Convergence and Approximation of Semigroups of Nonlinear Operators in Banach Spaces

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A general convergence theorem for semigroups of nonlinear operators in a general Banach space is proved. It is then applied to obtain an approximation theorem for such semigroups. These results extend the previously known results for semigroups of linear operators in Banach space.

1. Introduction

Let $X$ be a real Banach space with norm $| |$. If $S$ is a nonvoid subset of $X$ we define $$ || S || = \inf \{ | x | ; x \in S \}.$$ A subset $A$ of $X \times X$ is in the class $\mathcal{A}(\omega)$, $\omega \geq 0$ if for each $0 < \lambda < \omega^{-1}$ and $[x_1, y_1] \in A$, $i = 1, 2$ we have

$$ |(x_1 + \lambda y_1) - (x_2 + \lambda y_2)| \geq (1 - \lambda \omega)|x_1 - x_2|.$$ (1.1)

$A$ is called accretive if $A \in \mathcal{A}(0)$.

Consider the following initial-value problem

$$ \frac{du}{dt} + Au \geq 0 \quad \text{a.e. on } (0, +\infty), \quad u(0) = x; \quad (1.2)$$

where $A \in \mathcal{A}(\omega)$. A function $u(t)$ defined on $[0, +\infty)$ with values in $X$ is a solution of (1.2) if $u(t)$ is absolutely continuous in $t$, $u(t)$ is differentiable a.e. on $(0, +\infty)$, $u(t) \in D(A)$ a.e. on $(0, +\infty)$ and $u$ satisfies (1.2).

If (1.2) has a solution, it is unique; this is a simple consequence of $A \in \mathcal{A}(\omega)$. If (1.2) has a solution $u(t)$ for every $x \in D(A)$, we define $S(t)x = u(t)$. $S(t)$ is a continuous operator on $D(A)$ and extending it by continuity to $\overline{D(A)}$ we obtain a family $\{S(t), t \geq 0\}$ of operators $S(t) : \overline{D(A)} \rightarrow \overline{D(A)}$ satisfying

$$ S(0) = I, \quad S(t+s) = S(t) \cdot S(s) \quad \text{for } t, s \geq 0. \quad (1.3) $$

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\lim_{t \to 0} S(t)x = x \quad \text{for } x \in \overline{D(A)} \quad (1.4)

\|S(t)x - S(t)y\| \leq e^{\omega t} \|x - y\| \quad \text{for } t \geq 0, \quad x, y \in \overline{D(A)}. \quad (1.5)

Such a family is called a \textit{semigroup of type } \omega \text{ on } \overline{D(A)}. \text{ Concerning the existence of a solution of (1.2) for every } x \in D(A), \text{ many sufficient conditions are known; see e.g. [2, 3, 7, 8, 12], and their bibliographies. Recently, Crandall and Liggett [5] have proved the following}

\textbf{Theorem I.} \textit{If } A \in \mathcal{A}(\omega) \text{ and } R(I + \lambda A) \supset \overline{D(A)} \text{ for every } 0 < \lambda < \lambda_0, \text{ then}

\lim_{t \to +\infty} \left(I + \frac{t}{n} A\right)^{-n} x = S(t)x \quad (1.6)

\text{exists for every } x \in \overline{D(A)}, \quad t \geq 0 \text{ and the limit is a semigroup of type } \omega \text{ on } \overline{D(A)}.\n
The semigroup \( S(t) \) defined by (1.6) can be considered as a "generalized solution" of (1.2); this follows from a result of Brezis and Pazy [2].

\textbf{Theorem II.} \textit{Let } A \in \mathcal{A}(\omega) \text{ such that } R(I + \lambda A) \supset \overline{D(A)} \text{ for every } 0 < \lambda < \lambda_0, \text{ and let } x \in D(A). \text{ If the initial-value problem (1.2) has a solution } u(t), \text{ then } u(t) \text{ is given by the exponential formula (1.6).}^1

It may, however, happen that \( A \) satisfies the conditions of Theorem I but for no \( x \in D(A) \) does the initial-value problem (1.2) have a solution (see [5]).

Let \( A \in \mathcal{A}(\omega) \) such that \( R(I + \lambda A) \supset \overline{D(A)} \) for \( 0 < \lambda < \lambda_0 \), and let \( S(t) \) be the semigroup of type \( \omega \) associated with \( A \) through the exponential formula (1.6). We shall say that \( S(t) \) is \textit{generated by } \(-A\).

In the present paper we deal with relations between the convergence of the generators and the convergence of the corresponding semigroups (in the sense of Theorem I). For semigroups of linear operators, the results were obtained by Trotter [14] and Chernoff [4]. For semigroups of nonlinear operators in Hilbert spaces, the results are given in [1] (for \( \omega = 0 \)) and [13] (for \( \omega > 0 \)). Some results of similar nature for general Banach spaces were obtained by Miyadera [9, 10] and Miyadera Oharu [11]; most of them are included in our work.

Section 2 is devoted to some preliminaries. The main results are given in Section 3 and Section 4 is devoted to some applications.

\textsuperscript{1} This result is stated in [2] for \( \omega = 0 \), but the proof is carried over easily to the case where \( \omega > 0 \).
2. Preliminaries

Let $A \in \mathcal{A}(\omega)$ and let $\lambda$ be real, $J_\lambda$ will denote the set $(I + \lambda A)^{-1}$ and $D_\lambda = D(J_\lambda)$ its domain. The first lemma collects some elementary facts about the sets $J_\lambda$ and the sets $A_\lambda$ defined by $A_\lambda = \lambda^{-1}(I - J_\lambda)$.

**Lemma 2.1.** Let $A \in \mathcal{A}(\omega)$, $\lambda > 0$ such that $\lambda \omega < 1$, then

(i) $J_\lambda$ is a function and for $x, y \in D_\lambda$

$$| J_\lambda x - J_\lambda y | \leq (1 - \lambda \omega)^{-1} | x - y |$$

(ii) $A_\lambda$ is a function and, for $x \in D_\lambda \cap D(A)$,

$$| A_\lambda x | \leq (1 - \lambda \omega)^{-1} \| Ax \|$$

(iii) If $x \in D_\lambda$, $\lambda > 0$, and $\mu > 0$, then $(\mu/\lambda)x + (1 - \mu/\lambda) J_\lambda x \in D_\mu$ and

$$J_\lambda x = J_\mu \left( \frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda} \right) J_\lambda x \right).$$

For a proof of Lemma 2.1, see [5] or [7].

**Lemma 2.2.** Let $C \subset X$ be closed and convex and let $T : C \to C$ be lipschitz with constant $\alpha \geq 1$, then

(i) $(I + \lambda(I - T))^{-1}$ exists for $0 < \lambda < (\alpha - 1)^{-1}$ and $(I + \lambda(I - T))^{-1} : C \to C$.

(ii) $\rho^{-1}(I - T)$ is lipschitz and $\rho^{-1}(I - T) \in \mathcal{A}(\rho^{-1}(\alpha - 1))$ for $\rho > 0$.

**Proof.** (i) Let $x \in C$ and define

$$G(y) = \frac{1}{1 + \lambda} x + \frac{\lambda}{1 + \lambda} Ty,$$

then $G(y)$ is a strict contraction and $G : C \to C$. Therefore, $G$ has a unique fixed point $y \in C$. Thus $(I + \lambda(I - T))^{-1}$ is defined on $C$ and maps $C$ into $C$.

(ii) It is clear that $\rho^{-1}(I - T)$ is lipschitz since $T$ is lipschitz. Also

$$\left| x_1 + \frac{\lambda}{\rho} (I - T) x_1 - x_2 - \frac{\lambda}{\rho} (I - T) x_2 \right|$$

$$\geq \left(1 + \frac{\lambda}{\rho}\right) | x_1 - x_2 | - \frac{\lambda}{\rho} | T x_1 - T x_2 | \geq \left(1 - \frac{\lambda}{\rho} (\alpha - 1)\right) | x_1 - x_2 |,$$

and, therefore, $\rho^{-1}(I - T) \in \mathcal{A}(\rho^{-1}(\alpha - 1))$. 

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Lemma 2.3. Let $T$ be lipschitz with constant $\alpha \geq 1$ mapping a closed convex subset $C \subset X$ into itself; then $T - I$ generates a semigroup $S(t)$ of type $\alpha - 1$ on $C$ and

$$|S(m)x - T^mx| \leq \alpha^m e^{m(\alpha - 1)}\{m^2(\alpha - 1)^2 + m(\alpha - 1) + m\}^{\frac{1}{2}} |(I - T)x|$$

for every $x \in C$.

For a proof of Lemma 2.3, see Miyadera-Oharu [11, Appendix].

Lemma 2.4. Let $A \in \mathcal{S}(\omega)$ satisfying $R(I + \lambda A) \supset \overline{D(A)}$ (the strong closure of $D(A)$) for all $0 < \lambda < \lambda_0$. Let $S(t)$ be the semigroup generated by $-A$ (see Theorem 1) then, for every $x \in D(A),

$$|S(t)x - \int_{t/n}^t x| \leq 2tn^{-\frac{1}{2}} \|Ax\| \cdot \exp(4\omega t)$$

(2.5)

$$|S(t)x - S(\tau)x| \leq |t - \tau| \|Ax\| \cdot [\exp(2\omega(t - \tau)) - \exp(4\omega t)].$$

(2.6)

The proof of this lemma is contained in the proof of Theorem I of [5]; see, in particular, formulas (1.10) and (1.11).

3. The Main Results

We start with an extension of a theorem of Trotter [14]; see also [15, Chapt. IX.12].

Theorem 3.1. Let $A^\circ \in \mathcal{S}(\omega_\rho), A \in \mathcal{S}(\omega)$ such that

$$R(I + \lambda A^\circ) \supset \overline{D(A^\circ)}, \quad R(I + \lambda A) \supset \overline{D(A)}$$

for every $0 < \lambda < \lambda_0$. Let $S^\circ(t)$ and $S(t)$ be the semigroups generated by $-A^\circ$ and $-A$, respectively. If,

(i) $0 \leq \omega_\rho, \omega \leq \alpha < \infty$,

(ii) $\lim_{\rho \to 0} \int_{x}^\circ x = \int_{x} x$ for every $x \in \overline{D}$ and $0 < \lambda < \lambda_0$, where

$$D = \bigcap_{\rho > 0} \overline{D(A^\circ)} \cap D(A),$$

then

(iii) $\lim_{\rho \to 0} S^\circ(t)x = S(t)x$ for every $x \in \overline{D}$ and the limit is uniform on bounded $t$ intervals.

Proof. Since $A^\circ = \lambda^{-1}(I - J^\circ)$, we have $A^\circ x \to A_\lambda x$ for every
\( x \in \overline{D} \) as \( \rho \to 0 \). Let \( x \in D, \ 0 < \lambda < 1/2\alpha \), then there exists a \( \rho_0 \) depending on \( x \) and \( \lambda \) such that
\[
| A_{\lambda}^{\rho} x | \leq | A_{\lambda} x | + 1 \leq 2 \| A x \| + 1 \quad \text{for} \quad 0 < \rho < \rho_0
\]
(see Lemma 2.1, (ii)). Consider

\[
| S^o(t) x - S(t) x | \leq | S^o(t) J_{\lambda}^{\rho} x - S(t) x | + e^{\omega T} | J_{\lambda}^{\rho} x - x |
\]
(3.1)

\[
| S^o(t) J_{\lambda}^{\rho} x - S(t) x | \leq | S^o(t) J_{\lambda}^{\rho} x - (J_{t/n}^{\rho})^n J_{\lambda}^{\rho} x |
\]
\[
+ | | (J_{t/n}^{\rho})^n J_{\lambda}^{\rho} x - (J_{t/n}^{\rho})^n x |
\]
\[
+ | (J_{t/n}^{\rho})^n - J_{t/n}^{\rho} x | + | J_{t/n}^{\rho} x - S(t) x |
\]
(3.2)

Using Lemma 2.4, we obtain
\[
| S^o(t) J_{\lambda}^{\rho} x - (J_{t/n}^{\rho})^n J_{\lambda}^{\rho} x | \leq 2t_0 \| A^o J_{\lambda}^{\rho} x \| e^{4\omega t}
\]
\[
\leq 2t_0 \| A x \| e^{4\omega t} (2 \| A x \| + 1)
\]
(3.3)

and
\[
| J_{t/n}^{\rho} x - S(t) x | \leq 2t_0 \| A x \| e^{4\omega t}.
\]
(3.4)

Since (by Lemma 2.1, (i)) \( J_{t/n}^{\rho} \) has the lipschitz constant \((1 - \omega \rho (t/n))^{-1}\), we have
\[
| (J_{t/n}^{\rho})^n J_{\lambda}^{\rho} x - (J_{t/n}^{\rho})^n x | \leq \left( 1 - \omega \rho \frac{t}{n} \right)^{-n} | J_{\lambda}^{\rho} x - x |
\]
\[
\leq (e^{\omega T} + 1) | J_{\lambda}^{\rho} x - x |
\]
(3.5)

provided that \( n \geq n_0 \). Finally,
\[
| J_{\lambda}^{\rho} x - x | = \lambda | A_{\lambda}^{\rho} x | \leq \lambda (2 \| A x \| + 1).
\]
(3.6)

Combining the inequalities (3.1)-(3.6) and assuming that \( 0 \leq t \leq T \), we obtain
\[
| S^o(t) x - S(t) x | \leq 2Tn^{-1} e^{4\omega T} (3 \| A x \| + 1)
\]
\[
\quad + \lambda (e^{\omega T} + 1)(2 \| A x \| + 1) + | (J_{t/n}^{\rho})^n x - J_{t/n}^{\rho} x |
\]
(3.7)

Given \( \epsilon > 0 \), we first fix \( 0 < \lambda < 1/2\alpha \) such that
\[
\lambda (e^{\omega T} + 1)(2 \| A x \| + 1) < \frac{\epsilon}{3}.
\]
Then we fix \( n \geq n_0 \) such that \( 2Tn^{-1/2} e^{4\omega T} (3 \| A x \| + 1) < \epsilon/3 \) and,
finally, we choose \( \rho < \rho_0 \) such that \( |(J_{t/n})^n x - J_t^2 x| < \varepsilon/3 \). Thus for every \( x \in D \), \( S^\rho(t)x \rightarrow S(t)x \). To prove that this limit is uniform in \( t \in [0, T] \), we use Lemma 2.4 and obtain

\[
|S(t)x - S^\rho(t)x| \leq |S^\rho(\tau) J_\lambda x - S(\tau)x| + e^{\alpha T} |J_\lambda x - x|
\]

\[
\leq |S^\rho(\tau) J_\lambda x - S(\tau)x| + 2e^{\alpha T} \|A^\rho J_\lambda x\| + \|Ax\| |t - \tau| + e^{\alpha T} |x - J_\lambda x|
\]

\[
\leq |S^\rho(\tau) x - S(\tau)x| + 2e^{\alpha T} (3 \|Ax\| + 1) |t - \tau| + 2e^{\alpha T} |x - J_\lambda x|
\]

\[
\leq |S^\rho(\tau) x - S(\tau)x| + 2e^{\alpha T} (3 \|Ax\| + 1) t - \tau| + 2e^{\alpha T} \cdot \lambda (2 \|Ax\| + 1).
\]  

(3.8)

Hence for every \( \varepsilon > 0 \) there exists \( \rho_0 > 0 \) depending on \( x \) and \( \varepsilon \) such that

\[
|S(t)x - S^\rho(t)x| \leq |S(\tau)x - S^\rho(\tau)x| + \varepsilon |t - \tau| + \varepsilon
\]

for \( 0 < \rho < \rho_0 \),  

(3.9)

where \( C = 2e^{\alpha T} (3 \|Ax\| + 1) \). This implies the uniform convergence of \( S^\rho(t)x \) to \( S(t)x \) for \( t \in [0, T] \) and \( x \in D \). Finally, since

\[
|S(t)x - S(t)y| \leq e^{\alpha T} |x - y|
\]

and

\[
|S^\rho(t)x - S^\rho(t)y| \leq e^{\alpha T} |x - y|,
\]

the result is true for every \( x \in \bar{D} \) and the proof is complete.


**Theorem 3.2.** Let \( \{T(\rho)\}_{\rho > 0} \) be a family of mappings from a closed convex subset \( C \) of \( X \) into itself. Let \( A \in \mathcal{A}(\omega) \) such that \( D(A) = C \) and \( R(I + AA) \supset D(A) \). Let \( S(t) \) be the semigroup generated by \(-A\) and assume that

\[
|T(\rho)x - T(\rho)y| \leq M(\rho)|x - y| \quad \forall x, y \in C, \quad \rho > 0
\]

with

\begin{enumerate}
\item[(i)] \( M(\rho) = 1 + \omega \rho + o(\rho) \) as \( \rho \rightarrow 0 \)
\item[(ii)] \( J_\lambda x = (I + (\lambda/\rho)(I - T(\rho)))^{-1}x \rightarrow J_\lambda x \) for every \( x \in \bar{D(A)} \)
\end{enumerate}

and \( 0 < \lambda < \lambda_0 \).
then
\[ \lim_{n \to \infty} T \left( \frac{t}{n} \right)^n x = S(t) x \quad \text{for every } x \in \overline{D(A)} \]  (3.10)
and the limit is uniform on bounded t intervals.

**Proof.** Let \( A^\rho = \rho^{-1}(I - T(\rho)) \). Then \( D(A^\rho) = \overline{D(A^\rho)} = C \), and by Lemma 2.2 \( R(I + \lambda A^\rho) \subseteq \overline{D(A^\rho)} \) for \( 0 < \lambda < \lambda_1 \), and \( A^\rho \in \mathcal{A}(\rho^{-1}(M(\rho) - 1)) \). We denote by \( S^\rho(t) \) the semigroup generated by \(-A^\rho\). By assumption (i) we have \( \rho^{-1}(M(\rho) - 1) = \omega + o(1) \). Thus for \( \rho < \rho_0 \), \( \rho^{-1}(M(\rho) - 1) \leq \omega + \varepsilon \). By assumption (ii), we have \( J^\rho x \to J^\lambda x \) for every \( x \in D(A) \) and \( 0 < \lambda < \lambda_0 \) and, therefore, by Theorem 3.1,
\[ \lim_{\rho \to 0} S^\rho(t) x = S(t) x \quad \text{for every } x \in \overline{D(A)} \]  (3.11)
and the limit is uniform on \([0, T]\). Now,
\[ | S^\rho(np) x - T(\rho)^n x | \leq K(\rho, n) | x - J^\rho x | + | S^\rho(np) J^\rho x - T(\rho)^n J^\rho x |, \]  (3.12)
where
\[ K(\rho, \eta) = e^{n(M(\rho) - 1)} + M(\rho)^n. \]
Let \( \tilde{S}^\rho(t) \) be the semigroup generated by \( T(\rho) - I \); then \( S^\rho(pt) = \tilde{S}^\rho(t) \). Using Lemma 2.3 we have
\[ | S^\rho(np) J^\rho x - T(\rho)^n J^\rho x | = | \tilde{S}^\rho(n) J^\rho x - T(\rho)^n J^\rho x | \leq H(\rho, \eta) | A^\rho J^\rho x | \leq \lambda^{-1}H(\rho, \eta) | x - J^\rho x |, \]  (3.13)
where
\[ H(\rho, \eta) = M(\rho)^n e^{n(M(\rho) - 1)[n^2(M(\rho) - 1)^2 + n(M(\rho) - 1) + n]^{1/2}} \cdot \rho. \]
Combining (3.12) and (3.13), we obtain
\[ | S^\rho(np) x - T(\rho)^n x | \leq (K(\rho, n) + \lambda^{-1}H(\rho, n)) | x - J^\rho x |. \]  (3.14)
Let \( x \in D(A) \) and \( 0 < \lambda < 1/2\omega \); then substituting \( \rho = t/n \) into (3.14) yields
\[ \left| S^{1/n}(t) x - T \left( \frac{t}{n} \right)^n x \right| \leq \left( K \left( \frac{t}{n}, n \right) + \lambda^{-1}H \left( \frac{t}{n}, n \right) \right) \| x - J^{1/n}_\lambda x \| \]
\[ \leq \left( K \left( \frac{t}{n}, n \right) + \lambda^{-1}H \left( \frac{t}{n}, n \right) \right) [2\lambda \| Ax \| + \| J^\lambda x - J^{1/n}_\lambda x \|] \]
\[ = 2\lambda K \left( \frac{t}{n}, n \right) \| Ax \| + 2H \left( \frac{t}{n}, n \right) \| Ax \| \]
\[ + K \left( \frac{t}{n}, n \right) + \lambda^{-1}H \left( \frac{t}{n}, n \right) \| J^\lambda x - J^{1/n}_\lambda x \| \]  (3.15)
From our assumption (i) it follows that \( M(t/n)^n \) and \( n(M(t/n) - 1) \) are uniformly bounded as \( n \to +\infty \) and \( t < T \). Therefore, \( K(t/n, n) \leq C_1 \) and \( H(t/n, n) \leq C_2 n^{-1/2} \) and

\[
\left\| S^{t/n}(t) x - T \left( \frac{t}{n} \right)^n x \right\| \leq 2\lambda C_1 \| Ax \| + 2C_2 n^{-\frac{1}{2}} \| Ax \|
+ (C_1 + C_2 \lambda^{-1} n^{-1}) | J^i x - J^{i/n} x | .
\]

Finally, for \( x \in D(A) \), we have

\[
\left| S(t) x - T \left( \frac{t}{n} \right)^n x \right| \leq \left| S(t) x - S^{t/n}(t)x \right| + 2\lambda C_1 \| Ax \| + 2C_2 n^{-\frac{1}{2}} \| Ax \|
+ (C_1 + C_2 \lambda^{-1} n^{-1}) | J^i x - J^{i/n} x | .
\]

Given \( \varepsilon > 0 \), we first choose \( \lambda \) so that \( 2\lambda C_1 \| Ax \| < \varepsilon/2 \), and then choose \( n \) so large that the sum of all the other terms is less than \( \varepsilon/2 \). Thus \( T(t/n)^n x \to S(t)x \) for \( x \in D(A) \) uniformly in \( t \in [0, T] \). Since

\[
\left| S(t) x - S(t) y \right| \leq e^{\omega T} \left| x - y \right|
\]

and

\[
\left| T \left( \frac{t}{n} \right)^n x - T \left( \frac{t}{n} \right)^n y \right| \leq C_0 \left| x - y \right|
\]

where \( C_0 \) is independent of \( n \), the result is true for every \( x \in \overline{D(A)} \).

4. Applications

We start this section by showing that Trotter's theorem for the convergence of semigroups of linear operators is a direct consequence of Theorem 3.1. For simplicity, we treat only the case \( \omega = \omega_0 = 0 \).

**Theorem (Trotter).** Let \( S^\rho(t) \) be a semigroup of contractions for \( \rho > 0 \) with generator \(-A^\rho\). Suppose that for some \( \lambda_0 > 0 \), \( \lim_{\rho \to 0} J^i_{\lambda_0} x \) exists for every \( x \in X \) and denote the limit by \( J^i_{\lambda_0} x \). If \( R(J^i_{\lambda_0}) = X \), then there exists a semigroup of contractions \( S(t) \) with generator \(-A\) such that \( J^i_{\lambda_0} x = (I + \lambda_0 A)^{-1} x \) and \( S^\rho(t) x \to S(t)x \) as \( \rho \to 0 \) uniformly on bounded intervals.

**Proof.** From the theory of semigroups of linear operators it is well known (see, e.g., [6]) that \( A^\rho \) are densely defined closed linear operators. Moreover, \( R(I + \lambda A^\rho) = X \) for every \( \lambda > 0 \). It is not
difficult to show that \( J^{\rho}x \) converges as \( \rho \to 0 \) for every \( \lambda > 0 \) and \( x \in X \) (see [15, IX.12]). Let \( A(\lambda) = \{ [J^x, \lambda^{-1}(x - J^x)] : x \in X \} \). Then, using the resolvent formula (2.3), one proves that \( A(\lambda) \) is independent of \( \lambda \) and is accretive. Let \( A \) be the closure in \( X \times X \) of \( A(\lambda) \); then \( A \) is accretive, \( R(I + \lambda A) = X \), and \( J_{\lambda_0}x = (I + \lambda_0 A)^{-1}x \). Therefore, \(-A\) generates a semigroup \( S(t) \) by Theorem I, and \( S(t)x \to S(t)x \) by Theorem 3.1.

Note that if \( A \) is linear, then \( S(t)x \) defined by Theorem I is differentiable for every \( t > 0 \) and \( x \in D(A) \) and satisfies the initial-value problem (1.2). Thus in the linear case there is no difference whether the semigroup \( A'(t) \) is related to its generator through Theorem I or through the initial-value problem (1.2).

Our next result shows that under certain restrictions, the convergence of \( A^\rho x \) to \( Ax \) in some sense implies the convergence of \( J^{\rho}x \) to \( J^x \).

**Theorem 4.1.** (i) Let \( A \in \mathcal{A}(\omega) \) be single valued and satisfy \( R(I + \lambda A) \supset D(A) \) for \( 0 < \lambda < 1/\omega. \) Then \( A \in \mathcal{A}(\omega) \) (\( A \) is the closure in \( X \times X \) of \( A \)) and \( R(I + \lambda A) \supset D(A) = D(A) \).

(ii) Let \( A^\rho \in \mathcal{A}(\omega_0) \) be single valued and satisfy

\[
R(I + \lambda A^\rho) \supset D(A^\rho) \quad \text{for} \quad 0 < \lambda < \frac{1}{\omega_0}.
\]

If \( D(A^\rho) \supset D(A) \), \( 0 \leq \omega_\rho \), \( \omega \leq \alpha < \infty \) and \( A^\rho x \to Ax \) for every \( x \in D(A) \), then

\[
\lim_{\rho \to 0} J^\rho x = J^x \quad \text{for every} \quad x \in D(A),
\]

where \( J^x = (I + \lambda A)^{-1} \).

**Proof.** (i) Clearly, \( D(A) \supset D(\bar{A}) \supset D(A) \), and therefore \( D(A) = D(A) \). Let \( f \in D(A) \supset R(I + \lambda A) \) and let \( f_n = x_n + \lambda Ax_n \) such that \( f_n \to f \). Since \( A \in \mathcal{A}(\omega) \), we have

\[
|f_n - f_m| = |x_n - x_m + \lambda(Ax_n - Ax_m)| \geq (1 - \lambda \omega)|x_n - x_m|.
\]

Therefore, if \( 0 < \lambda < 1/\omega \), then \( x_n \to x \). Hence,

\[
Ax_n = \lambda^{-1}(f_n - x_n) \to \lambda^{-1}(f - x),
\]

i.e., \( x \in D(\bar{A}) \) and \( \lambda^{-1}(f - x) \in \bar{A}x \) which is equivalent to \( f \in \mathcal{A} \) and \( \lambda \bar{A}x \in R(I + \lambda \bar{A}) \).
(ii) Using the assumption $A^o \in \mathcal{A}(\omega_o)$ at the point $J_\lambda x$ and $u \in D(A)$, we have

$$|x - (u + \lambda A^o u)| \geq (1 - \lambda \omega_o)|J_\lambda x - u| \quad \text{for } 0 < \lambda < \frac{1}{\alpha}.$$  

Thus

$$|J_\lambda x - J_\lambda x| \leq |J_\lambda x - u| + (1 - \lambda \alpha)^{-1} |x - (u + \lambda A^o u)|$$

and

$$\lim_{\rho \to 0} |J_\lambda x - J_\lambda x| \leq |J_\lambda x - u| + (1 - \lambda \alpha)^{-1} |x - (u + \lambda A^o u)| \quad \forall u \in D(A).$$

Since $\lambda^{-1}(x - J_\lambda x) \in \mathcal{A}(J_\lambda x)$, there is (by the definition of $\mathcal{A}$) a sequence $u_n \in D(A)$ such that $u_n \to J_\lambda x$ and $Au_n \to \lambda^{-1}(x - J_\lambda x)$. Hence,

$$\lim_{\rho \to 0} |J_\lambda x - J_\lambda x| = 0.$$  

As a direct consequence of Theorems 3.1 and 4.1, we obtain

**Corollary 4.2.** Let $A \in \mathcal{A}(\omega_o)$ be single valued satisfying $R(I + \lambda A) \supset D(A)$; then $-A$ generates a semigroup of type $\omega$ on $D(A)$. Let $A^o \in \mathcal{A}(\omega_o)$ be single valued and satisfying $R(I + \lambda A^o) \supset D(A^o)$ and let $S^o(t)$ be the semigroup of type $\omega_o$ generated by $-A^o$. If

(i)  \[ 0 \leq \omega_o, \omega \leq \alpha < \infty, D(A^o) \supset D(A) \quad \text{for every } \rho > 0. \]

(ii) $A^o x \to Ax$ for every $x \in D(A)$,

then

$$\lim_{\rho \to 0} S^o(t)x = S(t)x \quad \text{for every } x \in D(A)$$

and the limit is uniform on bounded $t$ intervals.

**Remark.** Corollary 4.2 is an extension of the main result of [10]. Theorem 1 of [10] follows directly from Corollary 4.2, assuming $D(A) = X$, for any reflexive Banach space, since in a reflexive Banach space $S(t)x$ given by Theorem I is differentiable a.e. in $t$ for every $x \in D(A)$. The assumption that $X^*$ is uniformly convex, assumed in Theorem 1 of [10], can, therefore, be replaced by $X$ reflexive. Corollary 4.2 also extends Theorems 2.2 and 2.3 of [9].

As a direct consequence of Theorem 3.2 and Theorem 4.1, we have
Corollary 4.3. Let $T(t)$ be Lipschitz with constant $M(t)$ mapping a closed convex subset $C$ of $X$ into itself. Let $A \in \mathcal{A}(\omega)$ be single valued such that $\overline{D(A)} = C$ and $R(I + \lambda A) \subseteq D(A)$. Then $-\overline{A}$ generates a semigroup of type $\omega$ on $C$. If

(i) $M(t) = 1 + \omega t + o(t)$ as $t \to 0$,
(ii) $t^{-1}(x - T(t)x) \to Ax$ as $t \to 0$ for every $x \in D(A)$, then

$$\lim_{n \to \infty} T\left(\frac{t}{n}\right)^n x = S(t) x \quad \text{for every } x \in \overline{D(A)}$$

and the limit is uniform on bounded $t$ intervals.

Remark. Corollary 4.3 is an extension of Theorem 1 of [11]. Assuming $C = X$, we obtain a generalization of Theorem 1 of [11]. It follows that the assumption that $X^*$ is uniformly convex, assumed in this theorem, is superfluous and it is sufficient to assume that $X$ is reflexive in order to obtain exactly the same results.

References

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