Harmonic Mappings Related to the Bounded Boundary Rotation

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Abstract

The aim of this paper is to give an investigation of the class of harmonic mappings related to the bounded boundary rotation. The class of bounded boundary rotation is generalized to the convex function.

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1 Introduction

Let $\Omega$ be the family of functions $\phi(z)$ which are regular in $\mathbb{D} = \{z\mid |z| < 1\}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Next, for arbitrary fixed numbers $A, B$, $-1 \leq B < A \leq 1$, denote by $\mathcal{P}(A, B)$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ regular in $\mathbb{D}$, such
that \( p(z) \) is in \( \mathcal{P}(A, B) \) if and only if
\[
p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)},
\]
for some functions \( \phi(z) \in \Omega \) and every \( z \in \mathbb{D} \). Geometrically, \( p(z) \) is in \( \mathcal{P}(A, B) \) if and only if \( p(0) = 1, p(\mathbb{D}) \) is inside the open disc centered on the real axis with diameter end points \( \mathbb{D}_1 = p(-1) = (1 - A)/(1 - B) \) and \( \mathbb{D}_2 = p(1) = (1 + A)/(1 + B) \). Special selection of \( A \) and \( B \) lead to familiar sets defined by the inequalities under the conditions \( p(0) = 1, M > \frac{1}{2} \) and \( 0 < \alpha < 1 \), we have:

- (i) \( \mathcal{P}(1, -1) \) is the set defined by \( \text{Rep}(z) > 0 \)
- (ii) \( \mathcal{P}(1 - 2\alpha, -1) \) is the set defined by \( \text{Rep}(z) > \alpha \)
- (iii) \( \mathcal{P}(1, 0) \) is the set defined by \( |p(z) - 1| < 1 \)
- (iv) \( \mathcal{P}(\alpha, 0) \) is the set defined by \( |p(z) - 1| < \alpha \)
- (v) \( \mathcal{P}(1, -1 + \frac{1}{M}) \) is the set defined by \( |p(z) - M| < M \)
- (vi) \( \mathcal{P}(\alpha, -\alpha) \) is the set defined by \( |\frac{p(z) - 1}{p(z) + 1}| < \alpha \)

Let \( \mathcal{A} \) be the class of analytic functions of the form \( s(z) = z + a_2z^2 + \cdots \) which are regular in \( \mathbb{D} \). Let \( \mathcal{C} \) denote the family of functions \( s(z) \in \mathcal{A} \) such that, \( s(z) \) is in \( \mathcal{C} \) if and only if
\[
1 + z \frac{s''(z)}{s'(z)} = p(z),
\]
for some \( p(z) \in \mathcal{P}(1, -1) \) and every \( z \in \mathbb{D} \). The class \( \mathcal{C} \) is called of convex functions and let \( s(z) \) be an element of \( \mathcal{A} \). If the equality
\[
z \frac{s'(z)}{s(z)} = p(z)
\]
is satisfied for some \( p(z) \in \mathcal{P}(1, -1) \) and every \( z \in \mathbb{D} \), then \( s(z) \) is called starlike functions, the class of starlike functions is denoted by \( \mathcal{S}^* \). Let \( \mathcal{S}^*(A, B) \) denote the family of functions \( s(z) \in \mathcal{A} \) such that, \( s(z) \) is in \( \mathcal{S}^*(A, B) \) if and only if
\[
z \frac{s'(z)}{s(z)} = p(z),
\]
for some \( p(z) \) in \( \mathcal{P}(A, B) \) and all \( z \) in \( \mathbb{D} \). The class \( \mathcal{S}^*(A, B) \) is called Janowski starlike functions, it is evidently \( \mathcal{S}^*(1, -1) = \mathcal{S}^* [2] \).
Moreover, let $\mathcal{P}_k(\rho)$ be the class of analytic functions $p(z)$ defined in $\mathbb{D}$ satisfying the properties $p(0) = 1$ and
\[
\int_0^{2\pi} \left| \frac{\text{Re}p(z) - \rho}{1 - \rho} \right| d\theta \leq k\pi,
\]
where $z = re^{i\theta}$, $k \geq 2$ and $0 \leq \rho < 1$. When $\rho = 0$, we obtain the class $\mathcal{P}_k = \mathcal{P}_k(0)$ defined in [7] and for $k = 2$, $\rho = 0$ we have the class $\mathcal{P}(1, -1)$ of functions with positive real part. We can write (1.5) as
\[
p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1 - 2\rho)ze^{-i\theta}}{1 - ze^{-i\theta}} d\mu(\theta),
\]
where $\mu(\theta)$ is a function with bounded variation on $[0, 2\pi]$ such that
\[
\int_0^{2\pi} d\mu(\theta) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\mu(\theta)| \leq k\pi.
\]
Also, for $p(z) \in \mathcal{P}_k(\rho)$, we can write from (1.5)
\[
p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z), \quad z \in \mathbb{D}
\]
where $p_1(z), p_2(z) \in \mathcal{P}(\rho)$. $\mathcal{P}(\rho)$ is the class of functions with positive real part greater than $\rho$, i.e., $\mathcal{P}(\rho) = \mathcal{P}(1 - 2\rho, -1)$ with $0 \leq \rho < 1$. Let $s(z)$ be an element of $\mathcal{A}$ and which maps $\mathbb{D}$ conformally onto an image domain of boundary rotation at most $k\pi$, the class of such functions is denoted by $V_k$. The concept of functions of bounded boundary rotation original from Loewner[3] in 1917, but he did not use the present terminology. Paatero[5], who systematically developed their properties and made exhaustive study of the class $V_k$. Paatero[5] has shown that $s(z) \in V_k$ if and only if
\[
s'(z) = \text{Exp} \left[ -\int_0^{2\pi} \log (1 - ze^{-i\theta}) d\mu(\theta) \right],
\]
where $\mu(\theta)$ is given (1.7). For a fixed $k \geq 2$ it can also be expressed as
\[
\int_0^{2\pi} \left| \text{Re} \left( \frac{z s'(z)'}{s'(z)} \right) \right| d\theta \leq k\pi, \quad z = re^{i\theta}.
\]
Clearly, if $k_1 < k_2$ then $V_{k_1} \subset V_{k_2}$, that is the class $V_k$ obviously expand as $k$ increases. $V_2$ is simply the class of $\mathcal{C}$ convex univalent functions and Paatero[5] showed that $V_4 \subset \mathcal{S}$, where $\mathcal{S}$ is the class of normalized univalent functions. Later Pinchuk[7] showed that functions $V_k$ are close-to convex in $\mathbb{D}$.
Let \( s(z) \) be an element of \( A \) and satisfying the condition
\[
1 + z \frac{s''(z)}{s'(z)} \in \mathcal{P}_k(\rho), \quad (0 \leq \rho < 1). 
\]
The class of such functions is denoted by \( V_k(\rho) \). When \( \rho = 0 \), we get the class \( V_k(0) = V_k \) of functions of bounded boundary rotation.

We note that through our paper \( \rho \) is defined in the following manner
\[
0 \leq \rho = \frac{1 - A}{1 - B} < 1
\]

Finally, a planar harmonic mapping \( f \) in the open unit disc \( \mathbb{D} \) is a complex-valued harmonic function which maps \( \mathbb{D} \) onto the same planar domain \( f(\mathbb{D}) \). Since \( \mathbb{D} \) is a simply connected domain, the mapping \( f \) has a canonical decomposition \( f = h(z) + g(z) \), where \( h(z) \) and \( g(z) \) are analytic in \( \mathbb{D} \) and have the following power series
\[
h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad a_n, b_n \in \mathbb{C}
\]
We call \( h(z) \) is analytic part of \( f \) and \( g(z) \) is co-analytic part of \( f \). An elegant and complete treatment theory of harmonic mappings is given in Duren’s monography[1]. Lewy [1] proved that the harmonic mapping \( f \) is locally univalent in \( \mathbb{D} \) if and only if its Jacobian \( J_f = (|h'(z)|^2 - |g'(z)|^2) \) is different form zero. In view of this result locally univalent harmonic mappings in the unit disc \( \mathbb{D} \) are either sense-preserving if \( |h'(z)| > |g'(z)| \) in \( \mathbb{D} \) or sense-reversing if \( |h'(z)| < |g'(z)| \) in \( \mathbb{D} \). In this paper, we will restrict ourselves to the study of sense-preserving harmonic mappings. We also note that \( f = h(z) + g(z) \) is sense-preserving in \( \mathbb{D} \) if and only if \( h'(z) \) does not vanish in \( \mathbb{D} \) and second dilatation \( w(z) = \frac{g'(z)}{h'(z)} \) has the property \( |w(z)| < 1 \) for all \( z \in \mathbb{D} \). Therefore the class of all sense-preserving harmonic mappings in the open unit disc \( \mathbb{D} \) with \( a_0 = b_0 = 0 \) and \( a_1 = 1 \) will be denoted by \( S_H \). Thus \( S_H \) contains standard class \( S \) of univalent functions. The family of all mappings \( f \in S_H \) with the additional property \( g'(0) = 0 \), i.e, \( b_1 = 0 \) is denoted by \( S_H^0 \). Hence, it is clear that \( S \subset S_H^0 \subset S_H \).

In this paper we will investigate the following class
\[
S_{HS^*(A,B)} = \{ f = h(z) + g(z) \mid w(z) = \frac{1}{h'(z)} g'(z) \in \mathcal{P}_k, \ h(z) \in S^*(A,B) \}
\]

## 2 Main Results

**Lemma 2.1.** Let \( p(z) \) be an element of \( \mathcal{P}_k(\rho) = \mathcal{P}_k(\frac{1-A}{1-B}) \), then
\[
|p(z) - \frac{1 + (2A - B - 1) r^2}{(1 - B)(1 - r^2)}| \leq \frac{k(A + B)r}{(1 - B)(1 - r^2)}
\]
Proof. Using the result of K.S. Padmanabhan and R. Parvatham [6] with \( \rho = \frac{1-A}{1-B} \) and after the simple calculations, we get desired result.

**Theorem 2.2.** Let \( f(z) = (h(z) + \overline{g(z)}) \) be an element of \( S_{HS^*(A,B)} \), then

\[
F(A,B,|b_1|,k,-r).C(r,-A,-B) \leq |g(z)| \leq F(A,B,|b_1|,k,r).C(r,A,B)
\]

\[
F(A,B,|b_1|,k,-r).\frac{1-A}{1-Br}C(r,-A,-B) \leq |g'(z)| \leq F(A,B,|b_1|,k,r).\frac{1+Ar}{1+Br}C(r,A,B)
\]

where

\[
C(r,A,B) = \begin{cases} 
  r(1+Br)^\frac{A-B}{Br}, & B \neq 0 \\
  re^{Ar}, & B = 0
\end{cases}
\]

\[
F(A,B,|b_1|,k,r) = \frac{|b_1|(1+k(A-B)r+(2A-B-1)r^2)}{(1-B)(1-r^2)}
\]

Proof. Using K. I. Noor and S. Mustafa’s result [4] we can write

\[
g(z) = b_1.h(z).p(z)
\]

(2.1)

\[
g'(z) = b_1.h'(z).p(z)
\]

(2.2)

If we use Janowski result [2] and Lemma 2.1 in the inequalities (2.1) and (2.2), we obtain desired result.

**Corollary 2.3.** Let \( f = (h(z) + \overline{g(z)}) \) be an element of \( S_{HS^*(A,B)} \), then

\[
\left( \frac{1-A}{1-Br} \right)^2.(C(r,-A,-B))^2.(1-F(A,B,|b_1|,k,-r))^2 \leq J_f \leq \\
\left( \frac{1+Ar}{1+Br} \right)^2.(C(r,A,B))^2.(1-F(A,B,|b_1|,k,r))^2
\]

Proof. Since

\[
J_f = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2(1 - \frac{|g'(z)|^2}{|h'(z)|^2})
\]

in this step we use Theorem 2.2, then we obtain desired result.
Corollary 2.4. If $f = (h(z) + \overline{g(z)}) \in S_{HS^*(A,B)}$, then
\[
\int \frac{1-Ar}{1-Br}(C(r,-A,-B))(1-F(A,B,|b_1|,k,r))dr \leq |f| \leq \\
\int \frac{1+Ar}{1+Br}(C(r,A,B))(1-F(A,B,|b_1|,k,r))dr
\]

Proof. Since
\[
(|h'(z)| - |g'(z)|)|dz| \leq |df| \leq (|h'(z)| + |g'(z)|)|dz| \Rightarrow \\
|h'(z)|(1 - |\frac{g'(z)}{h'(z)}|)|dz| \leq |df| \leq |h'(z)|(1 + |\frac{g'(z)}{h'(z)}|)|dz| \Rightarrow \\
\frac{1-Ar}{1-Br}(C(r,-A,-B))(1-F(A,B,|b_1|,k,r))dr \leq |df| \leq \\
\frac{1+Ar}{1+Br}(C(r,A,B))(1-F(A,B,|b_1|,k,r))dr
\]
then we integrate, we get desired result. \hfill \Box

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References


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