A Theory on Grasping Objects using Effectors with Curved Contact Surfaces and Its Application to Whole-Arm Grasping

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Abstract
This paper develops a theoretical framework for grasping objects using customized effectors that have curved contact surfaces and presents its application to the problem of robotic whole-arm grasping. We present a collection of immobilizing grasps and cages that can effectively restrain the mobility of a wide range of object shapes including polyhedra. Each of the grasps or cages is formed by at most three effectors with appropriate contact surfaces in contrast to customary point fingertips. We also discuss the morphology of the curved contact surfaces that can realize the grasps and cages; the surfaces can simply be a planar, cylindrical, or spherical one. Stable grasps are obtained by simple motion planning and control. Our theory is based on a conservative assumption that all contacts are frictionless, rigid, unilateral. Finally, we present a robotic system, comprised of a software suite and a modular reconfigurable manipulator outfitted with exchangeable end-effectors and arm links, demonstrating the theory and our approach to whole-arm grasping.

Keywords
Robotic grasping, robotic caging, whole-arm grasping, modular robot system

1. Introduction
This paper is concerned with developing a theoretical framework for robotic grasping using effectors (or “fingers”) with curved contact surfaces. Effectors with appropriate curvature properties can be effective for restraining the mobility of objects. For example, between the two grasps shown in Figures 1a and 1b, each of which has one point and one planar contact, the latter is showing a more restrictive grasp where the object cannot actually escape from the effectors due to the concavity of the effector surface contacting the vertex. We also present one application of the idea to a scenario of whole-arm grasping objects, implemented on a modular robot system with exchangeable end-effectors and arm links. Figure 1c is showing a grasp by the modular manipulator outfitted with two end-effectors that have curved contact surfaces.

It has been customary to employ the abstraction of point fingers in developing theories for robotic grasping and caging (see Section 2); the approach has the advantage of generality, but there are challenges in, for example, designing a physical system providing a sufficient number of independently controlled fingers that can realize the theories. Indeed, most practical solutions in autonomous grasping involve specially designed hardware and control algorithms that are tailored only to a
couple of objects to be handled or grasped. A robotic system that can grasp a wide variety of object shapes without many different types of effectors or complex multi-fingered hands can save time and cost in a wide range of scenarios from handling material in a warehouse to clearing rubble in an unstructured environment. In addition, it is imperative to develop planning algorithms that can guarantee the stability of the process of grasp acquisition and robustness to inevitable sensing/positioning errors; however, the complexity in modeling compliance or friction accurately is one of the challenges that make it hard to develop such algorithms.

We present a novel theory of three types of immobilizing grasps and cages that can effectively restrain the mobility of any object modeled as a polyhedron. The grasps and cages are formed by at most three effectors with curved contact surfaces, which can simply be a planar, cylindrical, or spherical one. The cages allow us to obtain stable grasps by simple motion planning and control in a manner robust to errors and uncertainties. In addition, the theory is extended with more types of grasps and cages. We apply the theory to developing hardware and software for stable, robust object grasping with a modular robot system, which can adapt to the sizes and shapes of a wide variety of objects. Our work is based on a conservative assumption that two bodies in contact can only push each other (unilateral contact) and there is neither friction nor compliance in contact.

This paper is organized as follows. In the following section, we review relevant literature in the areas of robotic grasping and caging. Section 3 introduces concepts and terminology necessary to develop our theory and algorithms. Section 4 discusses three types of immobilizing grasps using effectors with curved contact surfaces that can be applied to a wide range of object shapes including polyhedra. Section 5 discusses three types of cages derived from the immobilizing grasps and explains how to establish sufficient conditions for caging. Section 6 presents an algorithm for synthesizing the immobilizing grasps and cages. Section 7 presents one application of our theory to whole-arm grasping with implementation on a modular robot system. We conclude in Section 8, with suggestions for future work. In Appendix, we extend our theory by adding more types of three- and two-dimensional grasps and cages.

2. Related Work

Robotic grasping has been an active research area over the past few decades; see Bicchi and Kumar (2000) for a general survey. Section 2.1 outlines the literature focusing on immobilizing objects with multiple contacts. Section 2.2 introduces prior work on caging objects. Section 2.3 reviews modular robotic approaches to manipulation tasks. This paper builds on our earlier work presented in Seo et al. (2013).

2.1. Prehensile Approach

We begin with discussing the closure properties of grasps. A grasp is defined as **force closed** (Nguyen, 1988) if and only if it can resist any external wrench. If a grasp is force closed with frictionless contacts, it is said to be **form closed** (Lakshminarayana, 1978) or **immobilized** (Rimon and Burdick, 1998a). Trinkle (1992) proposed a quantitative test formulated
as a linear program for detecting form closure. Markenscoff et al. (1990) showed that it is possible to immobilize a three-dimensional object with seven frictionless point contacts, using first-order theories based on contact normals. Algorithms for synthesizing force or form closed grasps were presented by Ponce et al. (1997); Borst et al. (1999); Van der Stappen et al. (1999). Rimon and Burdick (1998a,b) developed a second-order mobility theory for rigid bodies in contact where the curvature properties at the contacts are taken into account. Czyzowicz et al. (1991) showed that \( n + 1 \) frictionless point contacts suffice to immobilize a general \( n \)-dimensional polytope by using the effects of relative curvature; thus, for a three-dimensional object, four frictionless point contacts suffice for immobilization.

It has been discovered that the stability of a grasp depends on the geometry of the grasp, contact forces, and material properties. Mason and Salisbury (1985) established a framework for testing the stability of a grasp: a grasp is stable if its stiffness matrix is positive definite. Cutkosky and Kao (1989) showed that grasp stability is a function of local geometry, fingertip models, and the compliance of the fingers. Nguyen (1989) proved that all force closed grasps can be made stable. Howard and Kumar (1996) established a framework for analyzing grasp stability that takes compliance, contact forces, and the local curvature properties of the bodies in contact into account. It was shown that immobilization implies dynamic stability with elastic contacts (Rimon and Burdick, 1998a,b).

In practice, having a large number of contacts can be beneficial to grasp stability; however, synthesizing such a grasp can be computationally intractable. Pollard (2004) presented an efficient algorithm for synthesizing many-contact grasps based on user-provided examples. Similar approaches can also be seen in the literature on whole-body grasping (Hsiao and Lozano-Perez, 2006) and enveloping grasping (Trinkle et al., 1988). Napier (1956) showed that there are two approaches to achieving stability in human grasping: power grip and precision grip. Whole-body grasps and enveloping grasps are in the same vein of the human power grip, where an object is held by a large number of contacts between the flexed fingers and the palm.

Since Hanafusa et al. (1977) presented one of the earliest examples of robotic hands, various robotic hands have been developed. Our work is relevant to the approach of building simple yet versatile end-effectors that can be seen in Jacobsen et al. (1986); Ulrich et al. (1988); Dollar and Howe (2010); Kragten et al. (2011); Mason et al. (2012). Another relevant approach can also be seen in the literature on modular fixturing (Brost and Goldberg, 1996; Ponce, 1996). Recently, there has been growing interest in developing robotic systems that can grasp/manipulate objects with some autonomy. Saxena et al. (2008) presented a vision-based approach to robotic grasping and demonstrated real systems that can grasp previously unknown objects using two-dimensional images; similar approaches can be seen in Morales et al. (2002); Bowers and Lumia (2003). Hudson et al. (2012) developed an autonomy system that can perform dexterous, high-precision tasks such as key insertion.

### 2.2. Non-Prehensile Approach

In contrast to the prehensile approach, the literature on caging investigates how to arrange “obstacles” (that is, robotic fingers or effectors) around an object so as to bound its mobility without necessarily making contact. Caging allows us to sidestep some difficult issues such as modeling contacts or optimizing contact forces although the caged object may have some freedom to move. Rimon and Blake (1999) formulated a technique for computing cages of two-fingered hands; Davidson and Blake (1998a) extended the result to three-fingered hands. Vahedi and van der Stappen (2008a,b,c, 2009) provided an algorithm for synthesizing cages of two and three fingers around polygonal objects and formalized the concepts of squeezing and stretching cages for polygonal objects, which were generalized by Rodriguez and Mason (2009) to address objects in Euclidean spaces of arbitrary dimension. Allen et al. (2012) presented a simpler algorithm for computing two-fingered cages for polygons based on contact space analysis. Wan et al. (2012) proposed a solution to synthesizing three-finger cages on the plane where two of the fingers are fixed. Zamfirescu (1995); Maehara (2011); Fruchard (2012) investigated
how to cage objects with a single circle. Rodriguez et al. (2012) discussed the relationship between caging and grasping; they investigated when a cage can be a useful waypoint to an equilibrium grasp.

Recently, there have been efforts to take advantage of caging to robustify robotic grasping tasks. Davidson and Blake (1998b) presented error-tolerant, vision-based planar grasping by closing fingers that form a cage. Gopalakrishnan and Goldberg (2002) presented a simple gripper with two vertical, parallel cylindrical jaws that can stably grasp objects by forming a cage on concavities. Diankov et al. (2008) proposed a motion planning algorithm for performing manipulation tasks with cages, relaxing task constraints. Yokoi et al. (2009) presented an approach to transporting objects using cages formed by not only robots but also the environment such as walls. Cappelleri et al. (2011a,b) employed cages formed by micro-manipulators for transporting and manipulating micro-scale polygonal parts. Dogar and Srinivasa (2011) showed that simple manipulation such as quasi-static pushing can help robots stably cage and grasp objects even in clutter. There is a body of literature featuring decentralized approaches to caging; we refer the reader to Section 2.3.

2.3. Modular Approach to Robotic Manipulation

The literature on multi-robot manipulation provides techniques for handling objects in a cooperative manner. When multiple robots manipulate a common object, it is necessary to control both the motion of the object and the internal forces exerted on the object (Murray et al., 1994). In addition, the dynamics of such systems is typically subject to unilateral constraints: the robots can only push or pull (for example, by cable tension) the object. Murray (1996); Sugar and Kumar (1999); Cheng et al. (2009); Fink et al. (2011); Bernard et al. (2011); Sreenath and Kumar (2013) provided solutions to the problem with applications to ground or aerial transport. A group of literature shows that robotic object handling can be facilitated by caging with a team of multiple robots: Kosuge et al. (1999); Pereira et al. (2004); Montemayor and Wen (2005); Fink et al. (2008) presented approaches to developing decentralized control algorithms. Some literature on cooperative transport took inspiration from the behaviors of social insect colonies (Kube and Bonabeau, 2000; Berman et al., 2011).

3. Preliminaries: Caging and Grasping

This section introduces concepts and terminology relating caging and grasping that we use in developing our theory and algorithms.

3.1. Caging

A cage around an object bounds its mobility (Figure 2a): the caged object cannot be moved arbitrarily far from its original position without penetrating the surrounding effectors forming the cage. Equivalently, a cage can also be defined in terms of the mobility of the surrounding effectors by regarding the object as an obstacle: according to Rodriguez et al. (2012), a cage is a configuration of effectors that lies in a compact, connected component of the free space (LaValle, 2006) of the system of the effectors (note that the system is assumed to move as a single rigid body).

Rodriguez et al. (2012) formalized the concept of an F-cage, which is generally stricter than that of a cage. Let $F$ be a scalar function defined on effector configurations. Then an $F$-cage is a configuration of the effectors that cages an object even if they have freedom to move while maintaining the value of $F$. An $F$-cage is an $F$-squeezing (stretching) cage if it still cages the object even if the effectors have freedom to move while decreasing (increasing) the value of $F$.

3.2. Contact

In this work, we are mainly concerned with an object modeled as a polyhedron in contact with effector surfaces. There can then be three types of contact geometry: point, line, and planar contact (Mason and Salisbury, 1985; Mason, 2001). For example, Figure 2c shows one point and one planar contact between the two planar effector surfaces and the object. A
contact can apply a contact wrench (Murray et al., 1994; Mason, 2001), contact force/moment pair, given by the positive linear combinations of unit contact wrenches, which are normalized, linearly independent vectors that span the vector space of the contact wrenches (see Figures 2b and 2c). Our work is based on an ideal yet conservative assumption that all contacts are frictionless, rigid, unilateral (unilateral contacts can only push an object); then, the unit contact wrenches are the vectors of the normalized screw coordinates only of the inward-pointing contact normals (Mason, 2001). If we assumed more realistic yet capable (for example, frictional, soft, or bilateral) contacts, more types of unit contact wrenches should be considered (Murray et al., 1994); this shows the conservativeness of our assumption.

A line contact can be made on a virtual edge (Peshkin and Sanderson, 1986), which is composed only of the vertices delimiting the edge without the interior (Figure 3). It can be seen that a “real” edge and a virtual edge are indistinguishable in terms of their ability to produce contact wrenches under the assumption of frictionless, rigid, unilateral contact if we are given a sufficiently large effector surface that can contact all the delimiting vertices: in Figure 3, even if the virtual edge were real (like edge RQ), the number of unit contact wrenches would still be the same, that is, two. Similarly, a planar contact can also be made on a virtual face.

3.3. Grasping

To grasp an object is to restrain its mobility by making contacts with effectors. A grasp can be in equilibrium if the net contact wrench, the sum of all contact wrenches, can be made zero in such a way that not all contact wrenches are equal to zero (Rimon and Burdick, 1998a). If there is no object twist, linear/angular velocity pair (Murray et al., 1994), consistent with the contacts of an equilibrium grasp, the object is said to be immobilized to the first order (Rimon and Burdick, 1998a) (or form-closed (Mason, 2001)), that is, the configuration of the object is an isolated point in the free space. Even if such a twist exists, any finite motion may be restricted by considering surface curvature effects. For example, in Figure 2b, the object may instantaneously rotate about its centroid (so a first-order kinematic analysis does not predict immobility), but any finite rotation results in penetrating the effectors. This idea is formalized using the concept of second-order immobility.
presented by Rimon and Burdick (1998a,b); they discovered that second-order immobility suffices for not only complete immobility, which means that the configuration of the grasped object is isolated from the free space, but also local, dynamic, asymptotical stability of the grasped object. Seven (four) point effectors are required to immobilize a polyhedral object to the first (second) order with frictionless, rigid, unilateral contacts (Markenscoff et al., 1990; Czyzowicz et al., 1991). Such immobility conditions are purely geometric; information on contact geometry is thus sufficient to investigate first- or second-order immobility. Our immobilizing grasps to be presented also exploit surface curvature effects as second-order immobility. An equilibrium grasp is called a grasping cage (Rodriguez et al., 2012) if it can also cage the object; a grasping cage is not necessarily an immobilizing grasp.

3.4. Clamping
Clamping (Bose et al., 1996), also known as parallel-jaw grasping, is one way to realize equilibrium grasps by holding an object between two parallel planar “jaws.” An object that is clamped can only move on the plane of the jaws, without penetrating them. If the jaws can exert frictional forces, the grasp can be force closed: an arbitrary contact wrench can be applied. Consider the antipodal pair of a convex polyhedral object, which is the intersection of the object with a pair of parallel support planes; an antipodal pair can thus be a vertex-vertex, vertex-face, edge-edge, edge-face, or face-face pair. According to Bose et al. (1996), all convex polyhedra can be clamped with parallel-jaw grippers. The grippers are then on one of the last four types of antipodal pairs, that is, vertex-face, edge-edge, edge-face, or face-face; such an element pair can determine the width of a polyhedron, the minimum distance between two parallel supporting planes. Figure 2c shows a clamp on a vertex-face pair.

4. Immobilizing Objects using Effectors with Curved Contact Surfaces
In this section, we discuss how to immobilize a three-dimensional object using at most three contacts, each of which can be a point, line, or planar contact made by an effector with a curved contact surface that can be represented as a two-dimensional manifold with boundary. In Sections 4.1, 4.2, and 4.3, we present three types of immobilizing grasps that can be applied to objects modeled as polyhedra; we also discuss the geometry of effector surfaces that can realize the grasps. In Section 4.4, we show that any polyhedron can be immobilized by the grasps and discuss how the assumption of polyhedral objects can be relaxed.

4.1. Immobilizing with Three Point Contacts
Here we show that it is possible to immobilize a polyhedron with three point contacts. We begin with choosing three vertices where the contacts can be made. See Figure 4a. Let $P$ and $Q$ be two vertices of the given polyhedron where two point contacts can be made with a pair of parallel planes perpendicular to $PQ$; the inward-pointing (toward the interior of the polyhedron) contact normals at $P$ and $Q$ point toward each other. Let $R$ be a vertex, not on $\xi$ (the line of $PQ$), where a point contact can be made with a plane such that the ray of the inward-pointing contact normal intersects $\xi$.

Three point contacts on $P$, $Q$, and $R$ made by curved effector surfaces can immobilize the polyhedron if (1) a neighborhood of the effector surface around the point contacting $P$ ($Q$), except for the point itself, can be embedded inside the ball whose diameter has endpoints $P$ and $Q$ (Figure 4b) and (2) a neighborhood of the effector surface around the point contacting $R$, except for the point itself, can be embedded inside the cylinder of radius $d(R, \xi)$ with axis $\xi$, where $d(R, \xi)$ denotes the shortest distance between $R$ and $\xi$. If condition (1) is satisfied, as can be seen in Figure 4b, the polyhedron cannot move at all except for rotating about $\xi$ because $PQ$ is at a configuration that is isolated from the free space: any finite displacement of $PQ$ results in penetrating the effectors at $P$ or $Q$. Furthermore, if condition (2) is satisfied, the polyhedron cannot rotate about $\xi$ because the rotation results in penetrating the effector at $R$. Two example grasps are
Fig. 4. (a) \((P, Q, R)\) is a vertex-vertex-vertex triple where three point contacts immobilizing the octahedron can be made. (b) The front view of the octahedron with the two curved effectors contacting \(P\) and \(Q\). The effector surfaces, except for the points contacting \(P\) and \(Q\), can locally be embedded inside the ball shown whose diameter has endpoints \(P\) and \(Q\). (c) An immobilizing grasp by the effectors with a spherical surface. (d) An immobilizing grasp by the cone-shaped effectors.

shown in Figures 4c and 4d: effectors with a spherical surface of a sufficiently small radius (Figure 4c) or effectors with a cusp (Figure 4d) can satisfy the two conditions.

4.2. Immobilizing with a Point, a Line, and a Planar contact

Here we show that it is possible to immobilize a polyhedron with one point, one line, and one planar contact. We begin with choosing a vertex, an edge, and a face where the contacts can be made. See Figure 5a. At \(P\) and \(QRST\), a point and a planar contact can be made with a pair of parallel planes. The ray of the inward-pointing contact normal at the vertex intersects the interior of the face; the inward-pointing contact normals at the face point toward the vertex. Among the edges incident to \(P\), choose the one with the least slope with respect to the face, that is, \(PS\).

Three (one point, one line, and one planar) contacts on the vertex \((P)\), edge \((PS)\), and face \((QRST)\) by curved effector surfaces can immobilize the polyhedron if (1) the contact area of the planar contact contains a neighborhood of \(P'\), the foot of perpendicular from \(P\) to the face (Figure 5b), (2) a neighborhood of the effector surface around the point contacting \(P\), except for the point itself, can be embedded inside the half-space below (toward the interior of the polyhedron) the plane contacting \(P\) (Figure 5b), and (3) a neighborhood of the effector surface around the line segment contacting \(PS\), except for the line segment itself, can be embedded inside the cone formed by rotating \(PS\) about \(\xi\) (the line of \(PP'\)). If conditions (1) and (2) are satisfied, as can be seen in Figure 5b, the polyhedron cannot move at all except for rotating about \(\xi\) because the two effectors at the vertex and face are not only clamping it but also restricting any translation. Furthermore, if condition (3) is satisfied, the polyhedron cannot rotate about \(\xi\) because the rotation results in penetrating the effector at \(PS\). Two example grasps are shown in Figures 5c and 5d.
4.3. Immobilizing with Two Line Contacts

Here we show that it is possible to immobilize a polyhedron with two line contacts. We begin with choosing two edges where the contacts can be made. See Figure 6a. \( \overline{PQ} \) and \( \overline{RS} \) are two skew edges of the given polyhedron where two line contacts can be made with a pair of parallel planes that are perpendicular to \( \xi \), the common perpendicular of the edges. \( \xi \) and the two edges intersect in the interior of the edges. The inward-pointing contact normals at one edge point toward the other edge.

![Diagram](image)

Fig. 6. (a) \( (\overline{PQ}, \overline{RS}) \) is an edge-edge pair where two line contacts immobilizing the tetrahedron can be made. (b) The front view of the object with the two curved effectors contacting the edges. The effector surfaces can locally be embedded inside the space between the two supporting planes (except for the line segments contacting the object). (c) An immobilizing grasp by the effectors with a cylindrical surface. (d) An immobilizing grasp by the two V-shaped effectors.

Two line contacts on the edges by curved effector surfaces can immobilize the polyhedron if a neighborhood of the effector surface around the line segment contacting \( \overline{PQ} (\overline{RS}) \), except for the line segment itself, can be embedded in the half-space below (above) the plane contacting it (Figure 6b). Note that the two contact areas (the line segments) and \( \xi \) should intersect. If the condition is satisfied, as can be seen in Figure 6b, the effectors are not only clamping the polyhedron but also restricting any planar motion: one of the effectors only allows the object to translate along the line of its contact, but such motion is not allowed by the other effector. Two example grasps are shown in Figures 6c and 6d.

4.4. Analysis

The three types of grasps discussed in Sections 4.1, 4.2, and 4.3 are complete: every polyhedron can be immobilized by applying at least one of the grasps. We break the analysis into two theorems.

First, we show that the grasp of three point contacts discussed in Section 4.1 suffices to immobilize all polyhedra.

**Theorem 1.** Every polyhedron can be immobilized by three frictionless, rigid, unilateral point contacts made by appropriately curved effector surfaces.

**Proof.** Every polyhedron has a vertex-vertex pair \( (P, Q) \) that admits two parallel supporting planes perpendicular to \( \overline{PQ} \) and contacting only \( P \) and \( Q \), respectively, in such a way that the interior of the polyhedron is between the planes. Consider the collection of the vertices of the polyhedron. Let \( (P, Q) \) be a pair of vertices determining the maximum distance between two vertices of the collection. Consider two planes \( \Pi_P \) and \( \Pi_Q \) perpendicular to \( \overline{PQ} \) and contacting the polyhedron respectively at \( P \) and \( Q \). No other vertex of the polyhedron can be located on \( \Pi_P \) and \( \Pi_Q \) because \( (P, Q) \) determines the maximum distance: \( \Pi_P (\Pi_Q) \) is supporting the polyhedron only at \( P (Q) \). Therefore, \( (P, Q) \) is a desired vertex pair.

Next, we can find an additional vertex, not on \( \xi \) (the line of \( \overline{PQ} \)), which admits a supporting plane whose inward-pointing normal at the vertex directly points to \( \xi \). Consider one vertex that is the most distant from \( \xi \) and denote the vertex as \( R \); let \( \xi' \) be a line parallel to \( \xi \) and passing through \( R \). If \( R \) is the only vertex of the polyhedron on \( \xi' \), \( R \) must be a pointed vertex that admits a supporting plane whose inward-pointing normal at \( R \) directly points toward \( \xi \). \( R \) is then the desired vertex. In case there are multiple vertices lying on \( \xi' \), first choose a point on \( \xi' \) as its origin, then pick the vertex that is the most distant from the origin, and denote the vertex as \( R \). It can be seen that \( R \) is the desired vertex.
Constructively as explained in Section 4.1, we get immobility with three appropriately curved effector surfaces respectively contacting $P$, $Q$, and $R$. Note that $P$, $Q$, and $R$ are on the convex hull of the polyhedron; otherwise, $P$ and $Q$ do not determine the maximum distance and $R$ is not the most distant from $\xi$, either.

We now show that the other two types of grasps discussed in Sections 4.2 and 4.3 are also complete in the sense that every polyhedron whose vertices are in general position, not admitting any pair of either two planes or a line and a plane that are parallel, can be immobilized by at least one of those grasps.

**Theorem 2.** Every polyhedron whose vertices are in general position such that there is no pair of either two planes or a line and a plane that are parallel, where each line (plane) is determined by a distinct collection of two (three) vertices, can be immobilized by either (1) one point, one line, and one planar contact or (2) two line contacts made by appropriately curved effector surfaces; the contacts are frictionless, rigid, unilateral.

**Proof.** Consider the convex hull of the given polyhedron. Because the convex hull does not have any pair of either two faces or an edge and a face that are parallel, it can be clamped at one of its antipodal vertex-face or edge-edge pairs. The convex hull (and thus the original polyhedron) can then be immobilized using the pair by applying the grasp discussed in Section 4.2 or 4.3. □

Note that the grasps employed in Theorem 2 might have contacts on virtual edges or faces that are on the convex hull but not real elements of the original polyhedron.

In Appendix, more types of two- and three-dimensional immobilizing grasps are presented, in addition to the collection of complete grasps.

**Remark:** In addition to all polyhedral shapes, the three types of grasps can actually be applied to immobilizing a wider range of shapes. Essentially, any given object shape can be immobilized using the grasps if it is possible to make the right set of contacts with effectors having the right curvature properties, as discussed in Sections 4.1, 4.2, and 4.3. For example, even if the vertices $P$, $Q$, and $R$ in Figure 4c were not actually pointed, it would be possible to immobilize the object with the three point contacts at $P$, $Q$, and $R$ in the way discussed in Section 4.1 as long as the actual object geometry around $P$, $Q$, and $R$ does not intersect the effector surfaces. Similarly, the place where a line (planar) contact is made does not have to be a perfect polyhedron edge (face) in order for the grasps presented in Sections 4.2 and 4.3 to be applied.

**5. Caging Objects using Effectors with Curved Contact Surfaces**

Based on the three types of immobilizing grasps discussed in Section 4, it can be seen that every polyhedron can be caged by two effectors.

**Corollary 1.** Every polyhedron can be caged by two appropriately curved effector surfaces, each of which is accommodating a single vertex, edge, or face.

**Proof.** According to Section 4.1, 4.2, or 4.3, it is possible to cage a polyhedron with two curved effector surfaces making contacts at a vertex-vertex, vertex-face, or edge-edge pair. For example, Figure 7 shows three types of cages obtained by the two effectors at the antipodal pairs in Figures 4c, 5c, and 6c, respectively. According to Theorems 1 and 2, every polyhedron can be caged by at least one of the three types of cages. □

Although the effectors are contacting the objects in Figure 7, a cage does not necessarily have to make contacts with the object. In order to see if an object is caged, it is necessary to take overall effector geometry into account because it is required to verify the compactness of the component of the free space to which the configuration of the object belongs,
which is a nonlocal property. For example, the distance between effectors is critical in our discussion on caging here. In contrast, recall that we need only local curvature properties in establishing the immobilizing grasps in Section 4.

In this section, we address how to establish sufficient conditions for caging three-dimensional objects in analytical (Section 5.1) and empirical (Section 5.2) manners. In Section 5.3, we present a theoretical analysis on how to acquire stable grasps from the cages and discuss how our assumption of polyhedral objects can be relaxed.

5.1. Analytical Method

In this section, we present three types of cages that are derived from the three types of immobilizing grasps. Considering simpler object geometry that can be inscribed in the original shape, we discuss how to establish sufficient conditions for caging in an analytical manner.

**Caging on a Vertex-Vertex Pair** Consider a vertex-vertex pair of a given polyhedron such as $(P, Q)$ in Figure 7a, allowing us to establish the immobilizing grasp discussed in Section 4.1. Here we investigate how to cage the two vertices assumed to be rigidly connected (see the line segment $PQ$ in Figure 8a).

Consider two curved effectors that can contain the vertices, as shown in Figure 8a. If the maximum clearance between the effectors is less than $d(P, Q)$, the distance between $P$ and $Q$, they can cage the line segment $PQ$. For example, consider a pair of effectors of the same hemisphere that are facing each other and only allowed to relatively translate on their common axis, $\eta$ (Figure 8b). The maximum clearance between the two effectors is then the distance between one point on the boundary of one effector and the foot of perpendicular from the point to the other effector, denoted as $c_{max}$ in the figure. If $\delta$, the distance between the two planes on which the boundaries of the effectors lie (Figure 8b), is small enough to guarantee $c_{max} < d(P, Q)$, $PQ$ (and thus the original object) is caged.
Caging on a Vertex-Face Pair  Consider a vertex-face pair of a given polyhedron such as $(P, □QRST)$ in Figure 7b, allowing us to establish the immobilizing grasp discussed in Section 4.2. Here we investigate how to cage a right circular cone that can be inscribed in the convex hull of the vertex and face (see the right circular cone with apex $P$ in Figure 7b).

![Fig. 9.](image)

**(a) The planar view shows the right circular cone in Figure 7b is contained between the two parallel planar effectors. (b) The curved and planar effectors are caging the cone. (c) A cage of the hemispherical and planar effectors. The hemispherical effector is only allowed to translate along its axis $\eta$.**

First, suppose that two infinitely large planar effectors are clamping the right circular cone at the apex and base. We now allow the effectors to move in such a way that they remain parallel to each other (Figure 9a). If their distance $\delta$ is less than $a$, the side length of the cone, the cone can stably be clamped again by decreasing $\delta$; moreover, the distance between the apex and the effector at the base is always larger than $h$, the height of the cone. It can then be seen that any curved effector surface containing the vertex in such a way that its maximum height is less than $a$ and the boundary lies below $h$ cages the cone along with the planar effector at the base (Figure 9b). For example, consider a hemispherical effector configured such that its boundary is parallel to the planar effector and only allowed to relatively translate along its axis, $\eta$ (Figure 9c). Then the cone (and thus the original object) is caged with the effectors if $\delta$, the distance between the planar effector and the plane of the boundary of the hemispherical effector, is small enough to guarantee (1) the apex is contained in the hemispherical effector, (2) $\delta + r_e < a$, and (3) $\delta < h$: (2) and (3) guarantee that the apex cannot escape from the hemispherical effector containing it by the analysis above. If the base belongs to a real face of the original object, the planar effector at the base only has to be as large as a disk of radius $r + r_e$, in order to support the base (Figure 9c). In case the face is virtual, the planar effector should be at least as large as the face itself.

Caging on an Edge-Edge Pair  Consider an edge-edge pair of a given a polyhedron such as $(PQ, RS)$ in Figure 7c, allowing us to immobilize the polyhedron as discussed in Section 4.3. Here we investigate how to cage a tetrahedron that can be inscribed in the convex hull of the two edges (see the tetrahedron $P'Q'R'S'$ in Figure 7c).

First, suppose that two infinitely large planar effectors are clamping the tetrahedron $P'Q'R'S'$. We now allow the effectors to move in such a way that they remain parallel to each other (Figure 10a). If their distance $\delta$ is less than $a$, the smallest value among $d(P'Q', R')$, $d(P'Q', S')$, $d(R'S', P')$, and $d(R'S', Q')$ (each term denotes the shortest distance between the edge and the vertex), the tetrahedron can stably be clamped again by decreasing $\delta$. Because of the stability, the heights of $P'$ and $Q'$ from the planar effector at the bottom has a lower bound, $a'$, which can be found by rotating the tetrahedron about $R'S'$ lying on the planar effector. It can then be seen that any curved effector surface containing edge $P'Q'$ in such a way that its maximum height is less than $a$ and the boundary lies below $a'$ (Figure 10b) cages the tetrahedron along with the planar effector at the bottom. Similarly, the heights of $R'$ and $S'$ are upper bounded; then, the planar effector at the bottom can also be replaced by an effector with a curved surface. For example, consider half-cylindrical effectors configured such that their boundaries are parallel to each other and only allowed to relatively translate along $\eta$ (Figure 10c).
Fig. 10. (a) The planar view shows tetrahedron $P'Q'R'S'$ in Figure 7c is contained between the two parallel planar effectors. (b) The planar effector at the bottom and the curved effector are caging the tetrahedron. (c) A cage of the two half-cylindrical effectors. They are only allowed to translate along their common perpendicular, $\eta$.

Then the tetrahedron (and thus the original object) is caged with the effectors if $\delta$, the distance between the two planes on which the boundaries of the effectors lie, is small enough to guarantee (1) the edges are contained in the effectors, (2) $\delta + r_1 + r_2 < a$, and (3) $r_1$ ($r_2$) is large enough to contain the lowest (highest) positions of $P'$ and $Q'$ ($R'$ and $S'$): (2) and (3) guarantee that the edges cannot escape from the effectors containing them by the analysis above. The half-cylindrical effectors should be at least as long as the edges of the original object that they are containing.

5.2. Empirical Method

Sufficient conditions for caging can also be established in an empirical manner by making use of off-the-shelf motion planning algorithms, particularly in case effector geometry is analytically challenging. We here employed a sampling-based motion planner implementing an RRT-based algorithm, available in ROS\(^1\); our approach is in the same vein of using sampling-based path planning for (dis)assembly planning (Sundaram et al., 2001; Ferre and Laumond, 2004; Le et al., 2009). In the experiment below, changing the distance between two effectors at intervals of 1 centimeter, we see if the motion planner can find paths for an object between the two effectors to reach random target configurations sufficiently far from the effectors; for each query distance, denoted by $\delta$ below, the effector pair was not considered a cage if it was possible for the object to escape at least once in 10 trials.

First, we consider a scenario where two torus-shaped effectors are caging an object on a vertex-vertex pair. Figure 11a shows how we set up experiments with the cone-like object. The effectors are only allowed to translate along $\eta$, their common axis of rotation. Let $\delta$ denote the distance between the two planes of the two circles, each of which represents the

\(^{1}\text{http://www.ros.org}\)}
center of each torus tube. When \( \delta = 0.86 \text{m} \) (Figure 11b), the motion planner failed to find such a path in 10 trials (each trial took 0.01 \( \sim \) 0.06 seconds on a 2.53GHz/4GB machine). Therefore, we can empirically conclude that the object is caged if \( \delta \leq 0.86 \text{m} \).

Second, we consider a scenario where a torus-shaped and a planar effector are caging an object on a vertex-face pair. Figure 11c shows our experimental setup with the cone object: the square-shaped planar effector is contacting the base of the cone; the torus-shaped effector is allowed to translate along \( \eta \), its axis of rotation collinear to the axis of the cone. When \( \delta = 0.5 \text{m} \) (Figure 11d), where \( \delta \) is the distance between the planar effector and the plane on which the circle representing the center of the torus tube lies, the motion planner failed to find an escaping path for the object in 10 trials (each trial took 0.01 \( \sim \) 0.04 seconds on a 2.53GHz/4GB machine). Therefore, we can empirically conclude that the object is caged if \( \delta \leq 0.5 \text{m} \).

Third, we consider a scenario where two open-ended, half-cylindrical effectors are caging an object on an edge-edge pair. In the experimental setup shown in Figure 11e, the effectors are only allowed to translate along \( \eta \), their common perpendicular. The motion planner could not find an escaping path for the object when \( \delta = 0.35 \text{m} \) (Figure 11f), where \( \delta \) is the distance between the two planes on which the boundaries of the effectors lie, in 10 trials (each trial took 0.02 \( \sim \) 0.04 seconds on a 2.53GHz/4GB machine). Therefore, we can empirically conclude that the object is caged if \( \delta \leq 0.35 \text{m} \).

In the trials above, the sampling-based motion planner did not have to try to take samples on a lower-dimensional manifold of the configuration space, which can be hard because such manifolds have measure zero, since our intention was to see how far the two effectors can be separated from each other while still caging the object.

5.3. Analysis

The sufficiency of our caging conditions allows us to address errors/uncertainties in not only fabricating the effectors but also sensing and control relating to grasp acquisition. We now discuss how to get a grasping cage (recall Section 3.3) from any of the cages discussed so far. We also discuss how to relax the assumption of polyhedral objects.

**Theorem 3.** For the cages discussed in Sections 5.1 and 5.2, a grasping cage is obtained if the two effectors are controlled such that the relative velocity is along \( \eta \) and \( \delta \) monotonically decreases until contact is established, which is assumed to be frictionless, rigid, unilateral (see Figures 8b, 9c, 10c, and 11 for \( \eta \) and \( \delta \)).

**Proof.** We first show that \( \delta \) is a grasping function (Rodriguez et al., 2012) for the cages. In each cage, the configuration space of the two effectors can be represented as \( \mathcal{M} = SE(3) \times SE(3) \); \( \delta \) is a semi-algebraic scalar function \( \delta : \mathcal{M} \rightarrow \mathbb{R} \) invariant with respect to the rigid transformations of the effectors as a whole in that it is the distance between the effectors. Furthermore, the preimages of \( \delta \) do not cage the object below (above) a certain value \( m (M) \) such that \( m < M \), for example, \( m = 0 \) and \( M = h \) in Figure 9c. Then \( \delta \) is a grasping function according to Rodriguez et al. (2012).

In addition, the cages are \( \delta \)-squeezing cages (Section 3.1) in that the object remains caged even if \( \delta \) decreases. Then, there exists a path in \( \mathcal{M} \) that leads the effectors into a configuration that can realize a grasping cage. Furthermore, in terms of the one-dimensional set representing the relative configuration space of the two effectors, \( \delta \) can be considered as a convex, that is, linear, function. Then, by the result of Rodriguez et al. (2012), we get to a grasping cage only by moving the effectors such that \( \delta \) monotonically decreases.

Rodriguez et al. (2012) discovered that the role of grasping functions in grasping is analogous to that of Lyapunov functions in stability analysis. Specifically, Theorem 3 shows that an object caged by any of our cages can be stabilized by simply moving the effectors closer to each other by relative translation. A translation that monotonically decreases \( \delta \) in the cages will be referred to as a squeezing motion in the remaining discussion. During a squeezing motion, we may add more contacts to further secure the grasp.
Remark: In addition to all polyhedral shapes, the three types of cages, each of which makes use of a vertex-vertex, a vertex-face, or an edge-edge pair, can actually be applied to a wider range of shapes. Essentially, if a polyhedron model of a given object is geometrically conservative such that the vertices, edges, and faces of the model are inscribed in the actual object geometry, then caging the polyhedron model guarantees that the actual object is also caged. The stability of squeezing motions is also guaranteed by the conservativeness. Note, however, that if the polyhedron approximation is too conservative, the cage might intersect the actual object geometry.

6. Synthesizing Grasps and Cages

In this section, we present an algorithm for finding element pairs/triples for the immobilizing grasps and cages discussed in the previous sections. Our pseudocode is presented in Algorithm 1. Given the model of an object, the output of the algorithm specifies where to place effectors in order to obtain the immobilizing grasps/cages. The following paragraphs elaborate each line of the algorithm.

Algorithm 1 Finding element pairs/triples for immobilizing/caging

Input: Object model: a polygonal mesh.

Output: Element pairs and triples labeled with instructions on placing effectors.

1: Compute the convex hull of the mesh.
2: For the convex hull, search for element triples (pairs) for the grasps of Sections 4.1 and 4.2 (Section 4.3).
3: Append instructions on placing effectors to each element pair or triple.

Line 1: We first compute the convex hull of the object model. The convex hull is essentially our object in the algorithm by considering that the vertices, edges, and faces lying on the convex hull suffice to guarantee the completeness of our grasps and cages (see Theorems 1, 2 and Corollary 1). It takes $O(n \log n)$ expected time to compute the convex hull of a given polyhedron, where $n$ is the number of the vertices (de Berg et al., 2000).

Line 2: We then compute the antipodal pairs of the convex hull that are vertex-vertex (for the grasps of Section 4.1), vertex-face (for the grasps of Section 4.2), and edge-edge pairs (for the grasps of Section 4.3). Given a polyhedron with $n$ vertices, its antipodal pairs can be computed in $O(n^2)$ time by applying a technique introduced by Brown (1979). Among them, choose the ones that admit supporting planes as illustrated in Figures 4a, 5a, and 6a; it takes $O(1)$ time to see if each pair satisfies the condition. For each vertex-vertex pair, we search for another vertex that is the most distant from the line of the two vertices (see the proof of Theorem 1); as a result, the pair is augmented into a triple of the three vertices in $O(n)$ time. For each vertex-face pair, we search for the edge with the least slope with respect to the face, among the ones incident to the vertex (recall Figure 5a); as a result, the pair is augmented into a triple of the vertex, edge, and face in $O(n)$ time.

Line 3: Finally, each element pair or triple from Line 2 is labeled with instructions on how to place effectors. First, the label shows if the constituent elements are real (lying on the original mesh) or virtual (lying only on the convex hull). Second, the label shows the positions and orientations at which effectors should aim. For a vertex-vertex-vertex triple, we specify the coordinates of the vertices and the inward-pointing normals of the three supporting planes. For a vertex-edge-face triple, for example, $P$, $PS$, and $QRST$ in Figure 5a, we specify the coordinates of $P$, the midpoint of $PS$, and the foot of perpendicular from $P$ to $QRST$; the inward-pointing normals of the two supporting planes; and the normal vector to $PS$ that directly points to $\xi$. For an edge-edge pair, for example, $PQ$ and $RS$ in Figure 6a, we specify the coordinates of the two intersections between $\xi$ and the two edges, along with the inward-pointing normals of the two supporting planes. Figure 12 shows such a label for an edge-edge pair. The labeling can be done in $O(1)$ time for each pair or triple.

Figure 13a shows an example object; Figures 13b, 13c, and 13d show some element pairs found by running the algorithm. The convex hull of the rock model has 162 vertices, 480 edges, and 320 faces; it took 0.02 seconds to find 21 vertex-vertex, 9 vertex-face, and 18 edge-edge pairs with our C++ implementation running on a 2.53GHz/4GB machine. The vertex-vertex
and vertex-face pairs (Figures 13b and 13c) can be augmented into the element triples for immobilizing grasps as explained in Line 2.

![Figure 13](image_url)

Fig. 13. (a) A polyhedral rock model with 1,000 faces (courtesy: Malcolm Lambert, Intresto Pty Ltd.). (b), (c), and (d) respectively show a vertex-vertex pair, a vertex-face pair, and an edge-edge pair found by running our algorithm. The reference frames are positioned and oriented such that the origin is at the contact position and the $z$-axis is along the contact normal, according to the label of each element pair.

A given object model can be immobilized by making contacts with any of the resultant element pairs or triples: we contact target elements with curved effector surfaces, whose curvature is sufficiently large, positioned and oriented as the label specifies. For caging purposes, we do not have to search for the third element in Line 2. Furthermore, effectors do not necessarily have to be controlled to make contacts with their target elements: the target positions for a pair of effectors caging an object can be moved away from the positions specified in the label as long as their distance $\delta$ remains less than $\delta^*$, the largest acceptable value of $\delta$ for the cage to be valid, along the line of the inward-pointing normals. Note, in fact, that the target positions can be moved away even farther because the caging conditions are sufficient.

Because all the elements found by Algorithm 1 (possibly except for the edge of a vertex-edge-face triple) are on the convex hull, that is, the outer frontier of the object, some of them can be virtual. An effector for a virtual edge or face should be large enough to contact all the vertices delimiting the virtual element. If it were not for an effector with a sufficiently large surface for a given virtual element, we would instead need to make contact with a smaller real element in the interior of the convex hull. Then, we may forgo computing the convex hull and proceed with the original mesh although it can take more time and the returned elements may be less easier to access in case they are in the interior of the convex hull.

**Remark:** It can be seen that closedness of input meshes, that is, whether an input mesh is homeomorphic to a closed manifold (compact manifold without boundary) or not, is not a precondition for computing antipodal pairs. This implies that Algorithm 1 may be applied to object models that are imperfectly perceived with holes due to, for example, visual occlusion.

7. Whole-Arm Grasping using End-Effectors with Curved Contact Surfaces

In this section, we address how our theory can be applied to a scenario where two collaborating manipulator arms, which may be the arm-torso chain of a humanoid robot or two collaborating industrial robot arms, are grasping objects using
end-effectors with curved contact surfaces. The stability of the two-effector cages discussed in Section 5 allows us to add more contacts not necessarily from the end-effectors; the scenario may then be called whole-arm grasping as can be seen in Figure 14. Whole-arm grasping can particularly be effective for grasping large, bulky objects such as rocks, with relatively small end-effectors. Without fabricating dedicated end-effectors, the curved shapes may be emulated in some ways, for example, cupping the fingers of a multi-fingered end-effector. Section 7.1 presents our algorithm for whole-arm grasping. Section 7.2 discusses the implementation of whole-arm grasping on a modular robot system. Section 7.3 presents a set of experiments.

Fig. 14. Two planar whole-arm grasps by the PR2. The robot is grasping the prismatic objects using the entire armchain.

7.1. Approach and Algorithm

Our approach to whole-arm grasping is composed of two phases: preshaping and squeezing. In the preshaping phase, a robot cages an object using its two end-effectors with curved contact surfaces as discussed in Section 5. In the squeezing phase, the robot performs a squeezing motion for the end-effectors. During the squeezing motion, not only the end-effectors but also other links can be made contact the object without adversely affecting the stability of the object if we assume that the robot is position controlled without compliance, in addition to the assumption of frictionless, rigid, unilateral contact:

**Corollary 2.** Suppose that the end-effectors of a robot, which is position controlled without compliance, are caging an object as discussed in Section 5. A grasping cage is obtained if the robot is controlled such that the end-effectors are performing a squeezing motion until contact, which is assumed to be frictionless, rigid, unilateral, is established.

**Proof.** The same argument as the proof of Theorem 3 can also be applied here by regarding (1) $\mathcal{M}$ as the configuration space of the robot itself and (2) $\delta$, the distance between the two end-effectors, as the grasping function again.

The corollary shows that as long as the end-effectors are squeezing, the final state is guaranteed to be a stable equilibrium grasp that can be composed of contacts from the whole body of the robot. Our approach can facilitate planning and control for grasping: in the preshaping phase, the robot can aim at any of the cages, whose collection is not a set of measure zero in the configuration space; the squeezing phase can be performed in a blind manner, only by position control, without direct feedback of the object pose. In fact, the two-phase approach has some similarities with multi-fingered grasping (Miller et al., 2003): approaching an object followed by “closing” the hand.

Our pseudocode is presented in Algorithm 2. The algorithm takes as input an initial configuration of the robot $c_i \in \mathcal{M}_R$, where $\mathcal{M}_R$ is the configuration space of the robot; it returns a reference trajectory for the robot to follow, $\gamma(s) : [0, 1] \rightarrow \mathcal{M}_R$, where $s$ is a non-dimensional parameter increasing with time. The following paragraphs elaborate each line of the algorithm.

**Line 1:** We first construct $c_p, c_s \in \mathcal{M}_R$ that are supposed to describe configurations at which preshaping and squeezing should aim, respectively (Figure 15). They can thus be interpreted as desirable waypoints for whole-arm grasping. $c_p$ is constructed such that the two end-effectors cage the object in a kinematically feasible manner. $c_s$ is constructed such that the robot deliberately intersects the object. These tasks are essentially inverse kinematics problems.
Algorithm 2: Motion planning for whole-arm grasping

**Input:** Robot’s initial configuration, \( c_i \in \mathcal{M}_R \)

**Output:** Reference trajectory for the robot, \( \gamma(s) : [0, 1] \to \mathcal{M}_R \)

1. Plan a trajectory \( \gamma_{ip} \) from \( c_i \) to \( c_p \) for the preshaping.
2. Plan a trajectory \( \gamma_{ps} \) from \( c_p \) to \( c_s \) for the squeezing (possibly in parallel with Line 2).
3. Concatenate \( \gamma_{ip} \) and \( \gamma_{ps} \) into \( \gamma \), the resultant trajectory from \( c_i \) to \( c_s \) via \( c_p \).

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**Fig. 15.** The robot is currently in configuration \( c_p \), where the curved end-effectors of the armchain are caging the object. The wireframe shows the configurations of the end-effectors at \( c_s \), after the squeezing motion.

**Line 2:** We plan for a trajectory from \( c_i \) to \( c_p \). During the motion, we do not want the robot to interact with the object; thus any collisions should be avoided. This can be addressed by ordinary path planning algorithms which are commonly available. Ultimately, some manipulations such as quasi-static pushing (Mason, 2001) may be needed to reach \( c_p \).

**Line 3:** We now plan for a trajectory from \( c_p \) to \( c_s \) realizing a squeezing motion. The geometry of the object can be ignored; however, the robot should be treated as a closed kinematic chain in the sense that the two end-effectors are only allowed to approach to each other by relative translation. This is generally a hard problem due to the kinematic closure, but can be solved efficiently for planar chains as will be discussed in Section 7.2.

It is sufficient to control the robot to follow the resultant trajectory in a quasi-static manner, by considering that only the relative configuration of the object and robot matters in caging and squeezing. In fact, the motion of the robot is necessarily interrupted on the way because it is planned to “collide” with the object. Under the assumption of being frictionless, rigid, with unilateral contact, the configuration where the robot, which is position controlled without compliance, stops moving is an acceptable grasp that can realize a caged, equilibrium grasp by Corollary 2.

**Remark:** Algorithm 2 can also be applied to grasping with frictional, compliant contact as explained in the following paragraphs.

First, with friction, we can actually have more candidates for an acceptable grasp. When friction is present, the robot might get stuck on the way during a squeezing motion because nonzero friction can cause jamming and wedging (Mason, 2001). However, both phenomena imply force-closure, which in turn implies involved wrenches are in equilibrium. Thus a jammed or wedged configuration can also be an acceptable grasp. In conclusion, friction helps improve the stability of a grasp, as also discussed by Rimon and Burdick (1998a).

Second, nonrigid objects can also be grasped stably with the algorithm. If we assume a rigid robot moving with a stiff position control servo loop, a nonrigid object will be deformed during a squeezing motion by contact forces exerted by the robot. Since we take advantage of caging, the object will not be lost during the squeezing motion as long as the caging conditions are satisfied with the deformed geometry. The stability of a nonrigid object under a squeezing motion can be verified by online or offline computation. For online computation, we need to keep track of points of interest (for example, the vertices \( P \) and \( Q \) in Figure 7a) by, for example, visual deformation servoing (Navarro-Alarcon et al., 2014), and verify the corresponding caging conditions. For offline catalog work, we need to consider how to map the deformation of a wide
variety of nonrigid objects by, for example, measuring contact forces or joint efforts. A squeezing motion should terminate as soon as the stability of the object is jeopardized.

7.2. Implementing Whole-Arm Grasping

Here we present hardware/software that implements our approach to whole-arm grasping.

**Hardware**  Our idea of using effectors with curved contact surfaces in robotic grasping can be applied to a wide variety of robotic platforms; here, we employ a modular robot system, which can be a way to maximize adaptability and versatility. According to Sections 4 and 5, we may need a collection of end-effectors with a range of not only shapes but also curvature properties, for practical applications. Thus, a modular robot system with exchangeable end-effectors and attachable/detachable modular arm links allow us to quickly adapt to the sizes and shapes of a wide variety of objects.

Our robot is assembled with CKbot modules\(^2\) (Park and Yim, 2009; Davey et al., 2012), our chain style modular robot system. Each CKbot module can be used as an one degree of freedom swivel or elbow joint (Figure 16a). Figure 16b shows three subassemblies: one spine and two (left and right) arms. The spine, composed of two swivel joints and one elbow joint, provides all the three rotational degrees of freedom by realizing $z$-$y$-$z$ Euler angles. The arms are planar manipulators composed only of elbow joints assembled such that their axes of rotation are parallel. Although Figure 16b shows arms composed of three modules, the modular architecture allows us to assemble more links easily.

![Fig. 16.](image_url) (a) Two types of CKbot modules providing one rotational degree of freedom. The left one (swivel joint) provides continuous rotation; the right one (elbow joint) is limited to 180 degrees rotation. (b) Two 3 degrees of freedom planar arms and a 3 degrees of freedom spine between them. (c) A finished two-armed robot. (d) 3D printed end-effectors that can be docked to the robot.

The two arms are attached to the top of the spine such that the whole armchain is again a planar open kinematic chain. The torque capacity of the spine essentially prevents adding more arms. Although it may not be straightforward to form the immobilizing grasps of Section 4 featuring up to three contacts with the two-armed configuration that is usually supposed to have two end-effectors, the planar chain actually suffices to realize the immobilizing grasps for the following reason. For each point, line, or planar contact, consider the net wrench that is the combination of all the contact wrenches exerted at the contact. The three (or two) net wrenches in any of our immobilizing grasps should be coplanar (from a planar pencil); otherwise, it is impossible to even guarantee equilibrium, which is necessary for immobility. Thus, three effectors appropriately positioned on the planar armchain can form the immobilizing grasps. Our cages and squeezing motions discussed in Section 5 can also be realized by the planar architecture because two end-effectors suffice for caging and they are only required to approach to each other. In Figure 16c, the finished two-armed robot is shown. Figure 16d shows 3D printed end-effectors compatible with CKbot; the effector shapes are derived from our theory discussed in Sections 4 and 5.

**Software**  Our software implementing whole-arm grasping for the hardware platform is organized as ROS\(^3\) packages written in C++ and Python. Figure 17 shows the architecture of our software. The move_group node in the center, organized

\(^2\)http://www.modlabupenn.org/ckbot
\(^3\)http://www.ros.org
as MoveIt\textsuperscript{4} packages, works as an integrator between the hardware and a human user. The software provides a user with high-level methods for grasp synthesis (Algorithm 1) and motion planning for preshaping and squeezing (Algorithm 2); the robot is then position controlled to follow the resultant reference trajectory. In the following paragraphs, we further explain how Algorithm 2 is implemented for our robot with the planar armchain.

First, the inverse kinematics solver for Line 1 of Algorithm 2 is based on the well-studied closed-form solutions of the inverse kinematics of a planar 3R manipulator (planar manipulator with three revolute joints); see Figure 18a. For an arm with three CKbot modules (exactly a planar 3R manipulator), there can be two solutions commonly known as the “elbow-up” and “elbow-down” configurations. The method can also be applied to solving the inverse kinematics of an arm with $n$ CKbot modules, where $n > 3$, by converting the complete problem into appropriate subproblems that the solver can address.

Second, the squeezing in Line 3 of Algorithm 2 is based on energy-based motion planning. An algorithm presented by Iben et al. (2009) generates an interpolation sequence without any self-intersections between two simple polygons. In the algorithm, link lengths are allowed to be monotonically changed; this allows us to implement squeezing motions. The resultant motion basically reconfigures the two polygons “towards each other” according to a metric defined between a pair of polygons, for example, the $\ell^2$-norm on the vector of vertex coordinates. Whenever the direct reconfiguration increases the value of an energy function such as

\begin{equation}
E = \sum \frac{1}{d(p_i, p_{j+1})^2}
\end{equation}

where the denominator can be the squared shortest distance between joint $p_i$ and link $p_jp_{j+1}$ connecting the joints $p_j$ and $p_{j+1}$ ($i \neq j, j + 1$) (see Figure 18a for the notation), we follow the downward gradient of $E$ to avoid collisions. This energy-based planning can be performed efficiently even for a hyper-redundant arm.

Extending Iben et al.’s Algorithm The energy-based motion planning algorithm explained in the previous paragraph is applicable to line segment links without joint limits, that is, $\theta_i \in [-\pi, \pi]$ where $\theta_i$ is the angle of joint $p_i$ ($\theta_i = 0$ when

\textsuperscript{4}http://moveit.ros.org
the two incident links are collinear). We further discuss how to adapt the algorithm so as to address joint limits and link shapes that are not line segments.

First, given narrower joint ranges ($\theta_i \in [-\ell_i, u_i]$ where $0 < \ell_i, u_i < \pi$ for each $\theta_i$), suppose that at a certain instant there are one or more joint angles close to their limit, that is, $\theta_j \in [-\ell_j, -\ell_j + \varepsilon_1]$ or $\theta_j \in [u_j - \varepsilon_2, u_j]$ for some $\varepsilon_1, \varepsilon_2 > 0$, and $j$. If the armchain is described as a concave polygon at that instant, we propose to apply an expansive motion (Connelly et al., 2003) to unfolding all joints such that each $\theta_j$ can return to the “safe” range, that is, $\theta_j \in [-\ell_j + \varepsilon_1, 0]$ or $\theta_j \in [0, u_j - \varepsilon_2]$. During an expansive motion for a closed chain, every joint is monotonically unfolded ($|\theta_i|$ is decreasing for all $i$) until the chain is convexified. Such a motion always exists as long as the chain is described as a simple, concave polygon and is computed by convex optimization (Connelly et al., 2003). In case the armchain is described as a convex polygon at the instant, there also exists such an angle-monotone motion to bring the joint angles back to the safe range (Aichholzer et al., 2001).

Second, in order to avoid collisions among links with nonzero volume, we use $(d(p_i, p_jp_{j+1}) - \delta_j)^2$ as the denominator of each term of $E$ where $\delta_j$ is determined for each link $p_jp_{j+1}$ such that the collection of $x$’s on the level set $d(x, p_jp_{j+1}) - \delta_j = 0$ can address the collision hull of the link (Figure 18b). With the adaptation, however, two adjacent links overlap each other around the joint connecting them; in practice, this issue can be mitigated by having links move on different planes.

7.3. Experiments

A set of experiments were run to evaluate our hardware/software implementation and its performance on the task of grasping objects. We first see how accurately our robot can position its end-effectors; we then proceed into grasping objects.

End-Effector Positioning  Positioning end-effectors is critical to successful grasping; thus, we first evaluated the positioning accuracy of our hardware/software implementation. We assembled a robot with two arms; each arm is composed of three link modules as already shown in Figure 16c. We set up three target configurations for the robot such that (1) the tips of the right and left arms are level at the same height (Experiment 1), (2) the tip of the right arm is higher than that of the left arm (Experiment 2), and (3) the tip of the right arm is lower than that of the left arm (Experiment 3). For each target configuration, a total of 25 trials were conducted and we measured the actual, final positions of the tip of the right arm using the Vicon motion capture system\(^5\). Figure 19 illustrates the results; Table 1 enumerates the data points measured.

The average positioning error of the 75 trials was 3.73 centimeters.

<table>
<thead>
<tr>
<th>$x$-, $y$-, $z$-error (mm) frequency</th>
<th>$x$-, $y$-, $z$-error (mm) frequency</th>
<th>$x$-, $y$-, $z$-error (mm) frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-17, -16, -24) 6</td>
<td>(-19, -32, -20) 19</td>
<td>(-19, -3, -31) 19</td>
</tr>
<tr>
<td>(-17, -15, -24) 3</td>
<td>(-19, -32, -21) 2</td>
<td>(-19, -4, -30) 3</td>
</tr>
<tr>
<td>(-18, -17, -23) 3</td>
<td>(-19, -31, -19) 1</td>
<td>(-19, -4, -31) 2</td>
</tr>
<tr>
<td>(-17, -16, -23) 3</td>
<td>(-19, -32, -19) 1</td>
<td>(-19, -3, -30) 1</td>
</tr>
<tr>
<td>(-17, -15, -23) 2</td>
<td>(-19, -31, -20) 1</td>
<td></td>
</tr>
<tr>
<td>(-17, -16, -23) 2</td>
<td>(-20, -32, -21) 1</td>
<td></td>
</tr>
<tr>
<td>(-18, -16, -23) 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-17, -17, -22) 1</td>
<td></td>
<td></td>
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<tr>
<td>(-18, -18, -23) 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-17, -17, -23) 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-18, -17, -24) 1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1. The results of Experiments 1, 2, and 3 are summarized in the first, second, and third column, respectively. Each entry of the column shows a triple of numbers that represent $x$-, $y$-, and $z$-directional positioning errors, with respect to the reference frame shown in Figure 19, measured in millimeters with the number of times it appeared.

\(^5\)http://www.vicon.com
Fig. 19. The figure illustrates the results of Experiments 1, 2, and 3. In (a), (b), and (c), the simulated robot in the upper panel shows the target configuration for Experiments 3, 1, and 2, respectively; the real robot was controlled to the targets as shown in the lower panels: the ‘△’, ‘○’, and ‘□’ marks represent data points showing the actual, final positions of the tip of the right arm in Experiments 3, 1, and 2, respectively, with respect to the reference frame as shown attached at the tip of the right arm of the simulated robots. In principle, the data points were expected to coincide with the origin of the frame (see the ‘*’ mark at the top of the graph).

In computer simulations, there were no errors in positioning end-effectors; thus, the errors in the real experiments are mostly due to hardware such as tolerance, mechanical play, motor backlash, or compliance. Obviously, the positioning errors are not negligible by considering that each arm is approximately 30 centimeters long. The arm length of the robot can be compared with the arm length of infants (the mean of the upper arm length of infants aged 3-5 months is 12.8 centimeters (McDowell et al., 2008)); for comparison, it has been known that neonate infants can position their hands within 1.5 centimeters from objects, which they touch or occasionally grasp (Bower, 1970).

Whole-Arm Grasping  The positioning experiments suggest the appropriate sizes of end-effectors for successful grasping without position calibration. Figure 20 shows the CAD models of our end-effectors fabricated for grasping experiments. The dimensions of the effectors were determined to address the $x$- and $z$-directional positioning errors (recall Figure 19 for the axes and Table 1 for the error data). For example, for the torus-shaped effector in Figure 20a, the inner radius of the torus was determined to be 46 millimeters, larger than the maximum positioning error on the $xz$-plane from Experiments 1, 2, and 3; for the cylindrical effector in Figure 20b, the inner radius of the cylinder was also determined to be 46 millimeters for the same reason. The end-effectors are supposed to be squeezed along the $y$-axis; thus, the $y$-directional errors can be addressed by our approach that takes advantage of squeezing. Although the end-effectors appearing in Figure 20 look different from the examples in Section 5.1, the end-effectors are sufficient to grasp objects based on Theorem 3 and Corollary 2 in that their geometry is more conservative than the examples. For example, the torus-shaped end-effector in Figure 20a can be regarded as the boundary of the hemispherical effector in Section 5.1; moreover, the torus-shaped effector was empirically analyzed in Section 5.2.

We ran three sets of experiments to evaluate the capability of our system: grasping an object using its vertex-vertex pair (Experiment 4), vertex-face pair (Experiment 5), and edge-edge pair (Experiment 6). In each set of experiment, a total of
Fig. 20. (a) A torus-shaped end-effector. (b) A cylindrical end-effector; in the model, material usage was minimized to reduce weight and cost. The units are in millimeters.

25 trials were conducted and we verified if the robot could grasp an object, perceived by the Vicon system, via preshaping and squeezing without any position calibration. The results are summarized in Table 2.

<table>
<thead>
<tr>
<th>Grasp Type</th>
<th>Successes/Trials</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experiment 4: vertex-vertex grasping</td>
<td>22/25</td>
</tr>
<tr>
<td>Experiment 5: vertex-face grasping</td>
<td>23/25</td>
</tr>
<tr>
<td>Experiment 6: edge-edge grasping</td>
<td>25/25</td>
</tr>
</tbody>
</table>

Table 2. The results of Experiments 4, 5, and 6.

The snapshots in Figure 21 show the robot grasping objects via preshaping and squeezing, demonstrating Theorem 3 by showing that caged, equilibrium grasps can be obtained from our cages and verifying the soundness of our approach to whole-arm grasping (Corollary 2). The robot was controlled until joint torque saturation occurred during the squeezing. The final grasps shown in the rightmost panels of Figure 21 were obtained from cages on a vertex-vertex, a vertex-face, and an edge-edge pair discussed in Section 5. According to Corollary 1, the way the robot is grasping the box in Figure 21a can be applied to any polyhedral shape; almost all polyhedral shapes can be grasped in exactly the same way as grasping the tetrahedral object in either Figure 21b or 21c. In the failed trials, the two arms were moving in an asynchronized manner (possibly due to an unexpected communication error) or the end-effectors were dynamically interacting with the object (or the support under the object).

Finally, Table 3 shows the average time frame to perform preshaping and squeezing, for 25 trials. In the table, the planning times show how long it takes to run the respective software components (preshape() or squeeze(), recall Figure 17) on a 2.53GHz/4GB machine. The executing times show how long it takes for the robot to execute the computed plans. It can be seen that if preshaping and squeezing are planned in a parallel manner, as mentioned in Algorithm 2, they may be executed in a seamless manner without interruption because the time to preshape can be compared with the time to plan for squeezing. The methods preshape() and squeeze() were implemented in Python.

<table>
<thead>
<tr>
<th></th>
<th>Planning Time</th>
<th>Executing Time</th>
<th>Planning Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preshaping</td>
<td>min max</td>
<td>min max</td>
<td>min max</td>
</tr>
<tr>
<td>0.21s 0.41s</td>
<td>4.7s 6.4s</td>
<td>1.72s 6.41s</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. The time frame to perform the two phases of whole-arm grasping.
Fig. 21. (a) The robot is grasping the box with the two torus-shaped end-effectors that can cage and grasp the vertex-vertex pair. (b) The robot is grasping the tetrahedral object also with the two torus-shaped end-effectors that can cage and grasp the vertex-face pair; one of the torus-shaped effectors was used to contact the face. (c) The robot is grasping the tetrahedral object with the two cylindrical end-effectors that can cage and grasp the edge-edge pair.

8. Conclusion

We have presented three types of immobilizing grasps and cages and showed that they can be applied to a wide range of object shapes including polyhedra. Each of the grasps or cages is formed by at most three effectors with appropriately curved contact surfaces. The immobilizing grasps depend on the local curvature properties of the effectors, whereas the cages are formulated by global parameters, such as the distance between the effectors. A stable grasp can be obtained from any of our cages by a squeezing motion; it also allows us to get a whole-arm grasp. In Appendix that follows, the collection of the complete grasps and cages is extended to include more types of grasps and cages for two- or three-dimensional objects. Our theory was implemented on a modular robotic platform that can enhance versatility and adaptability needed in robotic grasping, compared to a customary manipulator with a fixed configuration. Not only pick-and-place tasks but also a variety of post-grasp manipulation tasks can potentially be facilitated by the secure grasps using the effectors with curved contact surfaces.

Our future research directions include extending theory, enhancing autonomy, and finding more applications. First, our theoretical framework can be extended in a number of directions. In order to guarantee performance under frictional, compliant contact, the effects of friction and compliance should be treated in a more quantitative manner. Approaches may include incorporating accurate and tractable models of finger-object contact, contact friction, and the compliance of the robot joints and links. Second, there are many great benefits to enhanced autonomy. For example, autonomous robotic systems can eliminate human errors and work safely in dangerous environments such as outer space or the ocean floor. Our objectives for enhancing autonomy include incorporating more sensing capabilities; for example, visual sensing will be beneficial to not only perceiving objects to grasp but also guaranteeing the safety of squeezing motions. The level of autonomy can also be greatly increased by enabling proper interaction with the environment, which will facilitate grasping an object in clutter. Another interesting direction is to develop hardware and software supporting efficient self-reconfiguration: such a system can adapt itself to grasp an arbitrary object by autonomously exchanging end-effector or attaching more arm links. Third, effectors with curved contact surfaces can be used as not only “hands” but also “feet” for walking or running. RHex (Saranli et al., 2001; Johnson and Koditschek, 2013) showed the versatility of single-bodied legs without any internal degrees of
freedom; leg stiffness will be an important factor as discussed by Galloway et al. (2011). Our approach can also be useful in manufacturing, eliminating the need of redesigning fixtures according to objects.

Appendix: Extending Our Theory

Here we discuss how to extend the collection of the grasps/cages presented in Sections 4 and 5: more types of grasps/cages can be added by employing other types of antipodal pairs. We also discuss grasping and caging two-dimensional objects with curved effectors on the plane.

Fig. 22. (a) An immobilizing grasp at an antipodal vertex-edge pair with the cone- and V-shaped effectors. The vertex, O, is the origin of the reference frame whose x-axis is parallel to the edge AB and y-axis is collinear to ξ. The arrows are involved unit contact wrenches that can be made with a pair of planar effectors under the assumption of frictionless, rigid, unilateral contact. (b) An immobilizing grasp at an antipodal edge-face pair. (c) An immobilizing grasp at an antipodal face-face pair.

We first show that it is possible to immobilize a polyhedron with one point and one line contact made by curved effector surfaces, which are frictionless, rigid, unilateral. See Figure 22a. O and AB are a vertex-edge pair of the given polyhedron where a point and a line contact can be made with a pair of parallel planes that are perpendicular to ξ, the line connecting the vertex and its foot of perpendicular to the edge (the foot of perpendicular is in the interior of the edge). The inward-pointing contact normal at the vertex points toward the edge and vice versa. With respect to the reference frame whose origin is O, the three unit contact wrenches (the arrows in Figure 22a) are:

\[
(s_1, s_{01}) = (0, -1, 0, 0, 0, 0)
\]
\[
(s_2, s_{02}) = (0, 1, 0, 0, 0, b)
\]
\[
(s_3, s_{03}) = (0, 1, 0, 0, 0, -a)
\]

For a differential twist \((t, t_0) = (t_1, t_2, t_3, t_4, t_5, t_6)\) to be consistent with the unit contact wrenches, each of the reciprocal products of the twist and the unit contact wrenches must be zero:

\[-t_5 = 0\]
\[t_5 + bt_3 = 0\]
\[t_5 - at_3 = 0\]

Then, the solutions are of the form:

\[(t, t_0) = (t_1, t_2, 0, t_4, 0, t_6)\]

because \(t_3\) and \(t_5\) must be zero. If \(O\) is being contacted by the apex of a cone-shaped effector, as can be seen in Figure 22a, the velocity of \(O\) cannot have nonzero \(x\)- and \(z\)-components with respect to the reference frame; otherwise, \(O\) will penetrate
the effector. In other words, $t_4$ and $t_6$ can also be made zero; then, the solutions must actually be of the form:

$$(t, t_0) = (t_1, t_2, 0, 0, 0, 0)$$

This is a zero-pitch screw (pure rotation) where the first three components give the direction of the rotation axis, which should pass through $O$. This rotation instantaneously moves $\overline{AB}$ in a direction perpendicular to the plane of $O$, $A$, and $B$, but can be restricted by contacting $\overline{AB}$ with a V-shaped effector, as can be seen in Figure 22a. The polyhedron is then immobilized with the point and line contacts on the vertex-edge pair; at the same time, the two effectors are also forming a cage, with their distance along $\xi$ as a grasping function similarly to the previous two-effector cages.

We additionally show that it is possible to immobilize a polyhedron with one planar and two point contacts made by curved effector surfaces, which are frictionless, rigid, unilateral. See Figure 22b. Consider an edge-face pair that determines the width of the convex hull of the given polyhedron, denoted as $\overline{AB}$ and $\square Q R S T$. At each vertex of the edge, consider a set of unit contact wrenches, each of which is perpendicular to one of the faces containing the vertex and points toward the interior of the polyhedron; then each of $A$ and $B$ has three such unit contact wrenches because there are three faces meeting at each vertex. Also consider a set of unit contact wrenches at the face, each of which is perpendicular to the face toward the interior at one vertex delimiting the face; then $\square Q R S T$ has four such unit contact wrenches at $Q$, $R$, $S$, and $T$.

The grasp of one planar and two point contacts shown in Figure 22b can always be made in equilibrium, by considering that the edge-face pair even admits a clamp. To be more specific, consider the following linear feasibility program

Find $\{c_i\}$ such that $\sum c_i \hat{w}_i = 0$, $c_i > 0$, and $\sum c_i = 1$  \hspace{1cm} (2)

where $\{\hat{w}_i\}$ is the collection of all the unit contact wrenches. The grasp can be made in equilibrium by configuring the effectors according to a solution to the program: the terms of $\sum c_i \hat{w}_i$ can be divided into three groups affiliated with the two vertices and face; each effector should be configured to exert the sum of one group of the wrenches, which is the positive linear combination of the unit contact wrenches at the element the effector is contacting. For example, in Figure 22b, the two hemispherical effectors are configured such that their axes of symmetry, $\xi_1$ and $\xi_2$, pass through $A$ and $B$, the points of contact, and intersect at $O$, whose foot of perpendicular to the plane of $\square Q R S T$ lies on the area of the planar contact. Under the equilibrium, the object can only move on the plane of the planar contact. If the two curved effectors on the two vertices are sufficiently concave, the object can in fact be immobilized because any finite motion of $\overline{AB}$ results in penetrating the effectors. This idea can also be applied to immobilizing a polyhedral object on a face-face antipodal pair (Figure 22c).

The objects in Figures 22b and 22c are also caged by the three effectors. One approach to establishing a grasping function here is to assume a control strategy that moves the three effectors as a three-fingered gripper system with one parameter that controls the opening of the gripper, as also discussed by Davidson and Blake (1998a) for three point fingers. Let $\delta$ denote the opening parameter of the gripper: as $\delta$ increases, the three effectors shown in Figures 22b and 22c monotonically move away from $O$ by translation along the axes ($\xi_1$, $\xi_2$, and $\xi_3$). In each of the grasps shown in Figures 22b and 22c, all the instantaneous motions of the object penetrate the effectors, which in turn guarantees that the configuration of the object is completely isolated from its free space (Rimon and Burdick, 1998a). Then it can be seen that the opening parameter $\delta$ is a grasping function similarly to Theorem 3. Constructing caging conditions for three effectors may be harder than two-effector cages; for point effectors, refer to Davidson and Blake (1998a).

Two-dimensional grasping: Our approach can also be applied to grasping two-dimensional objects with effectors that can be represented as one-dimensional manifolds with boundary on the plane. Every polygonal object can be immobilized on the plane by two such curved effectors of appropriately chosen dimensions. In Figure 23a, the two curved effectors are immobilizing the object by two point contacts; in Figure 23b, the object is immobilized by a point and a line contact. The
two types of grasps are complete: every polygon can be immobilized by two point contacts made by appropriately concave effectors on a pair of vertices determining the diameter (the maximum distance between two vertices); every polyhedron whose vertices are in general position such that there is no pair of two parallel lines, where each line is determined by a distinct collection of two vertices, can be immobilized by a point and a line contact made by appropriately concave effectors on a vertex-edge pair determining the width.

As a corollary, every polygon can be caged by two curved effectors: the effectors in Figures 23a and 23b are also caging the polygon. Sufficient conditions for caging can be derived by applying the results of Sections 5.1 and 5.2 (see the similarity between Figure 8 and Figure 23a (or Figure 9 and Figure 23b)).

We can also add more types of grasps and cages; for example, as shown in Figure 23c, two point effectors suffice to immobilize/cage some concave polygons as discussed by Vahedi and van der Stappen (2008a).

Acknowledgement We gratefully acknowledge the support of NSF 1328805, 1138847, and ARL Grant W911NF-10-2-0016.

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