n+1}{2}} = \prod_{n\geq 1} \frac{1+q^{2n-1}}{1-q^{2n-1}}$$

$$\sum_{n\geq 0} \frac{(-q;q)_n}{(q;q)_n} q^{\binom{n+1}{2}} = \prod_{n\geq 1} \frac{1-q^{4n}}{1-q^{n}}$$

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both of which are specializations of

Theorem

$$\sum_{n\geq 0} \frac{(-z;q)_n}{(q;q)_n} q^{\binom{n+1}{2}} = \prod_{n\geq 1} (1+q^n)(1+zq^{2n-1})$$

where $(z;q)_n = (1-z)(1-zq)\cdots(1-zq^{n-1})$.

Existing Combinatorial Proofs

K. Alladi and B. Gordon (1993), C. Bessenrodt (1994), M. Rowell (2007) have given bijective proofs of

$$\sum_{n\geq 0} \frac{(-z;q)_n}{(q;q)_n} q^{\binom{n+1}{2}} = \prod_{n\geq 1} (1+q^n)(1+zq^{2n-1})$$

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Our Goal:

Prove Lebesgue Identities by showing that both sides can be viewed as the generating function for the same collection of combinatorial objects, namely, weighted Pell tilings.

Pell Tilings

Definition

A Pell tiling is a covering of an infinitely long board:

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$\cdots$

using three different types of tiles:

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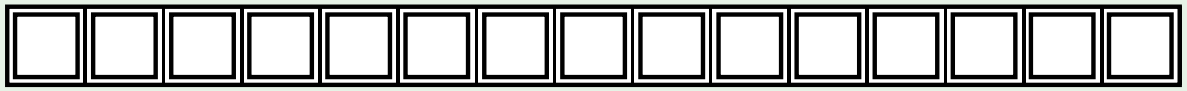

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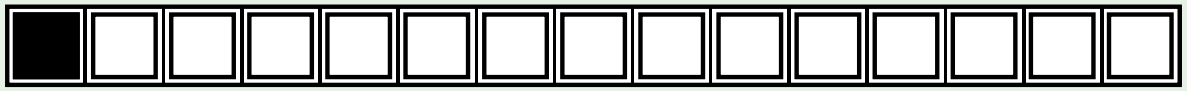

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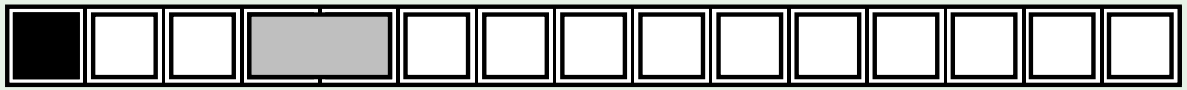

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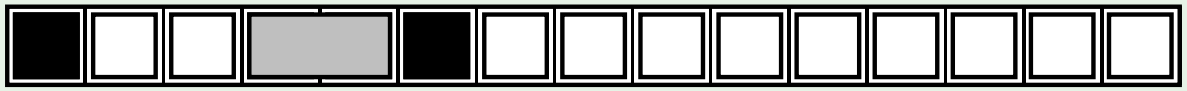

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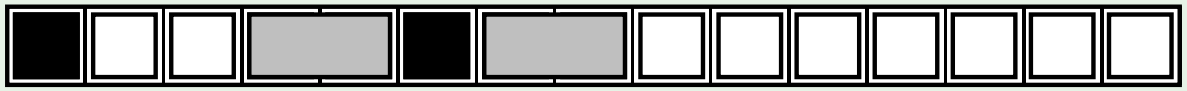

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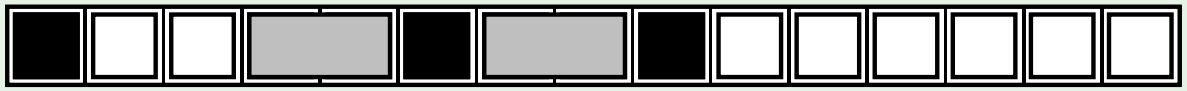

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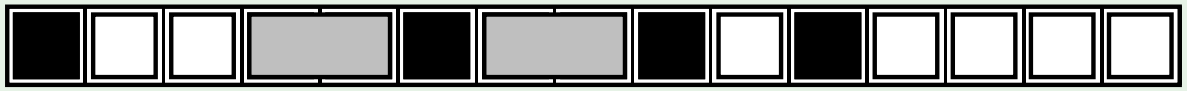

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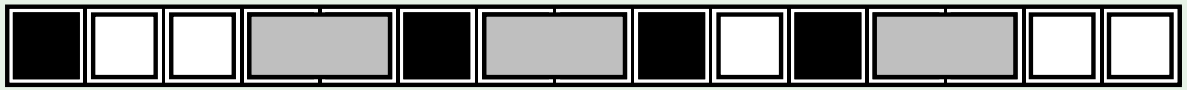

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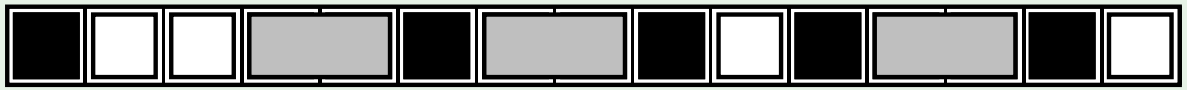

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Weighted Pell Tilings

The weight of tile t:

$$w(t) = \begin{cases} aq^i & \text{if } t \text{ is a black square in position } i \\ bq^i & \text{if } t \text{ is a gray domino in position } i \\ 1 & \text{if } t \text{ is a white square in position } i \end{cases}$$

The weight of tiling T:

$$w(T) = \prod_{t \in T} w(t)$$

Generating Function:

$$F_q(a, b) = \sum_{T \in \mathcal{T}} w(T)$$

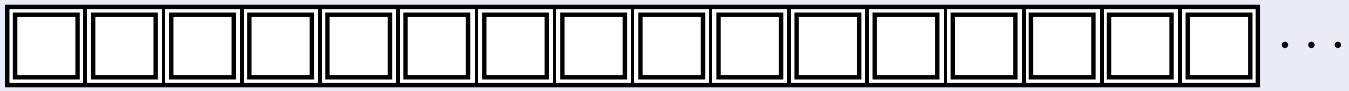
Series Expansion of $F_q(a,b)$

Theorem

$$F_q(a,b) = \sum_{n\geq 0} \frac{(a+b)(a+bq)\cdots(a+bq^{n-1})}{(q;q)_n} q^{\binom{n+1}{2}}$$

Proof.

STEP I: Choose n distinct places on the board, which will act as initial positions for black squares and dominoes.



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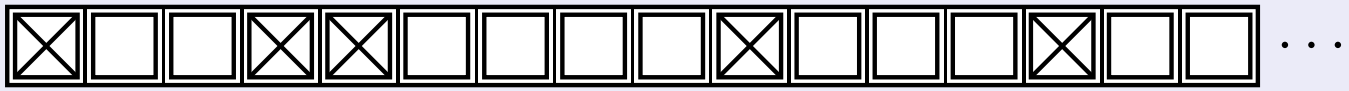

This accounts for a q-weight of

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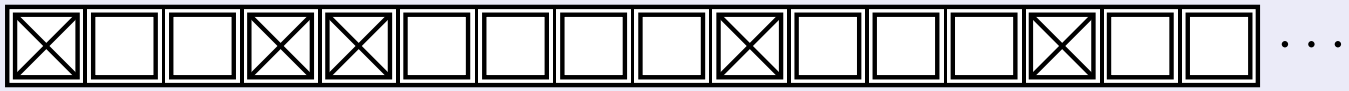

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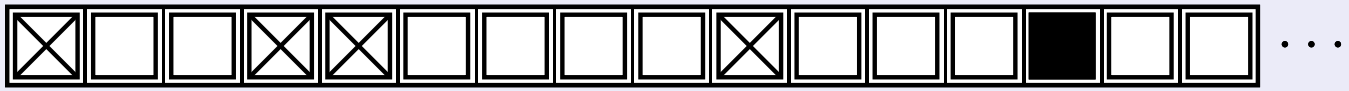

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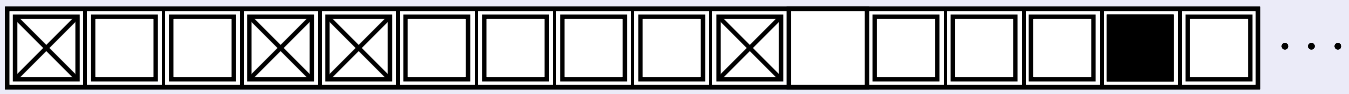

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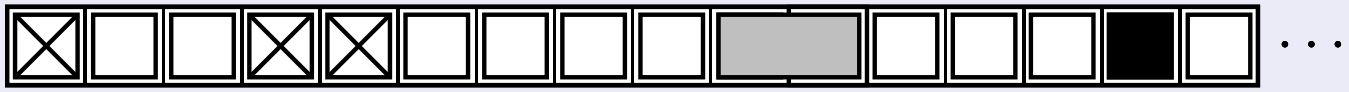

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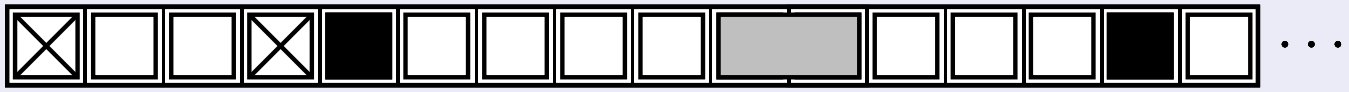

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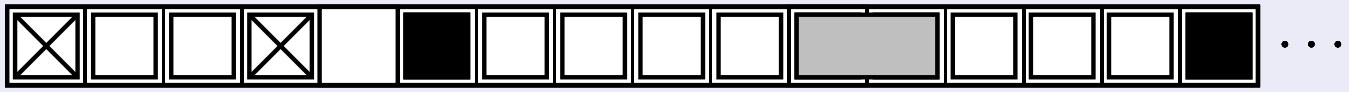

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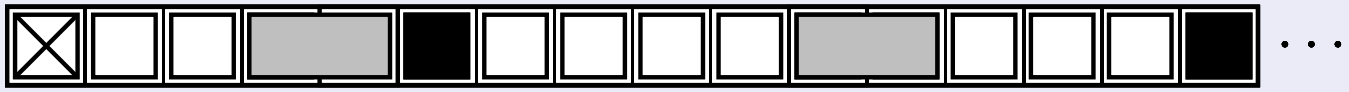

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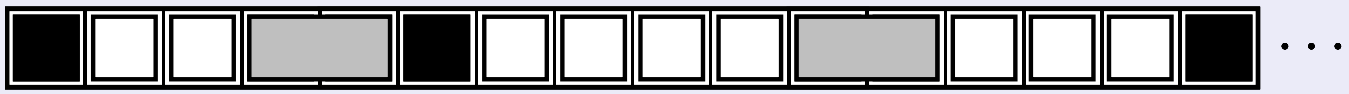

The choice of placing a black square or gray domino in the jth marked position from the right is represented by the factor $(a+bq^{j-1})$. □

A Theorem of Euler

Theorem

$$\sum_{n\geq 0} \frac{a^n q^{\binom{n+1}{2}}}{(q;q)_n} = F_q(a,0) = \prod_{n\geq 1}(1+aq^n)$$

Proof.

Setting $b=0$ eliminates any tiling that contains a domino.

Tilings can then be constructed by just deciding whether each position should be covered by a white square or a black square.

The factor $(1+aq^n)$ represents the choice of covering position n with a white square or a black square.

$\square$

Recursive Formulas Involving $F_q(a, b)$

Observations

- Replacing a with aq and b with bq has the effect of shifting all tiles one position to the right and placing a white square in position one.

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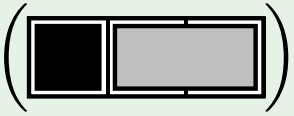
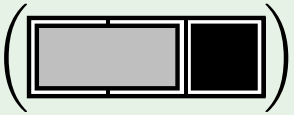
$w\Big($  $\Big) = abq^{2i+1}$

$w\Big($  $\Big) = abq^{2i+2}$

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- By working right-to-left, all black squares can be switched with the tile immediately to its right. This operation maintains relative order of white squares and gray dominoes.

Replacing a with aq

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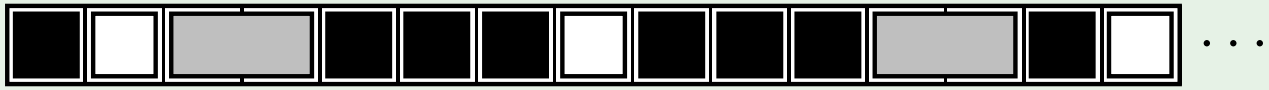

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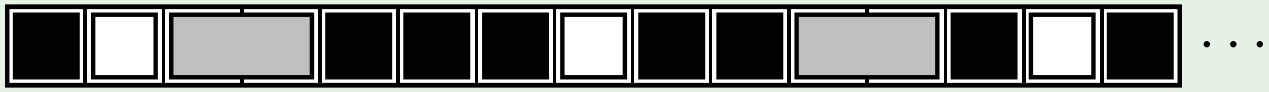

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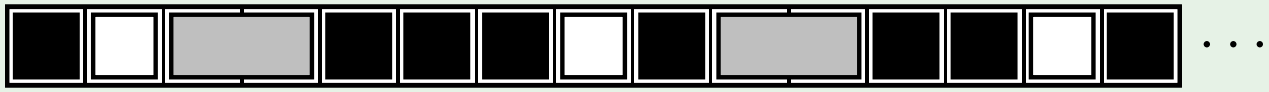

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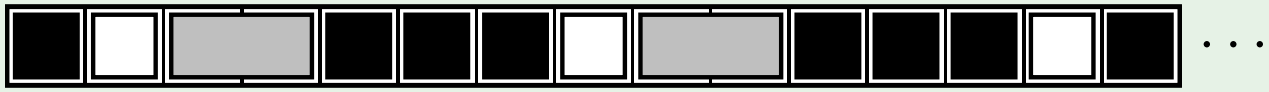

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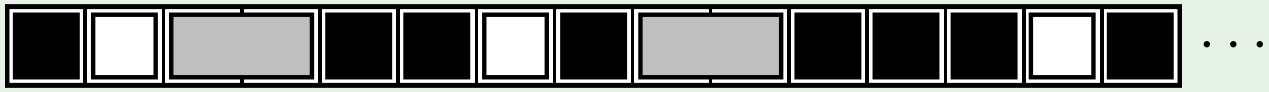

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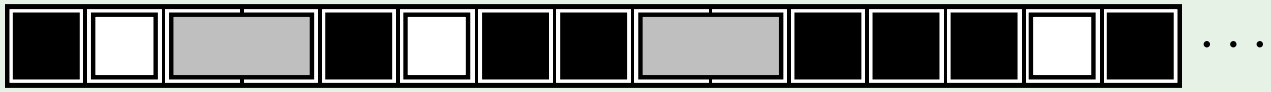

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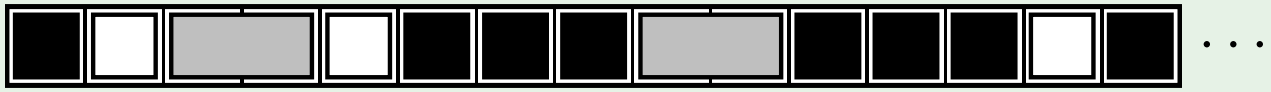

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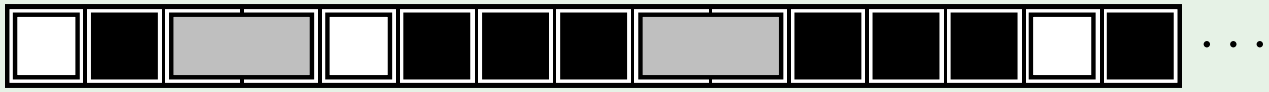

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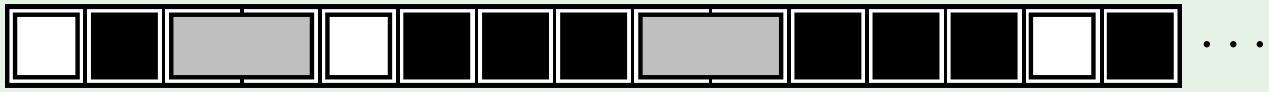

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$$F_q(aq, b).$$

Replacing a with aq

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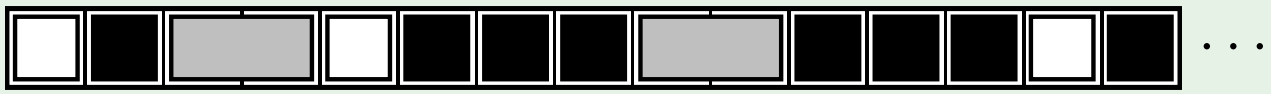

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The generating function for tilings where at least $n \geq 0$ white squares and/or dominoes appear before the first black square, if any, is given by

$$F_q(aq^n, b).$$

A Recursion

Lemma

$$F_q(a,b) = F_q(aq,b) + aqF_q(aq,bq)$$

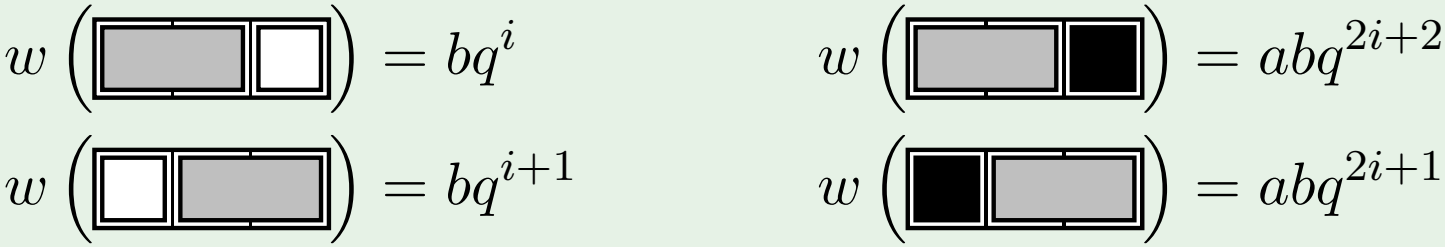
Proof.

$F_q(aq,b)$ counts tilings where at least one white square or domino occurs before the first black square.

$aqF_q(aq,bq)$ counts tilings where the first position is covered by a black square. $\square$

Observations

- Reversing the order of a domino with a square to its right cannot be interpreted as replacing b with bq.



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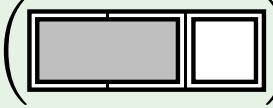
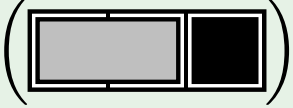
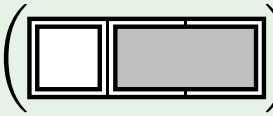
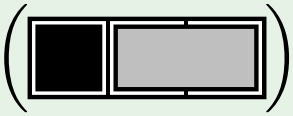
$w\Big($  $\Big) = bq^{i}$ $\qquad$ $w\Big($  $\Big) = abq^{2i+2}$

$w\Big($  $\Big) = bq^{i+1}$ $\qquad$ $w\Big($  $\Big) = abq^{2i+1}$

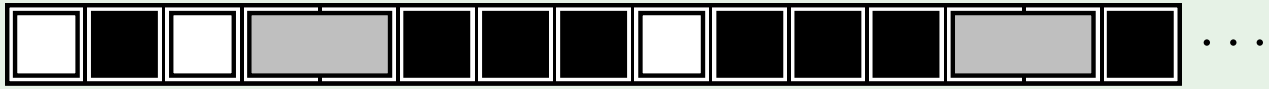
- Replacing b with bq can be achieved by the following two step process:
  1. Replace a with aq and b with bq (i.e. shift all tiles one position to the right)
  2. Second, replace a with a/q (i.e. switch each black square with the tile to its left, working left-to-right)

Replacing b with bq

Example

Replacing b with bq

Example



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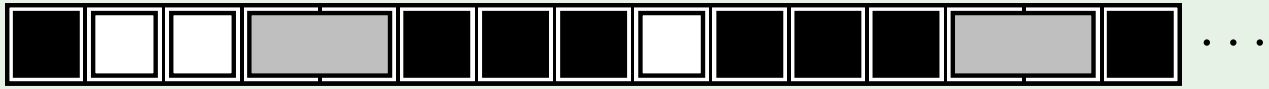

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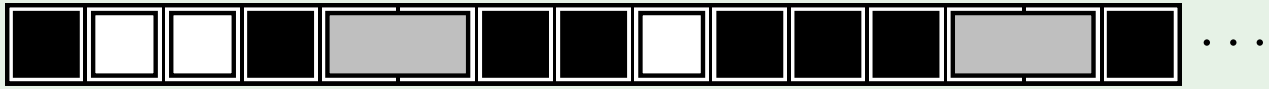

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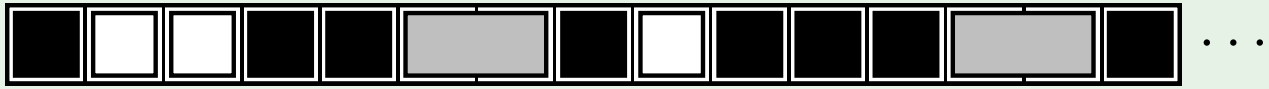

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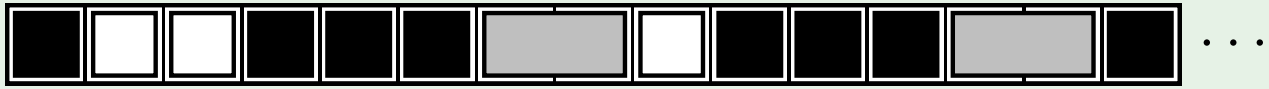

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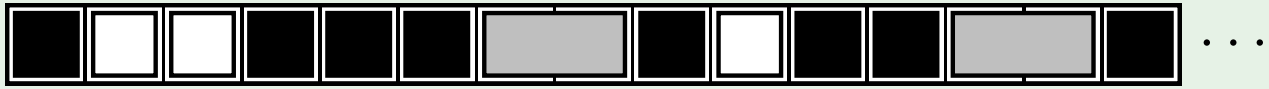

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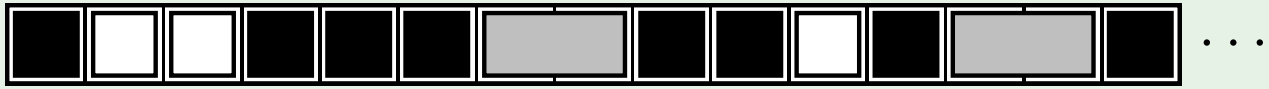

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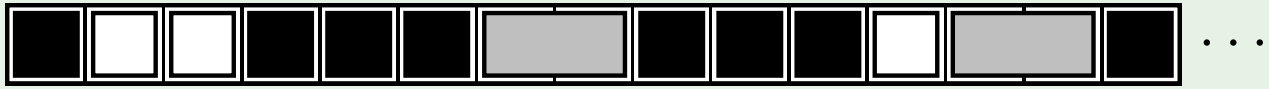

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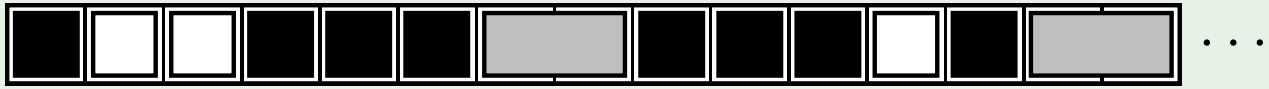

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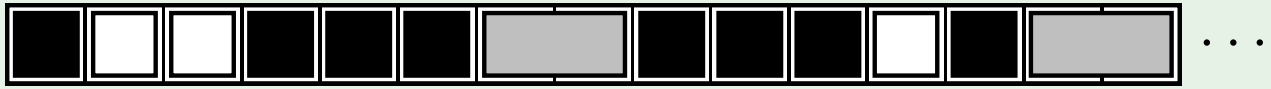

Lemma

The generating function for tilings where at least one white square appears before the first domino, if any, is given by

$$F_q(a, bq).$$

Replacing b with bq

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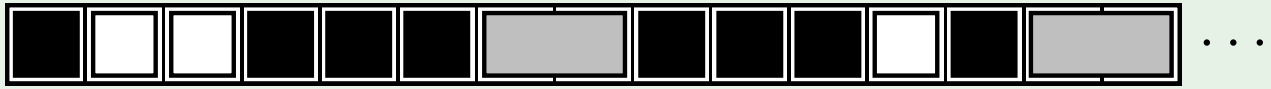

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The generating function for tilings where at least one domino appears before the first white square is given by

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Proof.

$F_q(a, bq)$ counts tilings where at least one white square appears before the first domino.

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This operation is equivalent to multiplication by bq, as desired. □

Another Recursion

Lemma

$$F_q(a,b) = F_q(a,bq) + bqF_q(aq,bq^2)$$

Proof.

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$bqF_q(aq,bq^2)$ counts tilings where at least one domino appears before the first white square. $\square$

Lemma

The generating function for tilings where exactly one white square appears before the first domino is given by

$$\frac{bq}{a}(F_q(a, bq^2) - F_q(aq, bq^2)).$$

Proof.

$F_q(a, bq^2) - F_q(aq, bq^2)$ counts tilings where at least two white squares appear before the first domino and the first position is covered by a black square.

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This operation is equivalent to multiplication by bq/a, as desired. □

The Recursion

Theorem: For $a \neq 0$,

$$F_q(a,b) = \left(1 + \frac{bq}{a}\right) F_q(a, bq^2) + bq\left(1 - \frac{1}{a}\right) F_q(aq, bq^2)$$

Proof.

Count tilings based on the number of white squares that appear before the first domino, if any.

$$\begin{aligned} F_q(a,b) &= F_q(a,bq^2) + bqF_q(aq,bq^2) + \frac{bq}{a}(F_q(a,bq^2) - F_q(aq,bq^2)) \\ &= F_q(a,bq^2) + \frac{bq}{a}F_q(a,bq^2) + bqF_q(aq,bq^2) - \frac{bq}{a}F_q(aq,bq^2) \end{aligned}$$

$\square$

Corollary (Lebesgue)

$$\sum_{n\geq 0} \frac{(-z;q)_n}{(q;q)_n} q^{\binom{n+1}{2}} = \prod_{n\geq 1} (1+q^n)(1+zq^{2n-1})$$

Proof.

The left–hand side of this equality is simply $F_q(1,z)$. Applying the previous recursion yields

$$F_q(1,z) = (1+zq)F_q(1,zq^2)$$

Iterating this recursion yields

$$F_q(1,z) = F_q(1,0) \prod_{n\geq 1} (1+zq^{2n-1})$$

provided $|q| < 1$. $\square$

Corollary (Göllnitz)

$$\sum_{n\geq 0} \frac{(-z;q^2)_n}{(q^2;q^2)_n} q^{n^2+n} = \prod_{n\geq 1} (1+q^{4n-2})(1+zq^{4n-2})(1+q^{4n})$$

Proof.

This follows from the Lebesgue identities by replacing q with q^2. $\square$

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Replacing q with q^2 in the Lebesgue identities results in a generating function for tilings where weighted tiles appear only in even–numbered positions and dominoes cannot be followed by another weighted tile.

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Replacing q with q^2 in the Lebesgue identities results in a generating function for tilings where weighted tiles appear only in even–numbered positions and dominoes cannot be followed by another weighted tile.

Setting $z = q$ and $z = 1/q$ amount to shifting every domino one position to the right or left, respectively.

Recap

Showed that both sides of the Lebesgue identity satisfy the same recursion.

$$\sum_{n\geq 0} \frac{(-z;q)_n}{(q;q)_n} q^{\binom{n+1}{2}} = \prod_{n\geq 1} (1+q^n)(1+zq^{2n-1})$$

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The infinite series expansion has a simple interpretation in terms of constructing tilings. How can we interpret the infinite product?

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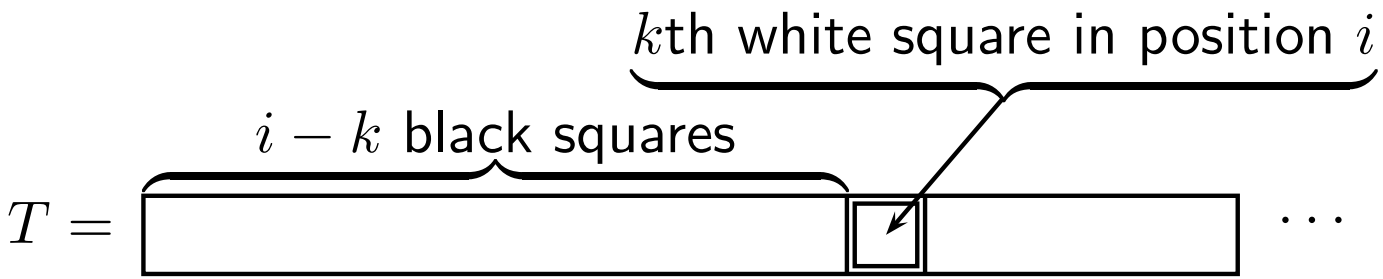
Question:

The infinite series expansion has a simple interpretation in terms of constructing tilings. How can we interpret the infinite product?

More specifically, we know how the factors of $(1+q^n)$ can be interpreted as placing black squares with $a=1$. Can we now interpret the factors of $(1+zq^{2n-1})$ as describing how to insert dominoes?

Inserting Dominoes



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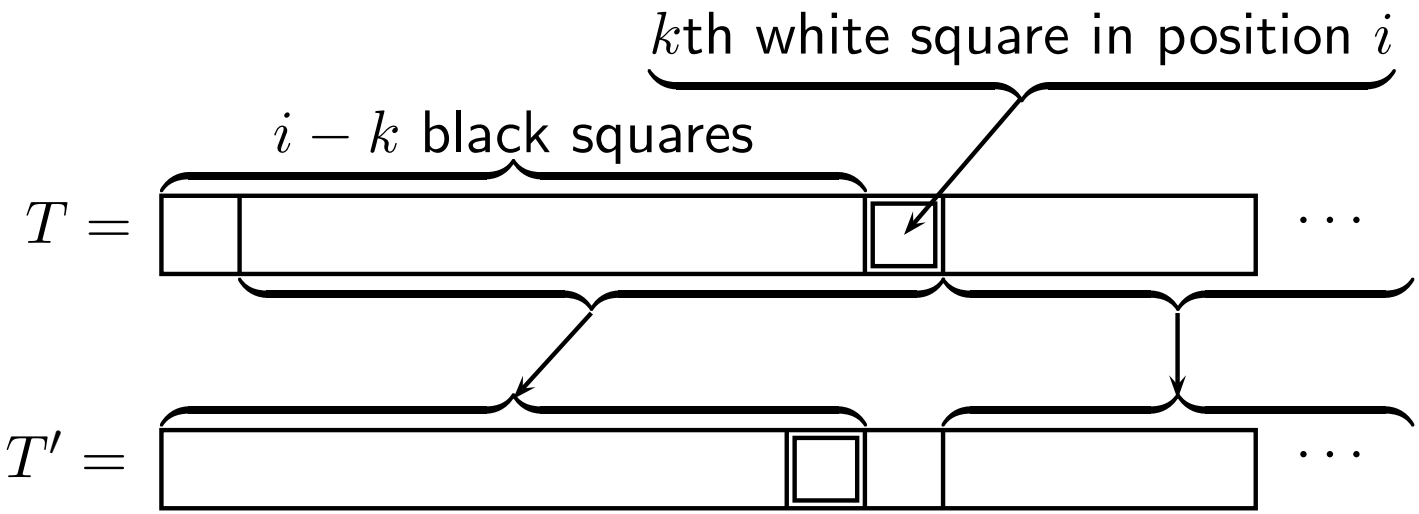

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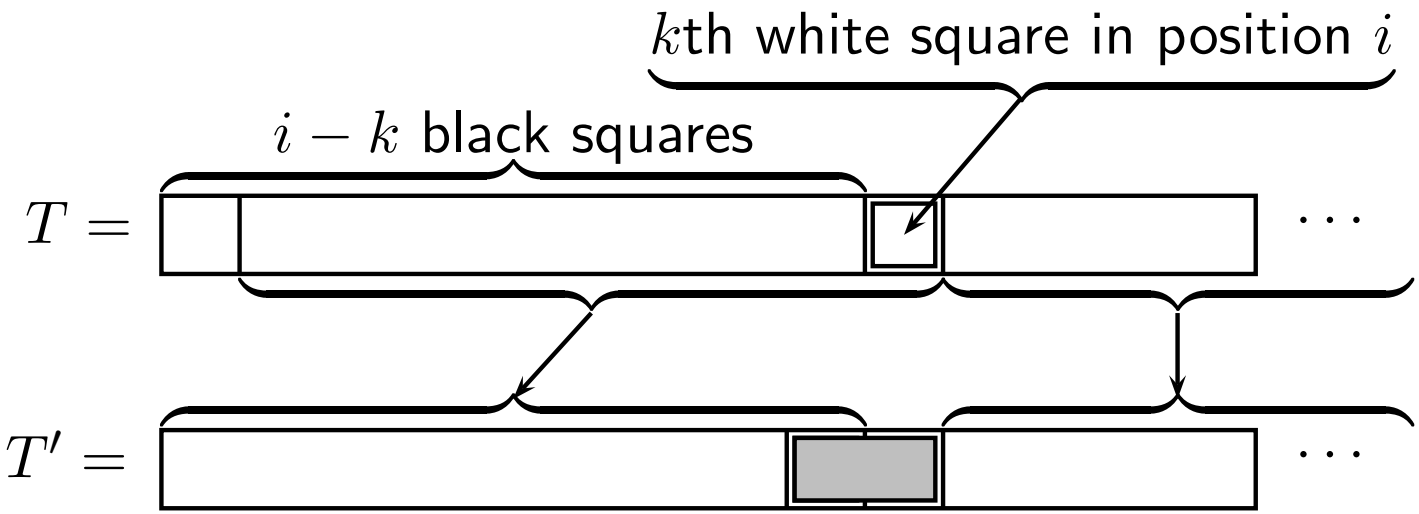

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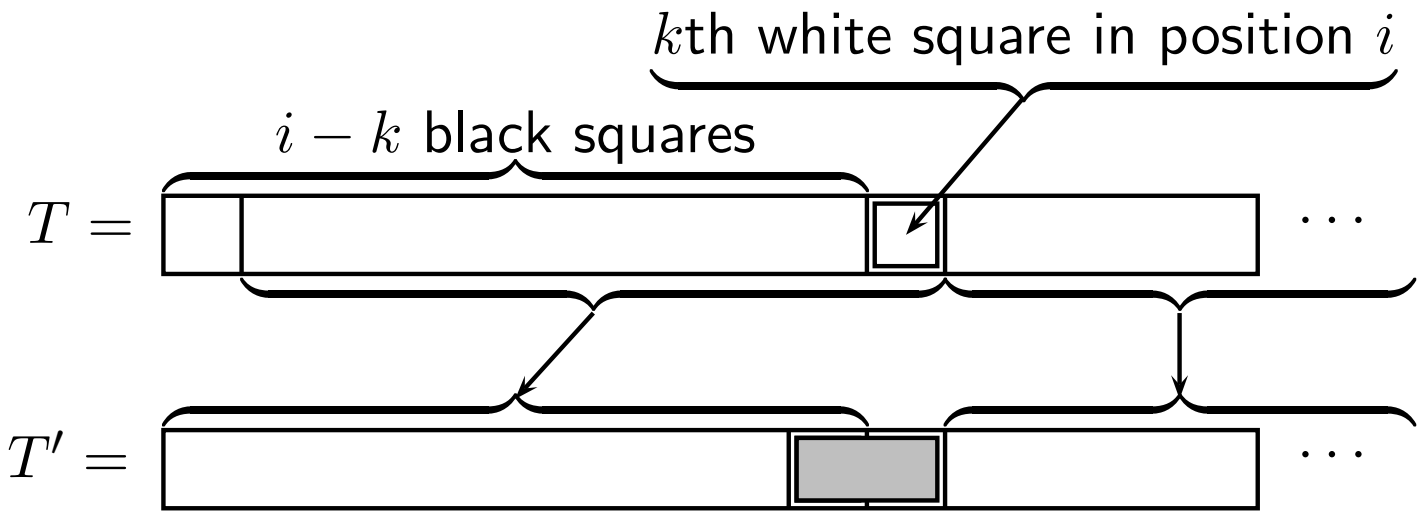

Lemma If the kth white square appears in position $i > 1$ of T, then

$$w(T') = bq^{k-1}w(T).$$

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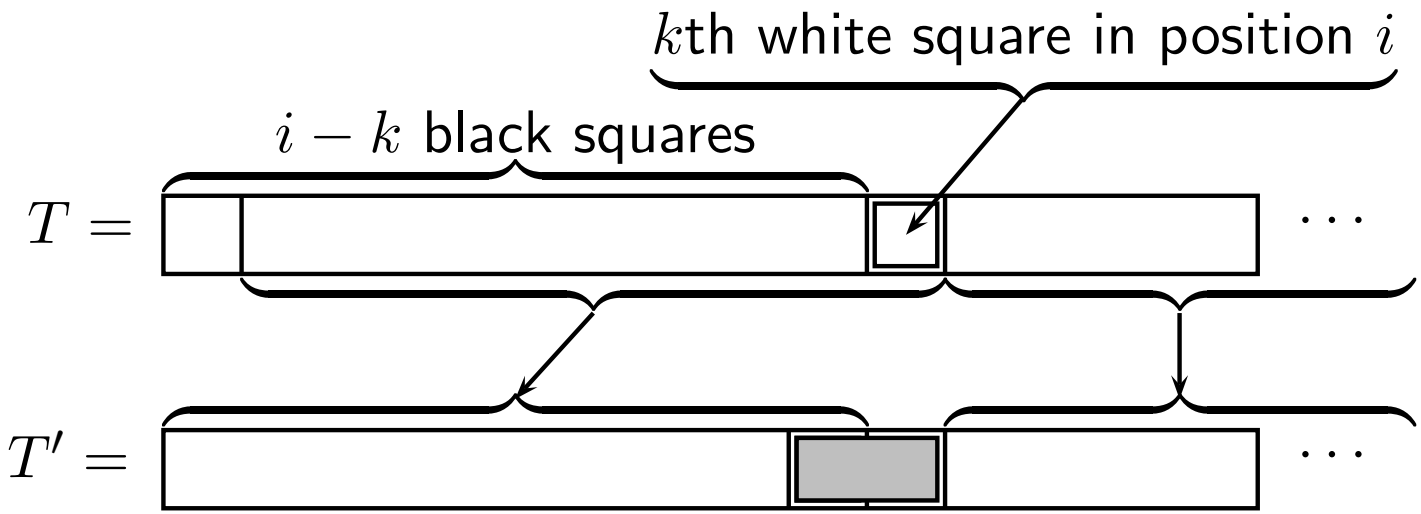

Lemma If the kth white square appears in position $i > 1$ of T, then

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Observation:

If the above operation is always performed when k is of the same parity, then the operation can be reversed.

Theorem

$$F_q(1,z) = \prod_{n\geq 1}(1+q^n)(1+zq^{2n-1}).$$

Proof.

Construct tiling $T^{(0)}$ by arbitrarily covering the board with squares. This explains each factor of $(1+q^n)$.

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Given $I = \{2n_1, 2n_2, \ldots, 2n_l\}$ with $n_1 > n_2 > \cdots > n_l > 0$, construct the following sequence of tilings

$$T^{(0)}, T^{(1)}, T^{(2)}, \ldots, T^{(l)}$$

where $T^{(i)}$ is obtained by inserting a domino at $k = 2n_i$ in $T^{(i-1)}$.

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Thus, each factor of $(1+zq^{2n-1})$ simply expresses the decision of whether or not to include $2n$ in I. The tiling $T^{(l)}$ is the final result of our construction.

Example

Consider the term

$$q^2 \cdot q^4 \cdot q^5 \cdot q^9 \cdot q^{12} \cdot q^{13} \cdot q^{15} \cdot bq^{2-1} \cdot bq^{6-1} \cdot bq^{8-1} = b^3 q^{73}$$

from the expansion of the product $\prod_{n \geq 1}(1 + q^n)(1 + bq^{2n-1})$.

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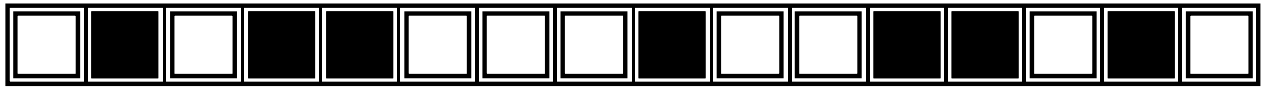
Place black squares in positions $2, 4, 5, 9, 12, 13,$ and 15

$T^{(0)} =$  $\cdots$

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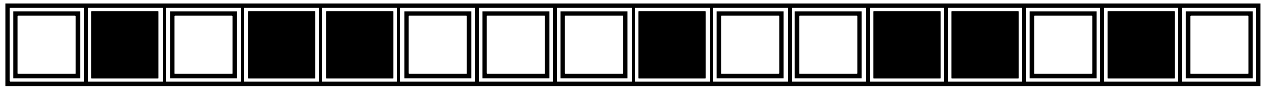
$T^{(0)} =$  $\cdots$

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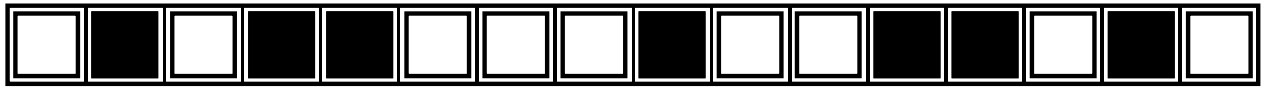
$T^{(0)} =$  $\cdots$

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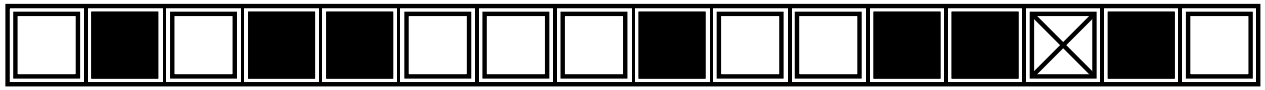
$T^{(1)} =$ 

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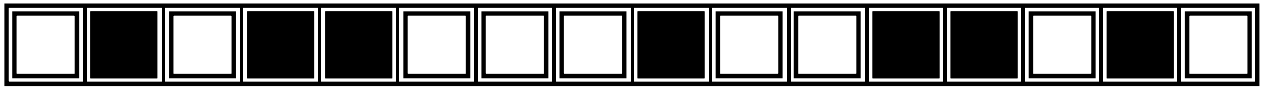
$T^{(0)} =$  $\cdots$

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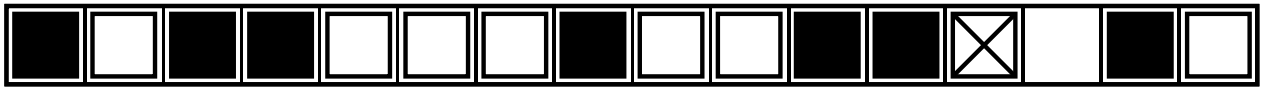
$T^{(1)} =$ 

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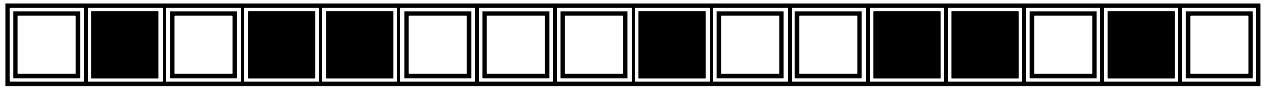
$T^{(0)} =$  $\cdots$

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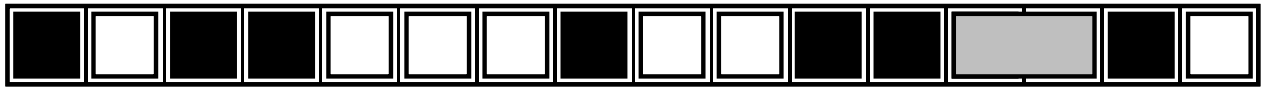
$T^{(1)} =$  $\cdots$

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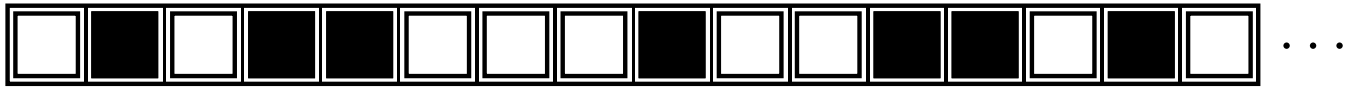
$T^{(0)} =$  $\cdots$

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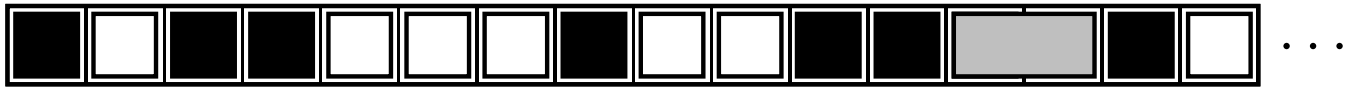
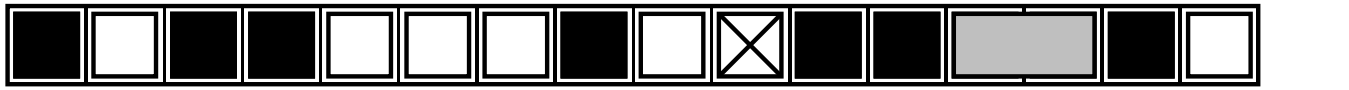
$T^{(1)} =$  $\cdots$

$T^{(2)} =$ 

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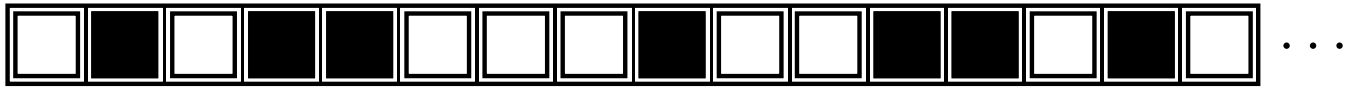
$T^{(0)} =$  $\cdots$

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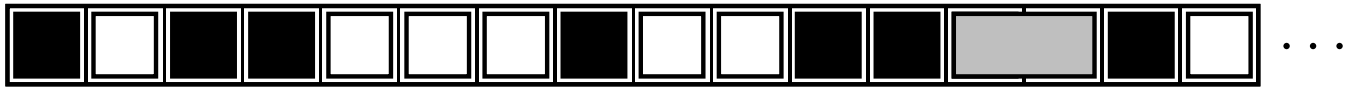
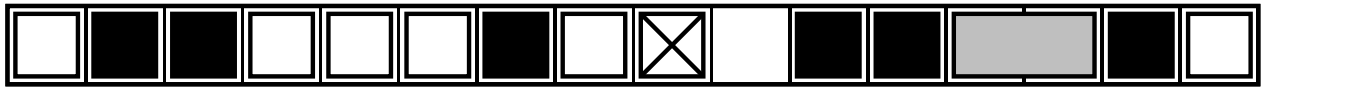
$T^{(1)} =$  $\cdots$

$T^{(2)} =$ 

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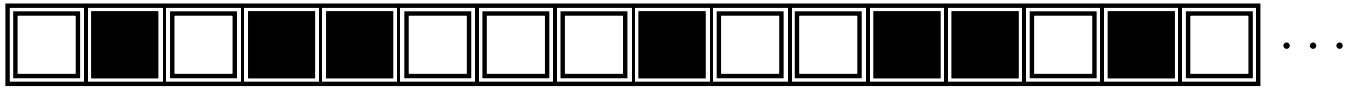
$T^{(0)} =$  $\cdots$

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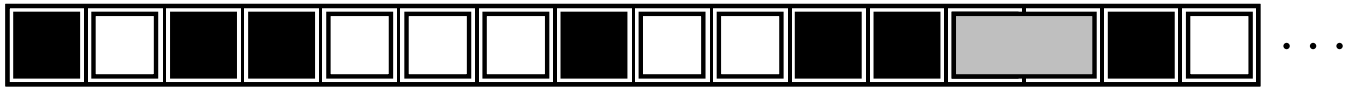
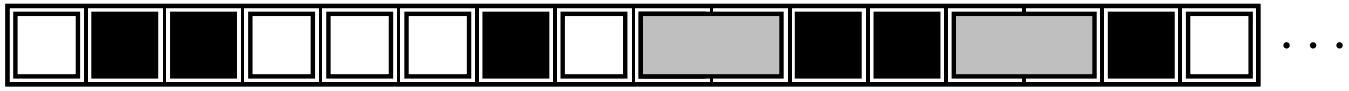
$T^{(1)} =$  $\cdots$

$T^{(2)} =$  $\cdots$

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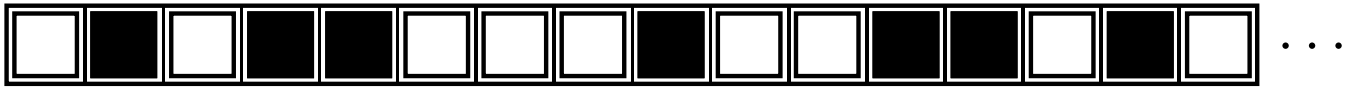
$T^{(0)} =$  $\cdots$

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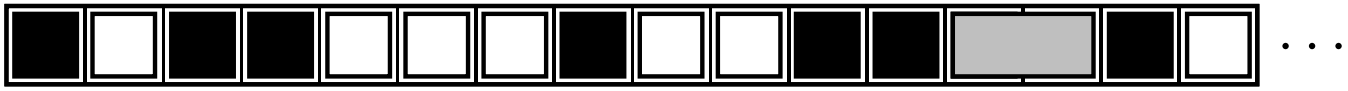
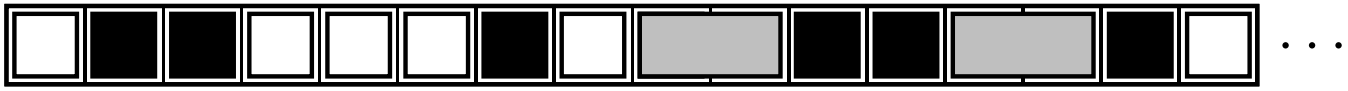
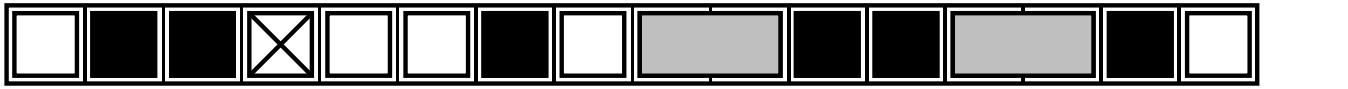
$T^{(1)} =$  $\cdots$

$T^{(2)} =$  $\cdots$

$T^{(3)} =$ 

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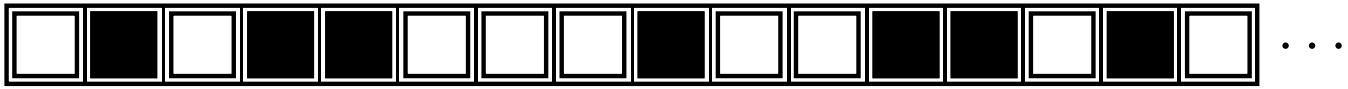
$T^{(0)} =$ 

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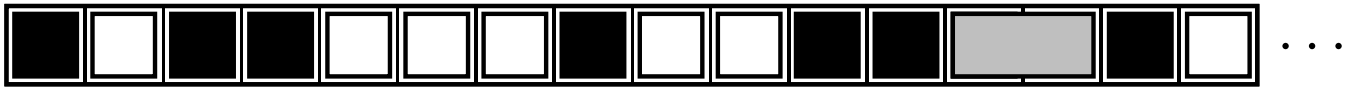
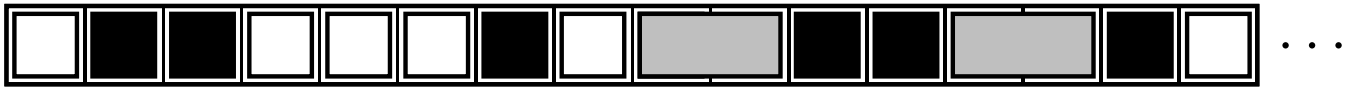
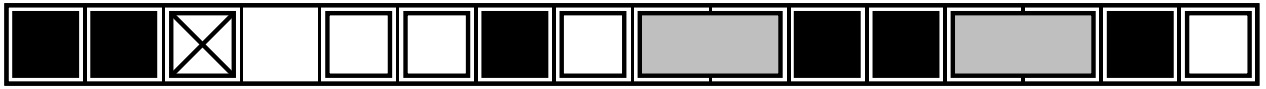
$T^{(1)} =$ 

$T^{(2)} =$ 

$T^{(3)} =$ 

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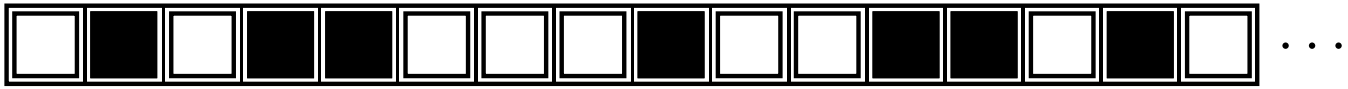
$T^{(0)} =$  $\cdots$

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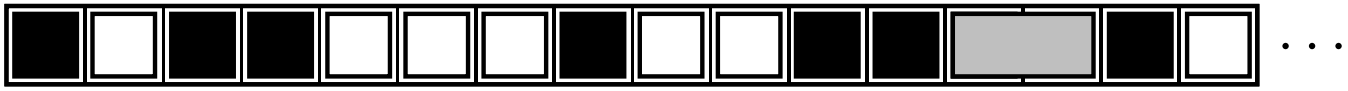
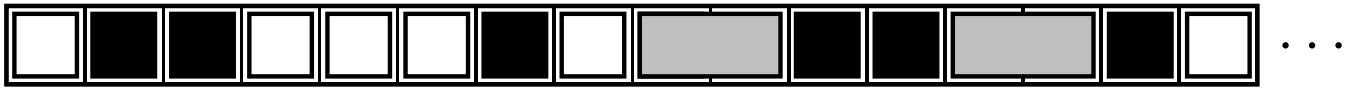
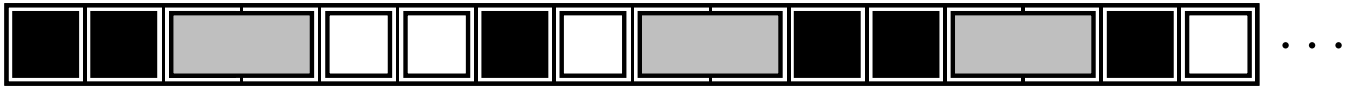
$T^{(1)} =$  $\cdots$

$T^{(2)} =$  $\cdots$

$T^{(3)} =$  $\cdots$

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