Biased SNR Estimation using Pilot and Data Symbols in BPSK and QPSK Systems

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Abstract: In wireless communications, knowledge of the signal-to-noise ratio is required in diverse communication applications. In this paper, we derive the variance of the maximum likelihood estimator in the data-aided and non-data-aided schemes for determining the optimal shrinkage factor. The shrinkage factor is usually the constant that is multiplied by the unbiased estimate and it increases the bias slightly while considerably decreasing the variance so that the overall mean squared error decreases. The closed-form biased estimators for binary-phase-shift-keying and quadrature-phase-shift-keying systems are then obtained. Simulation results show that the mean squared error of the proposed method is lower than that of the maximum likelihood method for low and moderate signal-to-noise ratio conditions.

Index Terms: Biased estimation, maximum likelihood (ML), phase shift keying, signal-to-noise ratio (SNR).

I. INTRODUCTION

Estimation of the signal-to-noise ratio (SNR) is essential in modern communication systems because various types of systems use algorithms that depend on a SNR estimate for proper operation. In turbo decoders, both the determined signal power and the noise power are required. Rate adaptive transmission systems often use the SNR to adapt the modulation scheme or the coding rate. A radio communication system is generally composed of multiple layers and the first layer, called physical layer or layer 1 (L1), is responsible for transmitting the binary information from a transmitter to a receiver through a radio channel. The channel being referred to is a physical transmission medium such as a wire or a multiplexed medium such as a radio channel. Depending on the channel conditions, different sets of L1 algorithms are developed such that they are robust against signal distortions caused by the propagation and interference conditions of the channel. In order to select the adequate decisions of L1 modes, link adaptation mechanisms require knowledge of the transmission quality over the channel conditions of the desired user. The symbol error rate, bit error rate, received signal strength and SNR can be used as metrics for determining the channel quality and the SNR has been the major point of interest in recent work. Code-division multiple-access (CDMA) systems, such as time division duplex CDMA (TDD/CDMA), time division synchronous CDMA (TDS/CDMA), frequency division duplex CDMA (FDD/CDMA), and CDMA 2000, utilize SNR estimation for power control to maintain the required link quality while using the minimum transmitted power. SNR estimation methods can be divided into data-aided (DA) methods and non-data-aided (NDA) methods. DA estimators rely on the insertion of pilot symbols in the data frame, whereas NDA estimators ignore the statistical information about the transmitted data, usually leading to poor performance when the SNR is low. Several studies on SNR estimation exist. Pauluzzi [1] investigated the SNR estimators of baseband M-ary phase-shift-keying (MPSK) TxDA and RxDA symbols. The TxDA method (DA method) uses a pilot signal with known training sequences, which provides enhanced performance. The drawback of TxDA is that additional equipment is required to process the training sequence. The RxDA method (NDA method), including the decision-directed method that uses the decision of the receiver instead of known training symbols, uses data symbols. Although the RxDA algorithm works well at high SNR regimes, its performance degrades at low SNR region. All estimators could be extended to complex modulation systems except for the split-symbol moments estimator. Further, the signal-to-noise variance (SNV) method, in which the average of the absolute values of the received samples is used for estimating the amplitude, has also been employed. The maximum likelihood (ML) and SNV methods were mostly efficient compared to the other algorithms, and the difference in performance between the TxDA and RxDA methods decreased with decreasing block length in the ML and SNV estimators. Alaghia [2] derived a Cramér-Rao lower bound (CRLB) for estimating the SNR of binary phase-shift-keying (BPSK) and quadrature phase-shift-keying (QPSK) modulated signals. A significant difference between the bounds for the NDA and DA methods was observed for low SNR conditions. However, under high SNR conditions, both methods showed identical bounds. Gappmair [3] extended Alaghia’s work to any M-ary PSK systems and linear modulation schemes for estimating the SNR using an NDA estimator, by exploiting the axis/halfplane symmetry of conventional modulation schemes. However, this method still depended on a numerical method for calculating the integrals. Bellili [4] developed an exact analytical CRLB for the NDA SNR estimator for square quadrature amplitude modulation (QAM) constellations. Chen [5] derived an ML SNR estimator that uses pilot and data symbols together, but it was restricted to the BPSK system and the mean squared error (MSE) performance was inferior to the second- and fourth-order moment (M2, M4) estimator for low SNR conditions. The performance of NDA-based methods tends to deviate considerably from the CRLB in low SNR
regimes. In order to solve this problem, recent activities focused on the code-aware (CA) algorithm. Wu [6],[7] extended the CA-based SNR estimator to higher order modulation schemes using the expectation maximization (EM) and ML SNR estimator for coded linearly modulated signals. The code-aided SNR estimator outperformed the NDA estimators for low-SNR conditions. Bergmann [8] extended the joint estimation of the carrier phase and SNR to the CA scheme, where a posteriori average of the symbol and symbol energy was introduced and a posteriori probability was obtained by LDPC or turbo codes. The CA joint solution improved the performance of the NDA methods in low SNR regimes. A closed-form SNR estimation method that uses the pilot and data symbols with different constellations [9] has been reported. The proposed method is distinguished from that in [9], because the method described in [9] is basically a moment-based method in the NDA case. Further, the method in [9] should be implemented in an iterative manner or the solution should be tabulated to attain a DA-NDA SNR estimator. In this paper, we propose a closed-form biased estimator for BPSK and QPSK modulation systems using the pilot and data symbols. The application of this estimator is restricted to BPSK and QPSK systems because the closed-form NDA ML SNR estimator cannot be obtained for other PSK and QAM systems [10], [11]. However, the proposed estimator has the advantage of being a closed-form estimator. In statistics and signal processing, the constraint of unbiasedness is often a practical one, as the MSE can be minimized over this class, i.e., the statistic does not depend on the unknown parameter. However, in some problems, restricting attention to unbiased approaches leads to unreasonable solutions, that may be independent of the estimated parameters [12], [13]. Further, the variance can be reduced at the expense of increasing the bias, while ensuring that the overall estimation error is reduced. Thus, unbiasedness does not necessarily lead to a small estimation error [14]. Consequently, the determination of estimators is typically subject to a biased estimator in which a trade-off between variance and bias is utilized. Biased estimation has been used in a various signal processing applications, such as image restoration, smoothing techniques in time series analysis, spectrum estimation, and wavelet denoising. In this paper, we propose a biased estimator in the context of SNR estimation. The proposed estimator outperforms the ML estimator in the low and moderate SNR regimes. To determine the optimal shrinkage factor, the variance of the ML SNR estimator in DA-NDA schemes for BPSK and QPSK systems is derived. It is shown via simulation results that the biased estimator employing pilot and data symbols is superior to the ML estimator in low SNR settings. The rest of our paper is organized as follows. Section II details the proposed algorithm, and Section III discusses the simulation results and evaluates the performance of the proposed algorithm. Finally, Section IV presents the conclusions.

II. PROPOSED BIASED SNR ESTIMATION METHOD USING PILOT AND DATA SYMBOLS

The proposed estimation procedure is derived for BPSK and QPSK communication systems. The measurement equation is represented as follows

\[ r_i = hx_i + n_i, \quad i = 1, 2, \cdots, P + L \]  

where \( h \) is the positive channel gain, \( x_i \in \{-1, 1\} \) is the transmitted BPSK signal of the \( i \)th symbol, and \( n_i \) is the noise component of the \( i \)th symbol, which is assumed to be a Gaussian random variable with \( N(0, \sigma^2) \). \( P \) is the number of pilot symbols and \( L \) is the number of data symbols. The pilot symbols are assumed to be \( x_i = 1 \) \((i = 1, 2, \cdots, P)\). By assuming \( n_i \) \((i = 1, 2, \cdots, P + L)\) are independent, the following likelihood function is derived from [5]

\[ f(r; h, \sigma^2) = \left( \frac{\gamma}{2\pi h^2} \right)^{P+L} \prod_{i=1}^{P+L} \cos \left( \frac{\gamma r_i}{h} \right) \eta \]  
(2)

where \( \eta = \exp \left( -\frac{2}{h^2} \sum_{i=1}^{P+L} r_i^2 - \frac{(P+L)\gamma}{2} + \frac{2}{\sigma^2} \sum_{i=1}^{P+L} r_i \right) \) and \( \gamma = h^2/\sigma^2 \). Equation (2) can be approximated using the high-SNR assumption as follows

\[ f(r; h, \sigma^2) \approx \left( \frac{\gamma}{2\pi h^2} \right)^L \left( \frac{1}{2} \right)^L \exp \left( \frac{\gamma \sum_{i=1}^{P+L} |r_i|}{h} \right) \eta. \]  
(3)

The aim is to estimate the SNR, i.e., \( \gamma = h^2/\sigma^2 \). Differentiating the log-likelihood function of (3) with respect to \( h \) and \( \gamma \), and solving for \( h \) and \( \gamma \) yields [5]

\[ \hat{\gamma}_{\text{ML}}_{\text{BPSK}} = \frac{1}{P+L} \sum_{i=1}^{P+L} |r_i|^2 \]  
(4)

\[ \gamma_{\text{ML}}_{\text{BPSK}} = \left( \frac{\hat{\gamma}_{\text{ML}}_{\text{BPSK}}}{h_{\text{ML}}_{\text{BPSK}}} \right)^2. \]  
(5)

Assuming perfect carrier and symbol timing recovery, the measurement equation in the QPSK scenario is the same as (1), except that \( x_i \in \{ \pm(1/\sqrt{2}) \pm j(1/\sqrt{2}) \} \) is the transmitted QPSK signal of the \( i \)th symbol, and \( n_i \) is the complex noise component of the \( i \)th symbol with independent real and imaginary parts each having zero mean and the same variance of \( \sigma^2/2 \). The pilot symbols are assumed to be \( x_i = \pm(1/\sqrt{2}) \pm j(1/\sqrt{2}) \) \((i = 1, 2, \cdots, P)\). The likelihood function for QPSK DA symbols is expressed as

\[ f(r_i; h, \sigma^2) = \frac{1}{\pi \sigma^2} \exp \left( -\frac{1}{\sigma^2} |r_i - hx_i|^2 \right), \quad i = 1, 2, \cdots, P. \]  
(6)

The likelihood function for QPSK NDA symbols is represented as [15, (4.24)]

\[ f(r_i; h, \sigma^2) = \frac{1}{\pi \sigma^2} \exp \left( -\frac{|r_i|^2}{\sigma^2} \right) \exp \left( \frac{h^2}{\sigma^2} \right) \]  
\[ \times \cosh \left( \frac{\sqrt{2}h}{\sigma^2} \text{Re} \{r_i\} \right) \cosh \left( \frac{\sqrt{2}h}{\sigma^2} \text{Im} \{r_i\} \right), \]
\[ i = P + 1, P + 2, \cdots, P + L \]  
(7)

where \( \text{Re}(\cdot) \) and \( \text{Im}(\cdot) \) denote the real part and imaginary part, respectively. The combined likelihood function is then obtained
under the high SNR condition as
\[
f(r; h, \sigma^2) \simeq \left( \frac{\gamma}{\pi h^2} \exp(-\gamma) \right)^{P+L} \left( \frac{1}{2} \right)^{2L} \times \exp\left( - \frac{\gamma}{h^2} \sum_{i=1}^{P+L} |r_i|^2 \right) \times \exp\left( 2\frac{\gamma}{h} \sum_{i=1}^{P+L} \text{Re}\{r_i x_i\} \right) + \frac{\sqrt{\gamma}}{h} \sum_{i=P+1}^{P+L} (|\text{Re}\{r_i\}| + |\text{Im}\{r_i\}|) \tag{8}
\]
where \( \gamma = h^2/\sigma^2 \) and the approximation that \( \cosh(\beta) \simeq \exp(\beta)/2 \) when \( |\beta| \gg 1 \) was adopted. Differentiating the log-likelihood function of (8) with respect to \( h \) and \( \gamma \), and solving for \( h \) and \( \gamma \) yields
\[
\hat{\gamma}_{\text{MLE}}^Q = \frac{1}{P + L} \times \left[ \sum_{i=1}^{P} \text{Re}\{x_i^* r_i\} + \frac{1}{\sqrt{2}} \sum_{i=P+1}^{P+L} (|\text{Re}\{r_i\}| + |\text{Im}\{r_i\}|) \right], \tag{9}
\]
\[
\hat{\gamma}_{\text{ML}}^{QPSK} \simeq \frac{\left( \hat{\gamma}_{\text{MLE}}^{QPSK} \right)^2}{\gamma + \sum_{i=1}^{P+L} |r_i|^2 - \left( \hat{\gamma}_{\text{MLE}}^{QPSK} \right)^2}. \tag{10}
\]
The MSE performance of the efficient estimator can be further improved by using biased estimation \cite{16-18}. The biased estimator makes use of the property that the variance can be decreased by increasing the bias, while reducing the total MSE, which is represented as the sum of the variance and squared bias. The following estimator is proposed to improve the efficient estimator in \cite{18}
\[
\hat{\vartheta}^r = (1 + \alpha) \hat{\vartheta}^e \tag{11}
\]
where \( \hat{\vartheta}^r \) is the biased estimator, \( \alpha \) is a constant to be determined to minimize the MSE, and \( \hat{\vartheta}^e \) is the efficient unbiased estimator. The efficient estimator makes use of the fact that the MSE performance approximates the unbiased CRLB and its bias is zero ( \( E[\hat{\vartheta}^e] = \theta \), where \( \theta \) is the parameter to be estimated) \cite{19}. As in \cite{18}, we also assume a linear bias; the biased SNR estimator can then be represented as
\[
\hat{\gamma}^b = (1 + \alpha) \hat{\gamma}^e \tag{12}
\]
where \( \hat{\gamma}^e \) is the efficient estimator for determining the SNR and the bias is \( E[\hat{\gamma}^b] = \gamma = \alpha \gamma \) because \( E[\hat{\gamma}^e] = \gamma \). In the proposed SNR estimation, the ML estimator will be used as the efficient unbiased estimator. In a BPSK system, \( B \) is represented as follows (see Appendix A for more details)
In a similar manner, \( B \) in the QPSK scheme is represented as follows
\[
B = \frac{1}{P + L} \sum_{i=1}^{P+L} r_i^2 - \left( \hat{\gamma}_{\text{MLE}}^Q \right)^2 \approx \frac{\sigma^2}{P + L} \times \left\{ \sum_{i=1}^{P} \left( \frac{r_i - \hat{\gamma}_{\text{MLE}}^Q}{\sigma} \right)^2 + \sum_{i=P+1}^{P+L} \left( \frac{r_i - \hat{\gamma}_{\text{MLE}}^Q}{\sigma} \right)^2 \right\}. \tag{13}
\]
where \( x_i \) is the pilot symbol. The biased estimator is determined using the ML estimator in (5) and (10), and the MSE criterion. The MSE of the biased estimator is represented as
\[
\text{mse}[\hat{\gamma}^b] = \text{var}[\hat{\gamma}^b] + \{\text{bias}[\hat{\gamma}^b]\}^2 = \text{var}[(1 + \alpha) \gamma] + \{(1 + \alpha) \gamma - \gamma\}^2 = (1 + \alpha)^2 \text{var}[\gamma] + \alpha^2 \gamma^2. \tag{15}
\]
The constant \( \alpha \) is determined by minimizing the MSE. Differentiating the MSE with respect to \( \alpha \) and setting it equal to zero yields
\[
\alpha = \frac{-\text{var}[\gamma]}{\text{var}[\gamma] + \gamma^2}. \tag{16}
\]
In (16), \( \text{var}[\gamma]_{\text{BPSK}} \) is obtained as follows (see Appendix D for more details)
\[
\text{var}[\gamma]_{\text{BPSK}} = E \left[ \left( \frac{\hat{\gamma}^Q_{\text{MLE}}}{B} \right)^2 \right] - E \left[ \left( \hat{\gamma}^Q_{\text{MLE}} \right)^2 \right]^2 \sim 2\gamma^2 \frac{(P + L)^2}{(P + L - 3)^2(P + L - 5)} + 6\gamma \frac{(P + L)}{(P + L - 3)(P + L - 5)}, \tag{17}
\]
where \( \text{var}[\gamma]_{\text{QPSK}} \) is obtained as follows (see Appendix D for more details)
\[
\text{var}[\gamma]_{\text{QPSK}} \approx 8\gamma^2 \frac{(P + L)^2}{(2(P + L) - 3)^2(2(P + L) - 5)} + 12\gamma \frac{(P + L)}{(2(P + L) - 3)(2(P + L) - 5)}. \tag{18}
\]
Basu’s theorem \cite{19} is used for deriving (17) and (18). Subsequently, \( \alpha \) is found by substituting (17) and (18) into (16)
\[
\alpha_{\text{BPSK}} = \frac{-2(P + L)^2 \gamma^2 - 6(P + L - 3)(P + L)\gamma}{Q_1}, \tag{19}
\]
\[
\alpha_{\text{QPSK}} = \frac{-8(P + L)^2 \gamma^2 - 12(2(P + L) - 3)(P + L)\gamma}{Q_2}. \tag{20}
\]
where $Q_1 = \{(P + L - 3)(P + L - 5) + 2(P + L)^2\} \gamma^2 + 6(P + L - 3)(P + L) \gamma$ and $Q_2 = \{(2(P + L) - 3)(2(P + L) - 5) + 8(P + L)^2\} \gamma^2 + 12(2(P + L) - 3)(P + L) \gamma$.

The shrinkage factor can be approximated as the constant (independent of SNR) by considering only the second-order terms in (19) and (20) in order to reduce the computational complexity. Compared to the computational complexity of the ML estimator, the computational complexity of the proposed method (SNR dependent shrinkage factor) was higher by 32 floating-point operations (flops) in the BPSK case and by 36 flops in the QPSK case.

### III. SIMULATION RESULTS

In this section, the performance of the proposed biased SNR estimator was compared with the ML estimator. In the first simulation, the sample size of the pilot and data symbols were fixed to 180 and 70, respectively, and the SNR was varied from 0 to 30 dB. The number of iterations was 5,000 for all simulations. Fig. 1 shows that the normalized MSE (NMSE) of the proposed method was smaller than that of the ML estimator over the entire SNR range. As the SNR increased, the differences between the proposed method and the ML estimator decreased. The MSE of all methods were much larger than that of the NCRLB at low SNRs, because the performance of the NDA methods degrades, and the high SNR assumption in the derivation of the ML SNR estimator is not valid at low SNR regimes. Note that the NMSE of the proposed method was smaller than that of the ML estimator by about 1 dB in the low SNR regimes (below 5 dB).

The MSE of the proposed method, in which a shrinkage factor depends on the SNR was adopted, was lower than that of the suboptimal method in which a constant shrinkage factor was used. However, the suboptimal method has an advantage in terms of the convenience of implementation. It can be seen that the MSE performance of the proposed method is slightly inferior to the NCRLB in the high SNR regimes due to the approximation used in (3). The following NMSE and NCRLB criterion (normalized with respect to the square of the true value of the SNR) was used in the simulation

$$\text{NMSE} = \frac{\sum_{i=1}^{q} (\gamma - \hat{\gamma}_b(i))^2}{q \gamma^2}$$  \hspace{1cm} (23)

where $\gamma = h^2/\sigma^2$ in the BPSK and QPSK case, $\hat{\gamma}_b(i)$ is the biased SNR estimation value in the $i$th Monte-Carlo iteration, and $q$ is the total number of Monte-Carlo iteration

$$\text{NCRLB} = \frac{\text{CRLB}}{\gamma^2}.$$  \hspace{1cm} (24)

The unbiased CRLB for the biased DA-NDA SNR estimator is represented as follows

$$\text{CRLB}(u) = \text{trace}([\text{F}_{DA}(u) + \text{F}_{NDA}(u)]^{-1})$$  \hspace{1cm} (25)

where

$$[\text{F}(u)]_{ij} = -E \left[ \frac{\partial \ln p(r; u)}{\partial u_i \partial u_j} \right], u = [\gamma \sigma^2]^T.$$

$\text{F}_{DA}(u)$ is obtained as the closed-form in (26) and $\text{F}_{NDA}(u)$ is found by adopting the iterative method [3].

$$\text{F}_{DA,BPSK}(u) = \begin{pmatrix} \frac{P}{4\gamma} & \frac{P}{4\alpha^2} & \frac{P}{4\sigma^2} \\ \frac{P}{4\sigma^2} & (2 + \gamma)P/4\sigma^4 \end{pmatrix}, \quad \text{F}_{DA,QPSK(M \geq 4)}(u) = \begin{pmatrix} \frac{P}{2\gamma} & \frac{P}{2\alpha^2} & \frac{P}{2\sigma^2} \\ \frac{P}{2\sigma^2} & (2 + \gamma)P/2\sigma^4 \end{pmatrix}.$$  \hspace{1cm} (26)

Meanwhile, the biased NCRLB of the SNR is represented as

$$\text{CRLB}(\gamma) = [G]_{1,1}$$  \hspace{1cm} (27)

where $G = (I + D(u))([\text{F}_{DA}(u) + \text{F}_{NDA}(u)]^{-1}(I + D(u)))$, $I$ is the identity matrix, $D(u) = \partial b(u)/\partial u$, $b(u) = [E[\gamma] - \gamma E[\sigma^2 - \sigma^2]^T$ and $[G]_{1,1}$ is the component of the 1st row and 1st column of matrix $G$. Fig. 2 shows that the NMSE of the proposed method was smaller than that of the ML estimator at low and moderate SNR. As the SNR increases, the difference between the NMSEs of the proposed method and the ML estimator decreases. Note that the NMSE of the proposed method was smaller than that of the ML estimator by about 0.7 dB in the low SNR regimes (below 5 dB). The MSE of the proposed method is slightly larger than the NCRLB in the high SNR condition.
because the approximation was utilized in the derivation of (8) in the same with the BPSK case. In the second simulation, the SNR is fixed to 5 dB, and the data symbol size \( L \) varies from 10 to 70. The number of iterations is 5,000 for all simulations. Figs. 3 and 4 show that the NMSE of the proposed method was smaller than that of the ML estimator with various data symbol sizes for an SNR of 5 dB. In all results, as the number of NDA symbols \( L \) increased, the NMSE performance was more inferior to the NCRLB. This is caused because the NDA symbols do not use a priori, i.e., training sequences and the SNR is comparatively low (the performance of the NDA SNR estimator is much degraded in the low SNR condition). Note that the NMSE of the proposed method was smaller than that of the ML estimator by about 1.5 dB in BPSK case and 1 dB in QPSK case in all regimes of the sample number. Fig. 5 shows the comparison of \( h^2 \) and \( A \) for BPSK and various SNR settings. Fig. 6 shows the comparison of \( h \) and \( D = L/(P + L)\sigma \sqrt{2/\pi} \exp (-h^2/2\sigma^2) \). It can be seen that \( h \) is much larger than \( D \); thus, \( D \) can be neglected in the computation of \( E[\tilde{h}_{ML}^2] \) [see Appendix (33), (38)]. Fig. 7 shows the variation of the shrinkage factor. In this simulation, \( P \) and \( L \) were set to 180
and 70, respectively. The shrinkage factor was found to increase as the SNR increases (noise level decreases). On the other hand, the shrinkage factor decreases as the SNR decreases, i.e., the estimate becomes more shrunken towards the origin. It is known that the shrinkage method has larger MSE reduction in low SNR regimes than in high SNR regimes. This phenomenon can be explained as follows. For a specific example, let us assume identical, independent observations \( x_i, 1 \leq i \leq n \), where each \( x_i \) is a Gaussian random variable with mean \( \theta \) and variance \( \sigma^2 \). Clearly, the efficient estimator (ML or MVUE) for \( \theta \) is the sample mean \( \overline{x} = (1/n) \sum_{i=1}^{n} x_i \). Then, the biased (shrinkage) estimator \( \hat{\theta}^b \) is represented as

\[
\hat{\theta}^b = (1 + \alpha) \overline{x} = (1 - \frac{\text{var}[\overline{x}]/(\text{var}[\overline{x}])}{\text{var}[\overline{x}]}) \overline{x} = ((\overline{x})^2/(\text{var}[\overline{x}])) \overline{x}.
\]

The MSE of the biased estimator is represented as

\[
\text{MSE}[\hat{\theta}^b] = \text{var}[\overline{x}] \left[ \left( \overline{x} \right)^2 + \theta^2 \text{var}[\overline{x}] \right] / \left( \left( \overline{x} \right)^2 \text{var}[\overline{x}] \right)^2
\]

\[
= \text{var}[\overline{x}] \left[ \left( \frac{\overline{x}}{\text{var}[\overline{x}]} \right)^2 + \theta^2 \right] / \left( \text{var}[\overline{x}] \right)^2 + 2 \text{var}[\overline{x}] / \left( \text{var}[\overline{x}] \right) + 1
\]

\[
< \text{var}[\overline{x}] \left[ \left( \frac{\overline{x}}{\text{var}[\overline{x}]} \right)^2 + \frac{\theta^2}{\text{var}[\overline{x}]} \right]
\]

\[
= \left[ \frac{n \overline{x}^2}{\text{var}[\overline{x}]} + \theta^2 \right]. \tag{28}
\]

From (28), the MSE of the biased estimator is smaller as \( \sigma^2/n \) is larger, i.e., when the noise variance is larger (the SNR is lower), the MSE reduction is more remarkable. Figs. 8 and 9 show the comparison of the NMSE by simulation [(23)] and mathematical analysis [(30), (31)]. The MSE for the mathematical analysis is represented as follows

\[
\text{NMSE}(\hat{\gamma}^b_{\text{BPSK}}) = \text{bias}^2[\hat{\gamma}^b_{\text{BPSK}}] + \text{var}[\hat{\gamma}^b_{\text{BPSK}}] / \gamma^2
\]

\[
= \left[ (1 + \alpha) \text{bias}[\hat{\gamma}^b_{\text{BPSK}}] - \gamma \right]^2 + (1 + \alpha)^2 \text{var}[\hat{\gamma}^b_{\text{BPSK}}] / \gamma^2. \tag{29}
\]

From (40), \( E[\hat{\gamma}^b_{\text{BPSK}}] = ((P + L)/(P + L - 3))\gamma \), \( \text{var}[\hat{\gamma}^b_{\text{BPSK}}] \approx 2\gamma^2 [(P + L)^2/((P + L - 3)^2)(P + L - 5)] \\ + 6\gamma [(P + L)/((P + L - 3)(P + L - 5))]. \)

Substituting \( E[\hat{\gamma}^b_{\text{BPSK}}] \) and \( \text{var}[\hat{\gamma}^b_{\text{BPSK}}] \) into (29) results in

\[
\text{NMSE}(\hat{\gamma}^b_{\text{BPSK}}) = \left\{ \frac{3 + (P + L)\alpha}{(P + L - 3)} \right\}^2 \frac{(P + L)}{(P + L - 3)(P + L - 5)} \left\{ \frac{P + L}{P + L - 3} + 3 \right\}. \tag{30}
\]

In the same manner, the NMSE in the QPSK case is obtained as follows

\[
\text{NMSE}(\hat{\gamma}^b_{\text{QPSK}}) = \left\{ \frac{3 + 2(P + L)\alpha}{(2(P + L) - 3)} \right\}^2 \frac{4(1 + \alpha^2)(P + L)}{(2(P + L) - 3)(2(P + L) - 5)} \left\{ \frac{2(P + L)}{2(P + L) - 3} + 3 \right\}. \tag{31}
\]

The NMSEs from the analysis results and simulation were approximately the same for the BPSK and QPSK schemes, except
for very low SNR regimes. The discrepancy in low SNR regimes is caused because some approximations, e.g., the computation of $E[\hat{h}_{ML}^2]$ and $E[(\hat{h}_{ML}^2)^2]$ (when $L$ is large compared to $P$); and the assumption of a high SNR condition in the derivation of $\hat{h}_{ML}$ for the NDA schemes, are not valid in the low SNR regime.

**IV. CONCLUSIONS**

A biased SNR estimation method was compared with the ML SNR estimator for BPSK and QPSK communication systems. The variance of the ML SNR estimator for BPSK and QPSK systems in the DA-NDA scheme was derived for determining the optimal shrinkage factor, and the biased SNR estimator was developed for BPSK and QPSK systems. The simulation results showed that the MSE average, employing the biased estimator, was smaller than that of the ML estimator, particularly for low and moderate SNR conditions.

**APPENDIX**

**A. Derivation of (13)**

\[
B = \frac{1}{P + L} \sum_{i=1}^{P+L} \sigma_i^2 - (\hat{h}_{ML})^2 \\
= \frac{1}{P + L} \sum_{i=1}^{P+L} (r_i - \hat{h}_{ML})^2 + \frac{2}{P + L} \sum_{i=1}^{P+L} (\hat{h}_{ML})^2 - (\hat{h}_{ML})^2 \\
= \frac{P}{P + L} \left\{ \frac{1}{P} \sum_{i=1}^{P} (r_i - \hat{h}_{ML})^2 \\
+ \frac{2}{P} \sum_{i=1}^{P} (r_i - \hat{h}_{ML}) (\hat{h}_{ML}) + (\hat{h}_{ML})^2 \right\} \\
+ \frac{L}{P + L} \left( \frac{1}{L} \sum_{i=P+1}^{P+L} (r_i - \hat{h}_{ML})^2 \\
+ \frac{2\hat{h}_{ML}}{L} \sum_{i=P+1}^{P+L} (r_i - \hat{h}_{ML}) x_i + (\hat{h}_{ML})^2 \right) - (\hat{h}_{ML})^2 \\
\approx \frac{\sigma^2}{P + L} \sum_{i=1}^{P} \left( \frac{r_i - \hat{h}_{ML}}{\sigma} \right)^2 + \frac{2P\hat{h}_{ML}}{P + L} \sum_{i=1}^{P} (r_i - \hat{h}_{ML}) \\
+ \frac{P}{P + L} (\hat{h}_{ML})^2 + \frac{\sigma^2}{P + L} \sum_{i=P+1}^{P+L} \left( \frac{r_i - \hat{h}_{ML}}{\sigma} \right)^2 \\
+ \frac{2L\hat{h}_{ML}}{P + L} \sum_{i=P+1}^{P+L} (r_i - \hat{h}_{ML}) x_i + \frac{L}{P + L} (\hat{h}_{ML})^2 \]

where $\hat{h}_{ML}$ is assumed to be close to the true parameter $h$, the time average is assumed to be approximately equal to the expected value $(E[y] \approx \frac{1}{T} \sum_{t=1}^{T} y_t)$, and $n_t$ and $x_t$ are independent.

**B. Derivation of $E[(\hat{h}_{ML})^2]$**

In the BPSK case,

\[
E[(\hat{h}_{ML})^2] = \frac{1}{(P + L)^2} E \left[ \left( \sum_{i=1}^{P} r_i + \sum_{i=P+1}^{P+L} |r_i| \right)^2 \right] \\
= \frac{1}{(P + L)^2} \\
E \left[ \left\{ \sum_{i=1}^{P} r_i \right\}^2 + \left\{ \sum_{i=P+1}^{P+L} |r_i| \right\}^2 + 2 \sum_{i=1}^{P} \sum_{i=P+1}^{P+L} |r_i| \right] \\
= \frac{1}{(P + L)^2} \\
E \left[ \left\{ \sum_{i=1}^{P} (h + n_t) \right\}^2 + \left\{ \sum_{i=P+1}^{P+L} |hx_t + n_t| \right\}^2 \right. \\
+ 2 \sum_{i=1}^{P} \sum_{i=P+1}^{P+L} |hx_t + n_t| \left. \right] \\
= \frac{1}{(P + L)^2} \left( Ph^2 + P\sigma^2 + P(P - 1)h^2 + Lh^2 + L\sigma^2 \right)
\]
where $\Phi(\cdot)$ can be approximated as $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-t^2/2)dt$. When $x_i = 1$, $E[|x_i + n_i|] = \sigma \sqrt{2/\pi} \exp(-h^2/2\sigma^2) + h[1 - 2\Phi(h/\sigma)]$. Because $1 \leq h/\sigma \leq 10$ in the simulation settings and $\Phi(h/\sigma)$ is nearly zero [$\Phi(1) = 0.8413$] in this range, $E[|x_i + n_i|]$ can also be approximated as $\sigma \sqrt{2/\pi} \exp(-h^2/2\sigma^2) + h$. Thus, $\sum_{i=1}^{P+L} E[|x_i + n_i|] \simeq Lh + L\sigma \sqrt{2/\pi} \exp(-h^2/2\sigma^2)$. 

D. Derivation of (17) and (18)

$\var[\hat{\gamma}^{ML}_{\text{het}}]$ in the BPSK scheme is obtained as follows

\[
\var[\hat{\gamma}^{ML}_{\text{het}}] = E\left[\frac{(\hat{\gamma}^{ML}_{\text{het}})^2}{B}\right] - \left[E\left\{\frac{(\hat{\gamma}^{ML}_{\text{het}})^2}{B}\right\}\right]^2
\]

where Basu’s theorem is used. By Basu’s theorem, the complete sufficient statistic ($\hat{\gamma}^{ML}_{\text{het}}$) and ancillary statistic ($B$) are independent [19].

\[
E[\hat{\gamma}^{ML}_{\text{het}}] = \var[\hat{\gamma}^{ML}_{\text{het}}] + \{E[\hat{\gamma}^{ML}_{\text{het}}]\}^2,
\]

\[
\var[\hat{\gamma}^{ML}_{\text{het}}] \simeq (2\hat{\gamma}^{ML}_{\text{het}})\var[\hat{\gamma}^{ML}_{\text{het}}]
\]

[delta method, $\var(g(\hat{\theta})) = \{g'(\hat{\theta})\}^2 \var(\hat{\theta})$]

\[
\simeq (2\hat{\gamma}^{ML}_{\text{het}})\{E[\hat{\gamma}^{ML}_{\text{het}}]\}^2 - E[\hat{\gamma}^{ML}_{\text{het}}]E[1/B]^2
\]

(36)

where $L/(P+L)\sigma \sqrt{2/\pi} \exp(-h^2/2\sigma^2)$ is neglected because it is much smaller compared to $h$ (see Fig. 6 for more details). Substituting (38) into (37) yields

\[
E[\hat{\gamma}^{ML}_{\text{het}}] = (2\hat{\gamma}^{ML}_{\text{het}})\frac{\sigma^2}{P+L} + \left(h^2 + \frac{\sigma^2}{P+L}\right)^2
\]

\[
= h^4 + 4(\hat{\gamma}^{ML}_{\text{het}})^2\frac{\sigma^2}{P+L} + \frac{2h^2\sigma^2}{P+L} + \frac{\sigma^4}{(P+L)^2}
\]

(39)

Substituting (39) into (36) results in

\[
\var[\hat{\gamma}^{ML}_{\text{het}}] \simeq (P+L)^2 \left\{ \frac{2\gamma^2}{(P+L-3)(P+L-5)} \right\}
\]

(40)

Given a normally distributed random variable $X$ with mean $\mu$ and variance $\sigma^2$, the random variable $Y = |X|$ has a folded normal distribution. It is known that $E[Y]$ is represented as

\[
E[Y] = \sigma \sqrt{2/\pi} \exp(-\mu^2/2\sigma^2) + \mu[1 - 2\Phi(-\mu/\sigma)].
\]
where $\gamma = h^2/\sigma^2$ and $\hat{\gamma}_{ML}$ is approximated as $h$. The variance $\text{var}[\hat{\gamma}_{ML}]$ of the QPSK scheme is obtained as (41) substituting $2(\hat{\gamma} + L)$ into (40) as follows:

$$\text{var}[\hat{\gamma}_{ML}] \approx 8\gamma^2 \frac{(P + L)^2}{(2(P + L) - 3)^2(2(P + L) - 5)} + 12\gamma \frac{(P + L)}{(2(P + L) - 3)(2(P + L) - 5)}.$$  

(41)

REFERENCES


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To Whom It May Concern,

This is to certify that Prof. Joon-Hyuk Chang is a corresponding author of the paper entitled “Biased SNR Estimation using Pilot and Data Symbols in BPSK and QPSK Systems” published in 2014 JCN December Issue.

Sincerely,

H. Vincent Poor
Editor-in-Chief, the JCN