A Hierarchy of Polyhedral Approximations of Robust Semidefinite Programs

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joint work with Eilyan Bitar

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Robust Semidefinite Program

Consider the robust semidefinite program (SDP)

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & \displaystyle\sum_{i=1}^{k} \xi_i \mathcal{A}_i(x) \ \in \ \mathbf{S}_+^{\boldsymbol{n}}, \quad \forall\, \xi \in \Xi, \end{array}$$

where

- $x \in \mathbf{R}^{\boldsymbol{m}}$ is the decision variable
- $\mathcal{A}_i : \mathbf{R}^{\boldsymbol{m}} \to \mathbf{S}^{\boldsymbol{n}}$ is an affine function of x, and
- $\Xi \subseteq \mathbf{R}^{\boldsymbol{k}}$ is the uncertainty set, a convex compact set

Some notation:

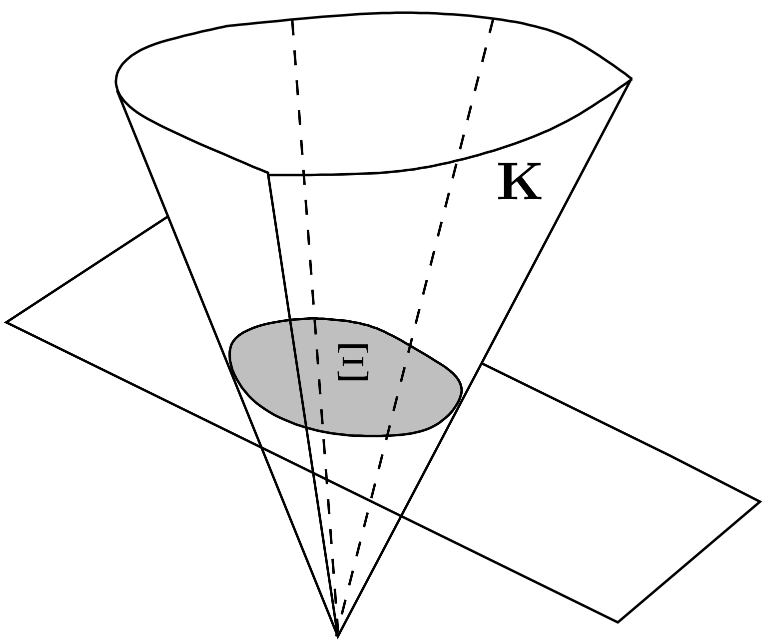
- $\mathbf{S}^{\boldsymbol{n}}$ $(\mathbf{S}_+^{\boldsymbol{n}})$: space of $n \times n$ symmetric (symmetric PSD) matrices

Uncertainty Model

The uncertainty set Ξ is defined

$$\Xi \triangleq \{\xi \in \mathbf{R}^k \mid \xi_1 = 1,\ B\xi \in \mathbf{K}\},$$

where $\mathbf{K}$ is a proper cone, e.g.,



- positive orthant
- second-order cone
- positive semidefinite cone

Taken from Jerome Malic's "Semidefinite Projections, regularization algorithms and polynomial optimization."

Robust SDPs : Challenges

The robust SDP is **NP-hard,** in general

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & \displaystyle\sum_{i=1}^{k} \xi_i \mathcal{A}_i(x) \ \in \ \mathbf{S}_+^{\boldsymbol{n}}, \quad \forall\, \xi \in \Xi, \end{array}$$

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$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & \displaystyle\sum_{i=1}^{k} \xi_i \mathcal{A}_i(x) \in \mathbf{S}^{\boldsymbol{n}}_{+}, \quad \forall\, \xi \in \Xi, \end{array}$$

The constraint holds if and only if

$$\min_{\xi \in \Xi} \underbrace{\lambda_{\min}\left(\sum_{i=1}^{k} \xi_i \mathcal{A}_i(x)\right)}_{\textbf{concave in } \xi} \geq 0.$$

$\lambda_{\min} : \mathbf{S}^n \to \mathbf{R}$ is the minimum eigenvalue function

A Robust LP Approach

Approximate the robust SDP with a robust linear program (LP)

Robust LP:

$$\begin{aligned} &\text{minimize} \quad c^\top x \\ &\text{subject to} \quad \sum_{i=1}^{k} \xi_i \mathcal{A}_i(x) \cancel{\in \mathbb{S}_+^n}, \quad \forall\, \xi \in \Xi, \end{aligned}$$

polyhedral cone

by **approximating the PSD cone by a polyhedral cone**

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by **approximating the PSD cone by a polyhedral cone**

Robust LPs: Admit **finite-dimensional reformulations** as conic convex programs over the cone $\mathbf{K}$ characterizing uncertainty set Ξ, e.g.

$\mathbf{K}$	Polyhedral Cone	Second-order Cone	Semidefinite Cone
Robust LP	LP	SOCP	SDP

Finite-Dim. Reformulations and Approximations

Exact Reformulations

- Ben-Tal, El-Ghaoui, Nemirovski, [‘00]
  - Ξ is *“Unstructured norm-bounded”*

Inner Approximations

- Ben-Tal, El-Ghaoui, Nemirovski, [‘00]
  - Ξ is *“Structured norm-bounded”*
- Scherer and Hol, [‘06]
  - Ξ is described by polynomial matrix inequalities

Other related work

- Packard et al. [‘93]
- El-Ghaoui et al. [‘97]
- Scherer [‘05]
- Dietz et al. [’08]
- Ben-Tal et al. [‘02]
- Oishi et al. [‘08]

Finite-Dim. Reformulations and Approximations

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Talk Outline

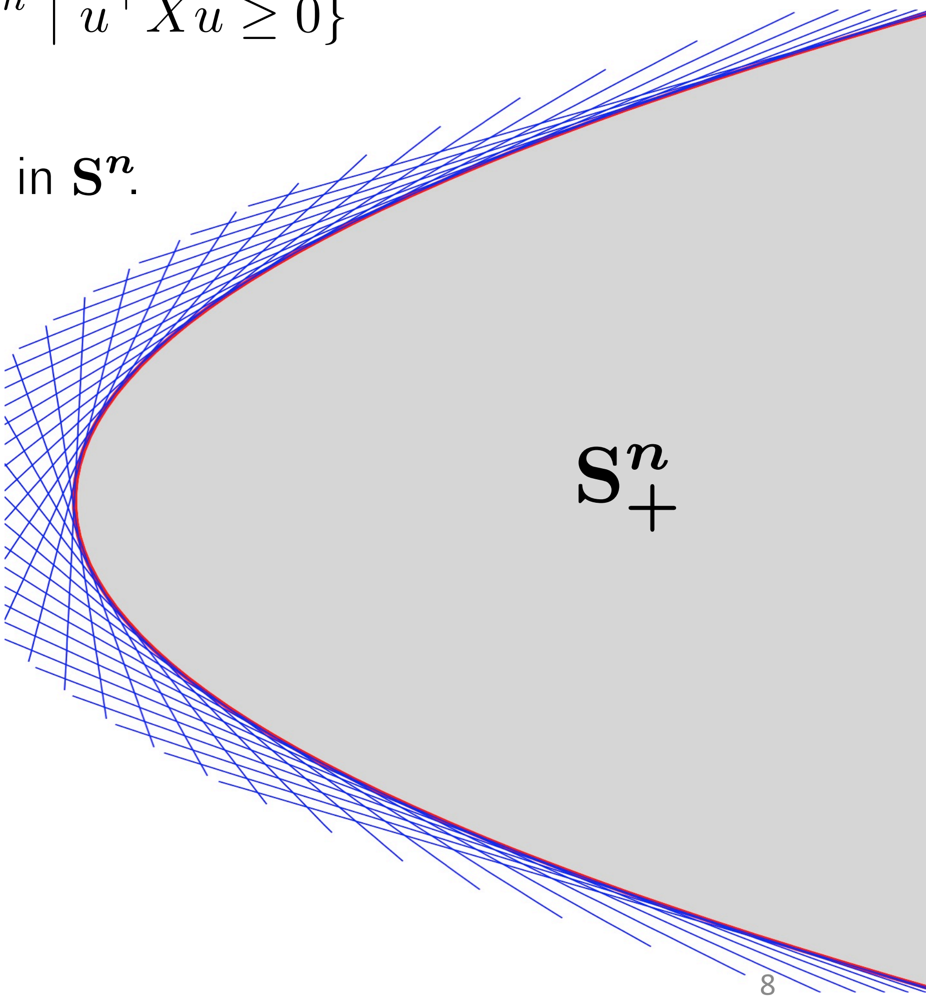
1. Introduction
2. Inner and Outer Polyhedral Hierarchies of the PSD Cone
3. Inner and Outer Hierarchies of Robust SDPs
4. Application : Robust Resistance Network Design Problem

Polyhedral Approximation of $\mathbf{S}^{\boldsymbol{n}}_{+}$

The positive semidefinite cone

$$\mathbf{S}^{\boldsymbol{n}}_{+} = \bigcap_{u \neq 0} \{X \in \mathbf{S}^n \mid u^\top X u \geq 0\}$$

is an infinite intersection of half-spaces in $\mathbf{S}^{\boldsymbol{n}}$.



Outer Polyhedral Approximation of $\mathbf{S}_+^{\boldsymbol{n}}$

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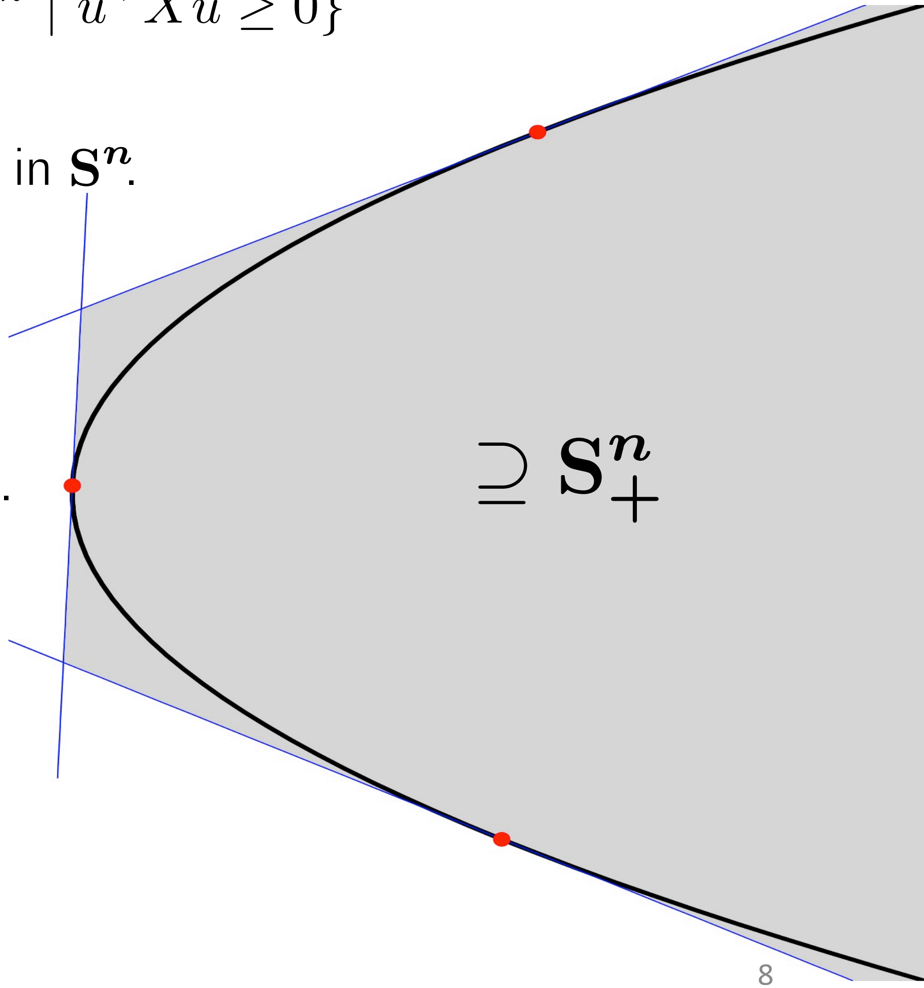
- A finite intersection of half-spaces yields an outer polyhedral cone to $\mathbf{S}_+^{\boldsymbol{n}}$.



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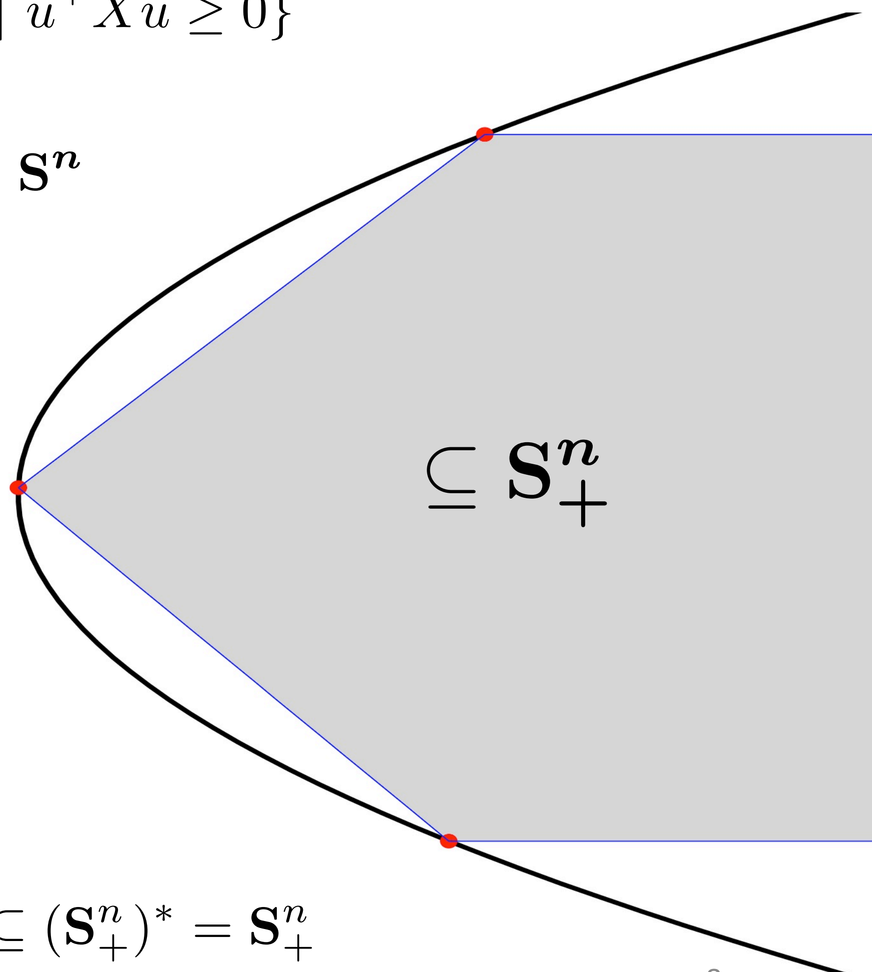


- A finite intersection of half-spaces yields an outer polyhedral cone to $\mathbf{S}_+^{\boldsymbol{n}}$

- The dual of the outer polyhedral cone is an inner polyhedral cone to $\mathbf{S}_+^{\boldsymbol{n}}$

$$\mathbf{S}_+^n \subseteq \text{polyhedral} \iff (\text{polyhedral})^* \subseteq (\mathbf{S}_+^n)^* = \mathbf{S}_+^n$$

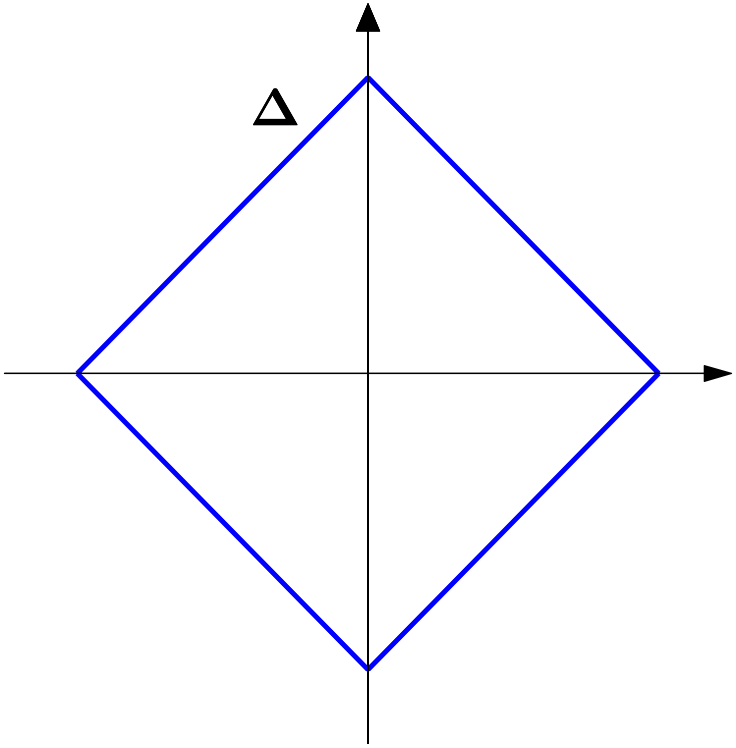
Construction of Outer Polyhedral Approximations

Let $\boldsymbol{\Delta}$ denote the boundary of the ℓ_1 norm ball in $\mathbf{R}^n$

$$\boldsymbol{\Delta} := \{x \in \mathbf{R}^n \mid \|x\|_1 = 1\}.$$

The PSD cone can be expressed as:

$$\mathbf{S}_+^{\boldsymbol{n}} = \bigcap_{u \in \boldsymbol{\Delta}} \{X \in \mathbf{S}^n \mid u^\top X u \geq 0\}$$



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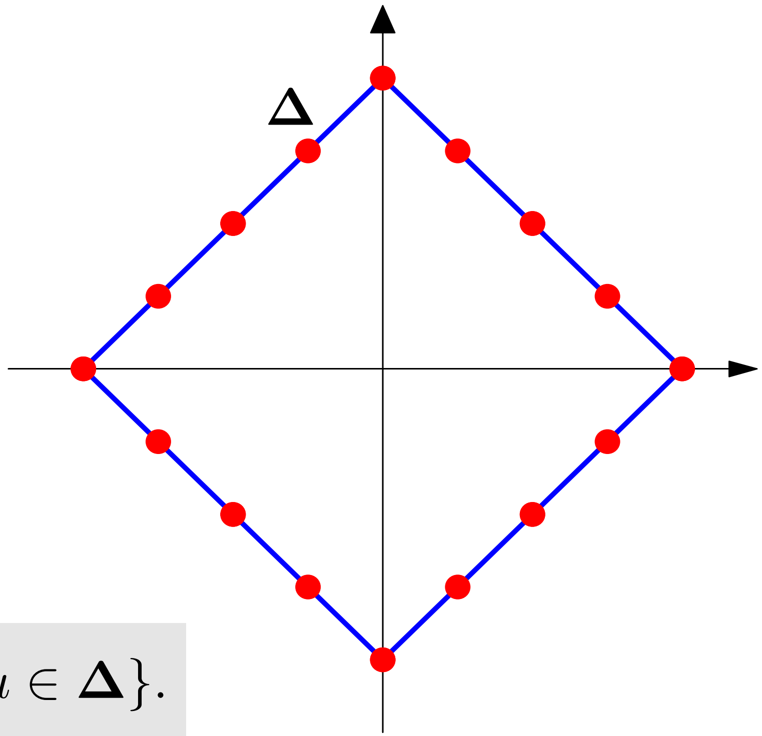

- An **outer polyhedral cone** to $\mathbf{S}_+^n$ arises by a **discretization of $\boldsymbol{\Delta}$**, i.e.,

$$\mathbf{S}_+^n \subseteq \{X \in \mathbf{S}^n \mid u^\top X u \geq 0, \ \textbf{for some } u \in \boldsymbol{\Delta}\}.$$

A Discretization Scheme of $\boldsymbol{\Delta}$

Fix $r \in \mathbb{N}$. Consider the following discretization of $\boldsymbol{\Delta} \subseteq \mathbf{R}^n$:

$$\boldsymbol{\Delta}_{\boldsymbol{r}} := \{u \in \boldsymbol{\Delta} \mid 2^{\boldsymbol{r}} u \in \mathbb{Z}^{\boldsymbol{n}}\}$$

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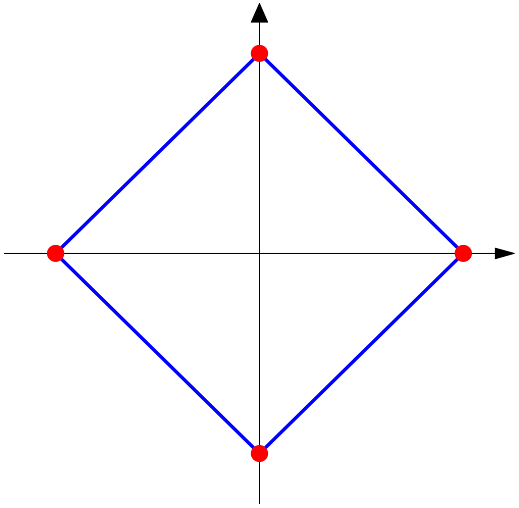
Examples: $n = 2$

$$\boldsymbol{\Delta_0} = \left\{ \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \pm 1 \end{bmatrix} \right\}$$



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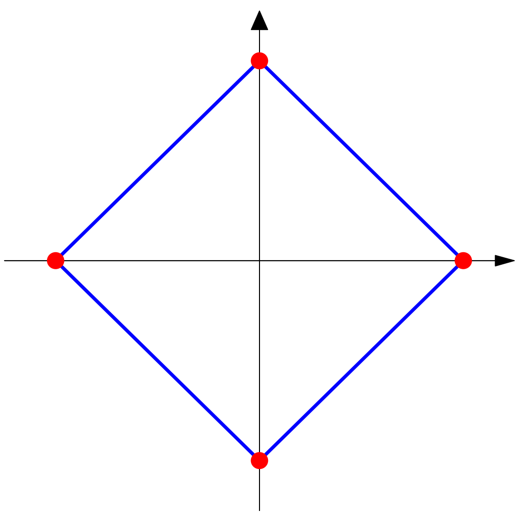

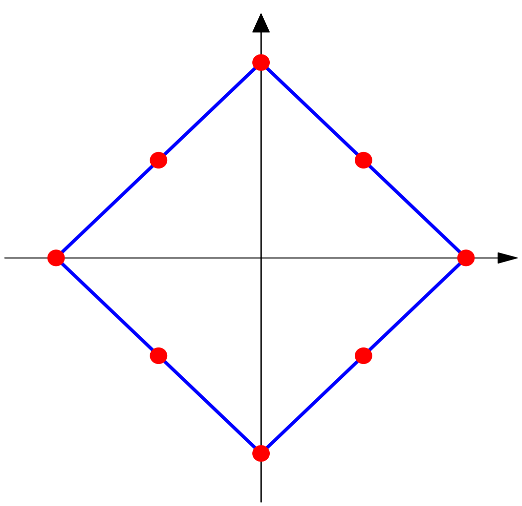

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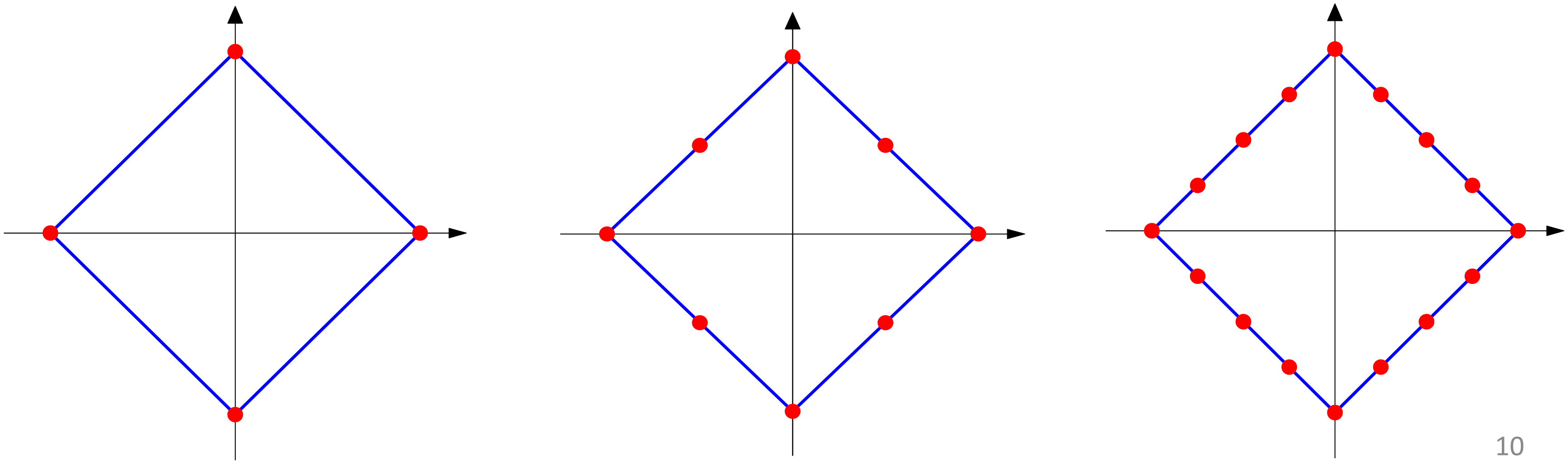

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Remarks: For any $r \in \mathbb{N}$, it holds that

$$\boldsymbol{\Delta_r} \subseteq \boldsymbol{\Delta_{r+1}}$$

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Remarks: For any $r \in \mathbb{N}$, it holds that

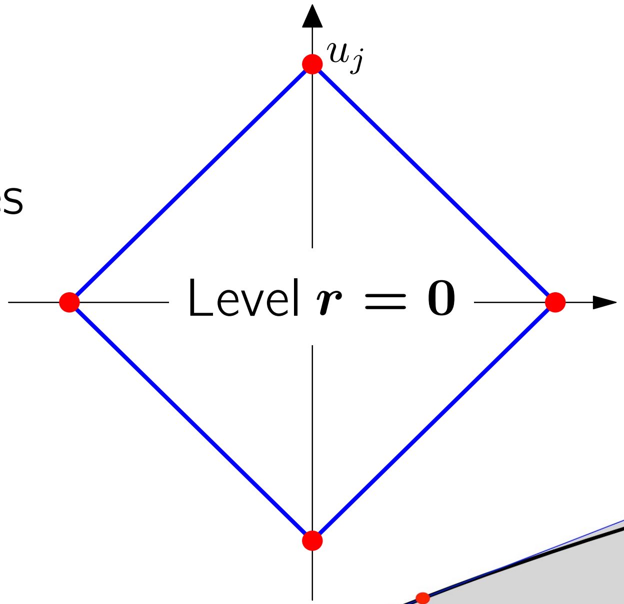
$$\boldsymbol{\Delta_r} \subseteq \boldsymbol{\Delta_{r+1}}$$

Some notation: The set $\boldsymbol{\Delta_r}$ has $p_{\boldsymbol{r}}$ elements denoted by:

$$u_1, u_2, \ldots, u_{p_{\boldsymbol{r}}}$$

Outer Polyhedral Hierarchies of $\mathbf{S}^{\boldsymbol{n}}_{+}$

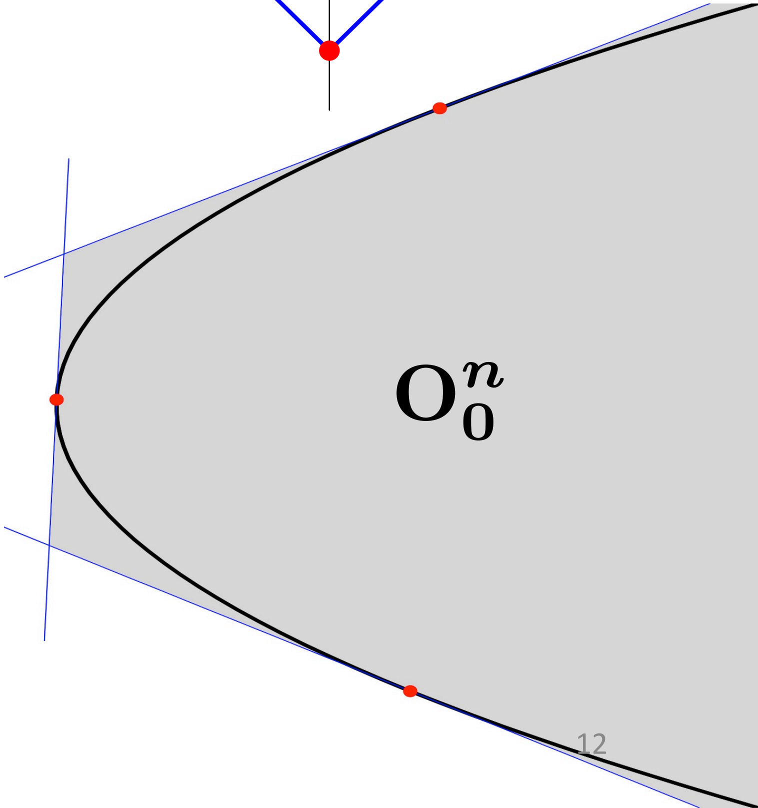
A **hierarchy of outer polyhedral cones** to $\mathbf{S}^{\boldsymbol{n}}_{+}$ arises by the following family of polyhedral cones



$$\mathbf{O}^{\boldsymbol{n}}_{\boldsymbol{r}} := \bigcap_{u \in \boldsymbol{\Delta}_{\boldsymbol{r}}} \{X \in \mathbf{S}^n \mid u^\top X u \geq 0\}$$

where $\boldsymbol{r} \in \mathbb{N}$. In particular

$$\mathbf{O}^{\boldsymbol{n}}_{\mathbf{0}} \supseteq \mathbf{S}^{\boldsymbol{n}}_{+}$$



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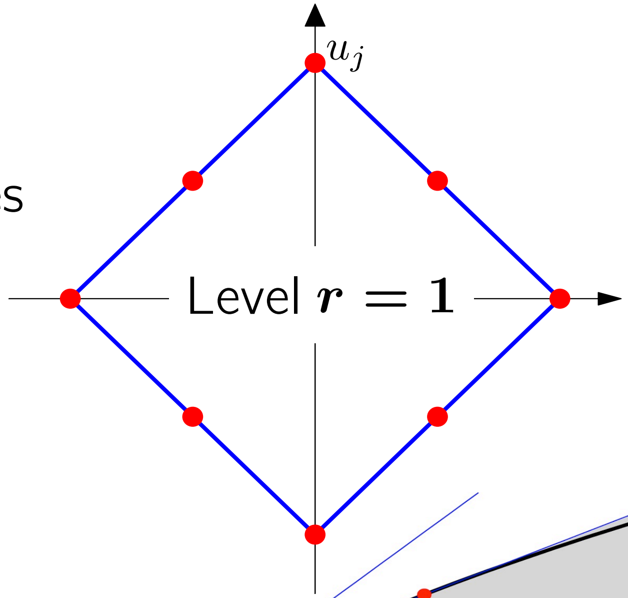

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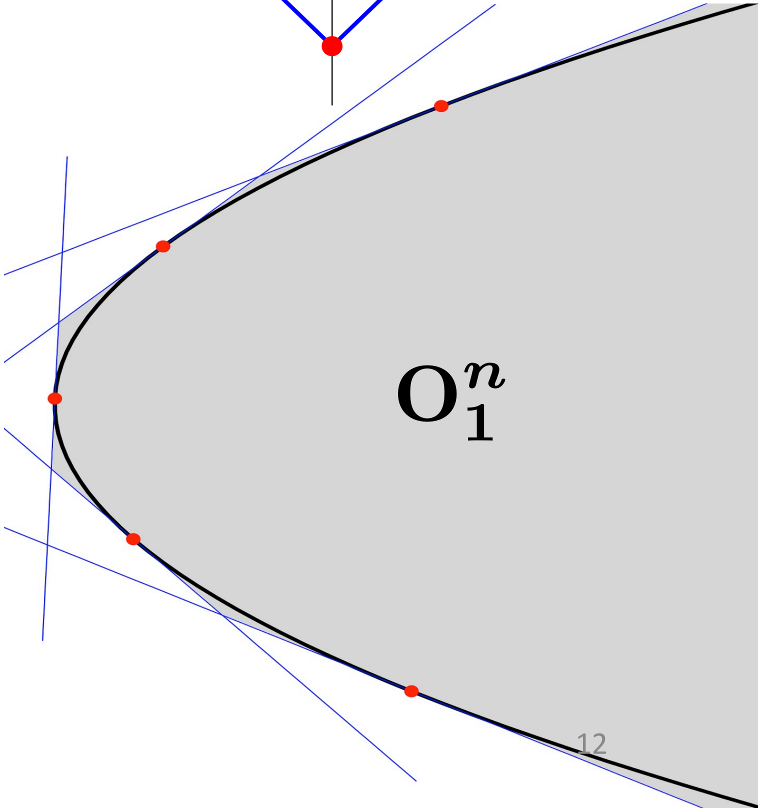
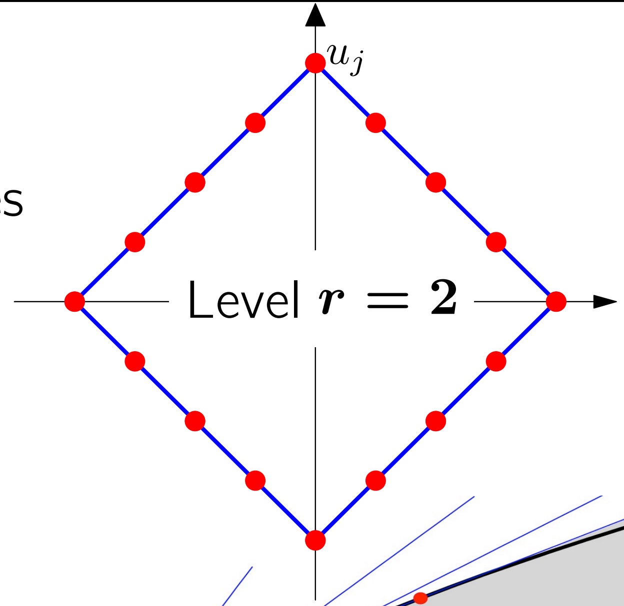
where $\boldsymbol{r} \in \mathbb{N}$. In particular

$$\mathbf{O}^{\boldsymbol{n}}_{\mathbf{0}} \supseteq \mathbf{O}^{\boldsymbol{n}}_{\mathbf{1}} \supseteq \mathbf{S}^{\boldsymbol{n}}_{+}$$

since $\boldsymbol{\Delta}_{\mathbf{0}} \subseteq \boldsymbol{\Delta}_{\mathbf{1}}$.



Outer Polyhedral Hierarchies of $\mathbf{S}_+^{\boldsymbol{n}}$



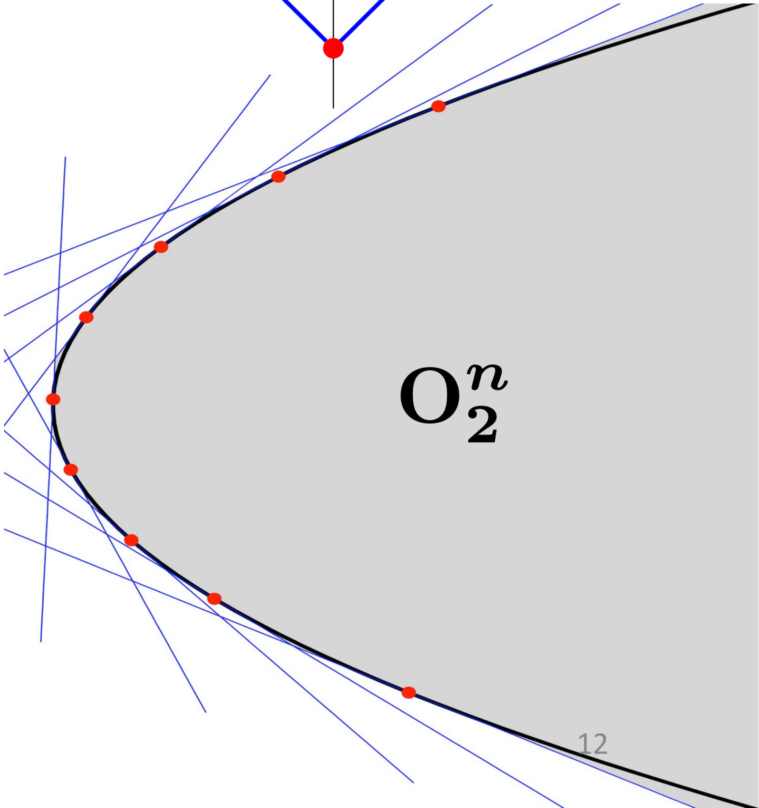
A **hierarchy of outer polyhedral cones** to $\mathbf{S}_+^{\boldsymbol{n}}$ arises by the following family of polyhedral cones

$$\mathbf{O}_{\boldsymbol{r}}^{\boldsymbol{n}} := \bigcap_{u \in \boldsymbol{\Delta}_{\boldsymbol{r}}} \{X \in \mathbf{S}^n \mid u^\top X u \geq 0\}$$

where $\boldsymbol{r} \in \mathbb{N}$. In particular:

$$\mathbf{O}_{\mathbf{0}}^{\boldsymbol{n}} \supseteq \mathbf{O}_{\mathbf{1}}^{\boldsymbol{n}} \supseteq \mathbf{O}_{\mathbf{2}}^{\boldsymbol{n}} \supseteq \mathbf{S}_+^{\boldsymbol{n}}$$

since $\boldsymbol{\Delta}_{\mathbf{1}} \subseteq \boldsymbol{\Delta}_{\mathbf{2}}$.



Outer Polyhedral Hierarchies of $\mathbf{S}^{\boldsymbol{n}}_+$

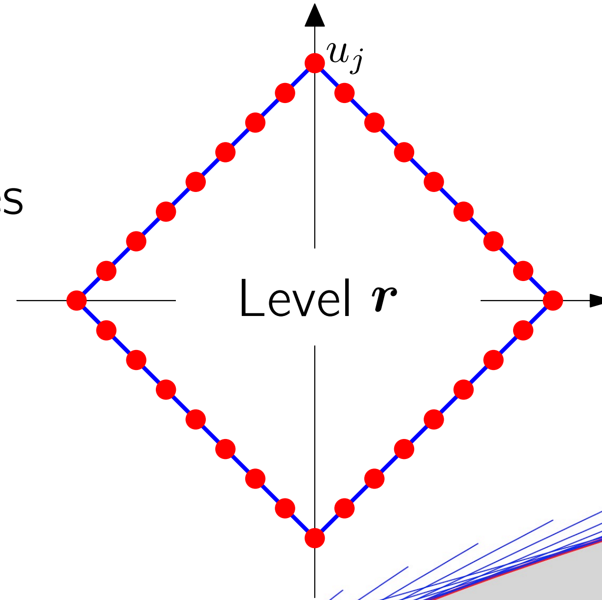
A **hierarchy of outer polyhedral cones** to $\mathbf{S}^{\boldsymbol{n}}_+$ arises by the following family of polyhedral cones

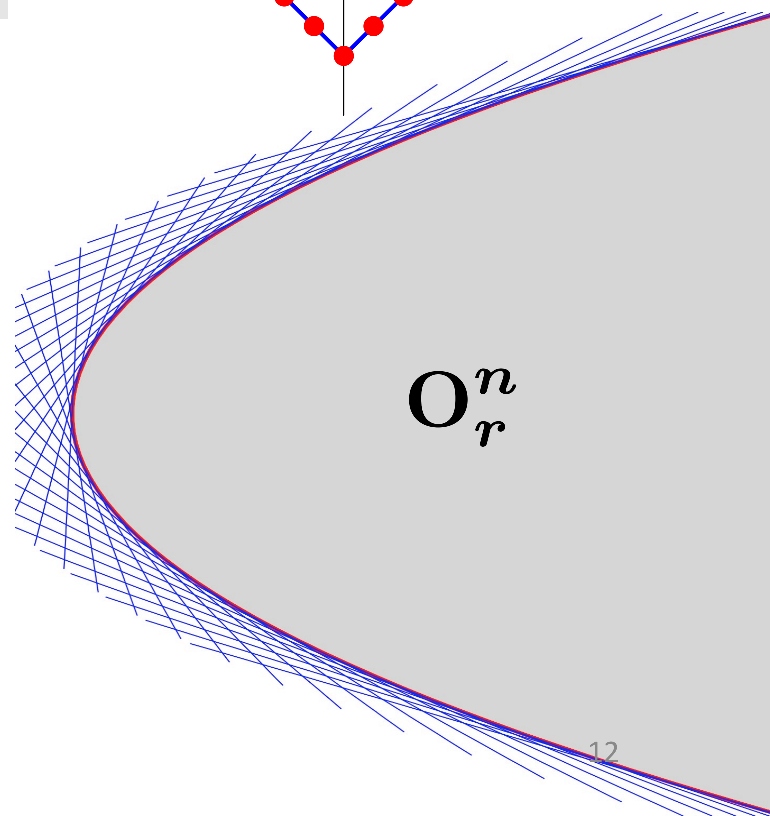
$$\mathbf{O}^{\boldsymbol{n}}_{\boldsymbol{r}} := \bigcap_{u \in \boldsymbol{\Delta}_{\boldsymbol{r}}} \{X \in \mathbf{S}^n \mid u^\top X u \geq 0\}$$

where $\boldsymbol{r} \in \mathbb{N}$. In particular:

$$\mathbf{O}^{\boldsymbol{n}}_{\mathbf{0}} \supseteq \mathbf{O}^{\boldsymbol{n}}_{\mathbf{1}} \supseteq \mathbf{O}^{\boldsymbol{n}}_{\mathbf{2}} \supseteq \cdots \supseteq \mathbf{S}^{\boldsymbol{n}}_+$$

since $\boldsymbol{\Delta}_{\boldsymbol{r}} \subseteq \boldsymbol{\Delta}_{\boldsymbol{r}+1}$, for all $\boldsymbol{r} \in \mathbb{N}$.





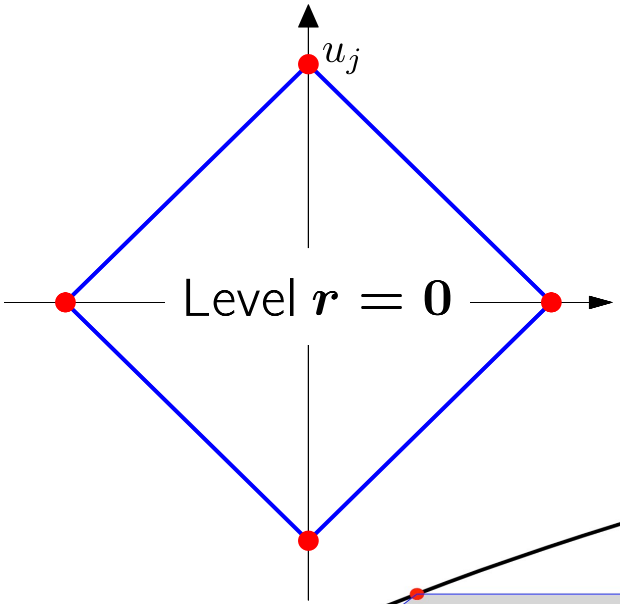
Inner Polyhedral Hierarchies of $\mathbf{S}^n_+$

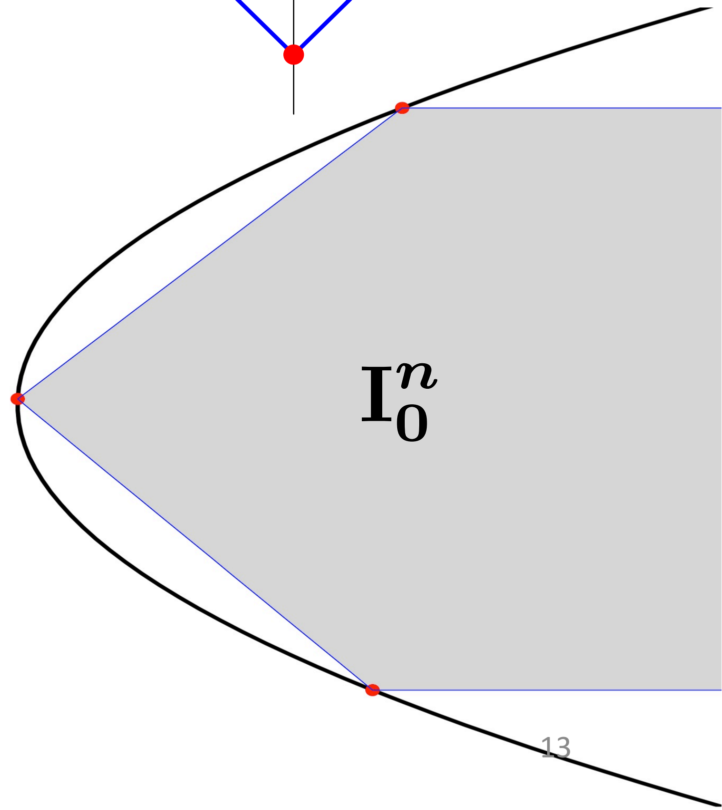
The dual cones to $\mathbf{O}^{\boldsymbol{n}}_{\boldsymbol{r}}$ give a **hierarch of inner polyhedral cones** to $\mathbf{S}^n_+$

$$\mathbf{I}^{\boldsymbol{n}}_{\boldsymbol{r}} = (\mathbf{O}^{\boldsymbol{n}}_{\boldsymbol{r}})^* = \mathrm{cone}\left\{u_1 u_1^\top, \ldots, u_{p_r} u_{p_r}^\top\right\}$$

where $\boldsymbol{r} \in \mathbb{N}$. In particular:

$$\mathbf{I}^{\boldsymbol{n}}_{\mathbf{0}} \subseteq \mathbf{S}^{\boldsymbol{n}}_+$$





Inner Polyhedral Hierarchies of $\mathbf{S}_{+}^{\boldsymbol{n}}$

The dual cones to $\mathbf{O}_{\boldsymbol{r}}^{\boldsymbol{n}}$ give a **hierarch of** $\mathbf{S}_{+}^{\boldsymbol{n}}$ **inner polyhedral cones** to $\mathbf{S}_{+}^{\boldsymbol{n}}$

$$\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}} = (\mathbf{O}_{\boldsymbol{r}}^{\boldsymbol{n}})^{*} = \operatorname{cone}\left\{u_1 u_1^\top, \ldots, u_{p_r} u_{p_r}^\top\right\}$$

where $\boldsymbol{r} \in \mathbb{N}$. In particular:

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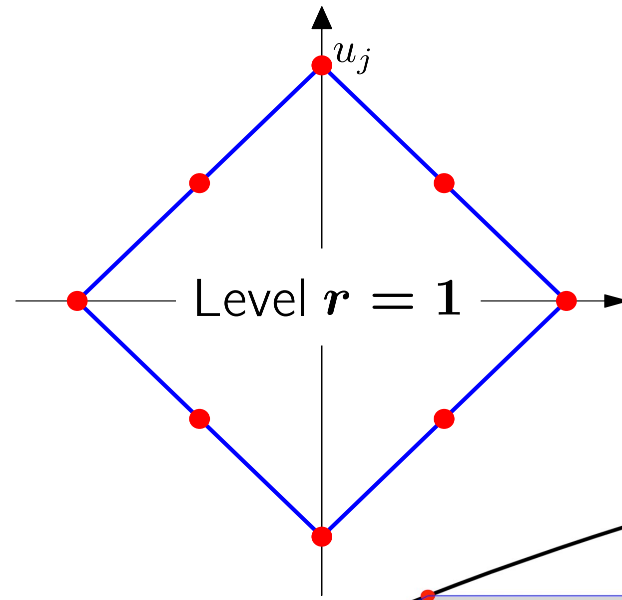

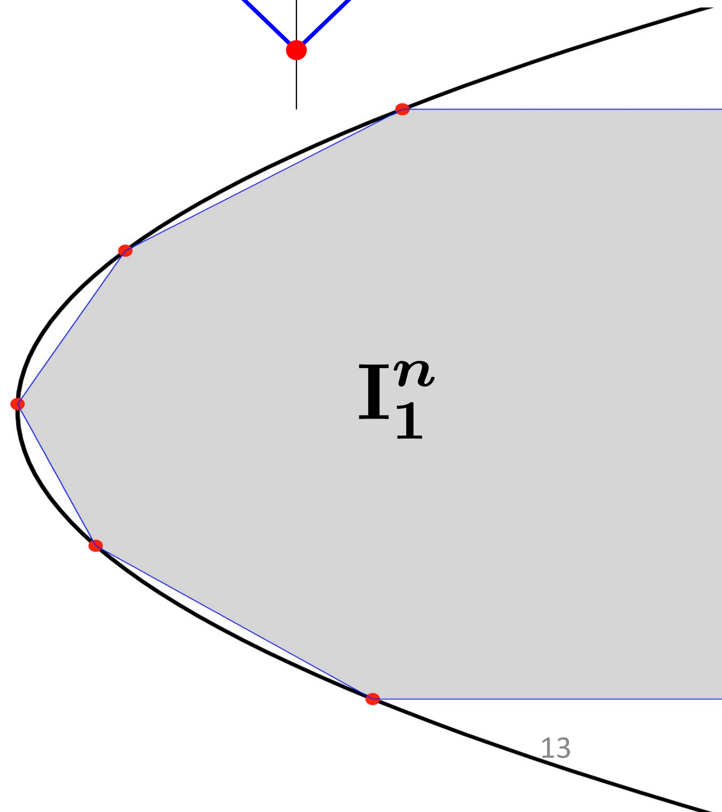

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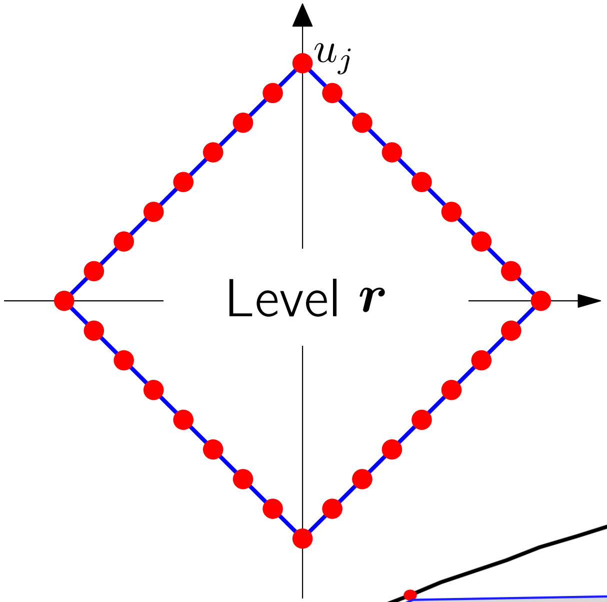
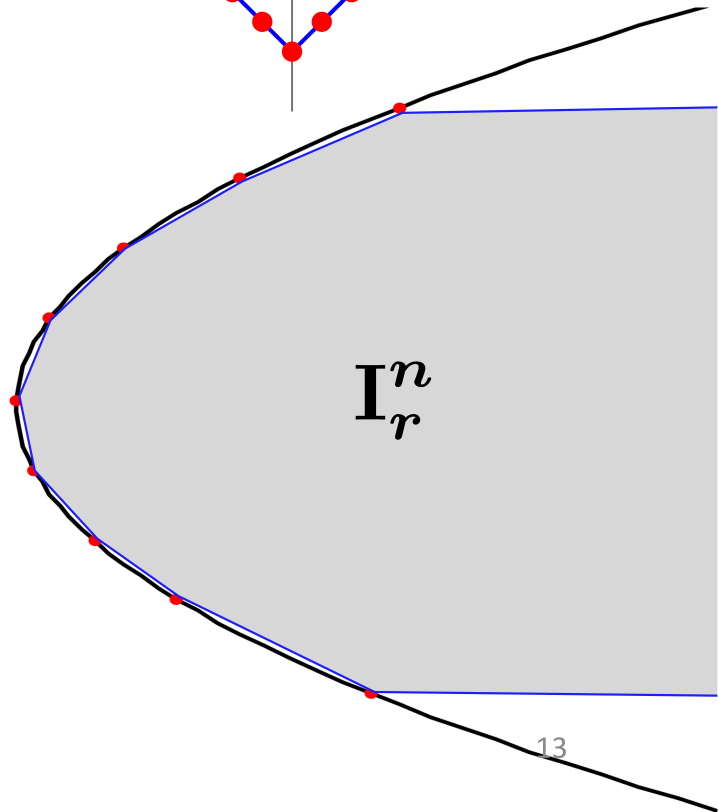
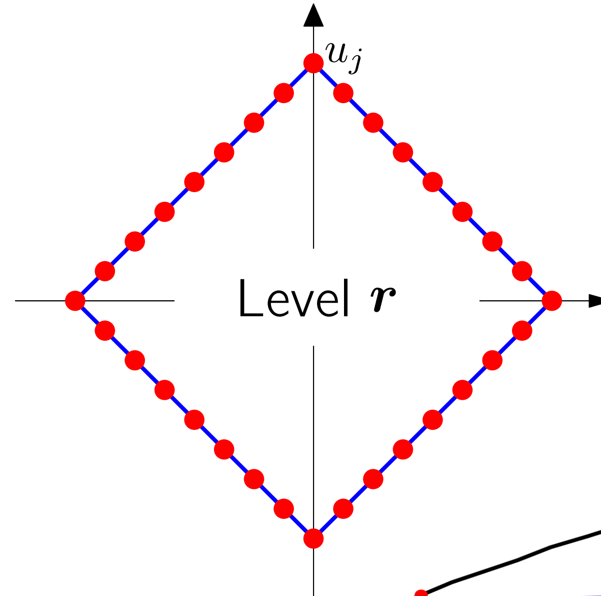




Inner Polyhedral Hierarchies of $\mathbf{S}_+^n$

The dual cones to $\mathbf{O}_r^n$ give a **hierarch of** $\mathbf{S}_+^n$ **inner polyhedral cones** to $\mathbf{S}_+^n$



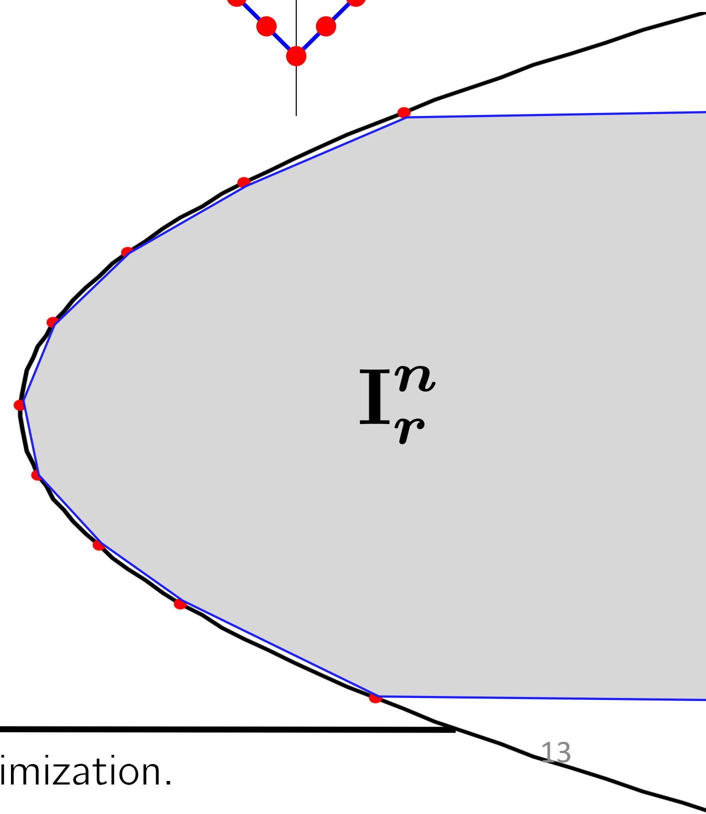
$$\mathbf{I}_r^n = (\mathbf{O}_r^n)^* = \mathrm{cone}\left\{u_1 u_1^\top, \ldots, u_{p_r} u_{p_r}^\top\right\}$$

where $r \in \mathbb{N}$. In particular:

$$\mathbf{I}_0^n \subseteq \mathbf{I}_1^n \subseteq \mathbf{I}_2^n \subseteq \cdots \subseteq \mathbf{S}_+^n$$

Examples

- $\mathbf{I}_0^n$: cone of nonnegative diagonal matrices
- $\mathbf{I}_1^n$: cone of diagonally dominant matrices with nonnegative diagonal entries.



Ahmadi et al.'16 : Application of $\mathbf{I}_1^n$ to Sums of Squares Optimization.

Polyhedral Hierarchies of the PSD Cone

Theorem: For each level $\boldsymbol{r} \in \mathbb{N}$,

1. $\mathbf{O}_{\boldsymbol{r}}^{\boldsymbol{n}} \supseteq \mathbf{O}_{\boldsymbol{r}+1}^{\boldsymbol{n}} \supseteq \mathbf{S}_{+}^{\boldsymbol{n}}$ and

$$\bigcap_{\boldsymbol{i} \in \mathbb{N}} \mathbf{O}_{\boldsymbol{i}}^{\boldsymbol{n}} = \mathbf{S}_{+}^{\boldsymbol{n}}$$

2. $\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}} \subseteq \mathbf{I}_{\boldsymbol{r+1}}^{\boldsymbol{n}} \subseteq \mathbf{S}_{+}^{\boldsymbol{n}}$ and

$$\mathrm{cl}\left(\bigcup_{\boldsymbol{i} \in \mathbb{N}} \mathbf{I}_{\boldsymbol{i}}^{\boldsymbol{n}}\right) = \mathbf{S}_{+}^{\boldsymbol{n}}$$

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$$\bigcap_{\boldsymbol{i} \in \mathbb{N}} \mathbf{O}^{\boldsymbol{n}}_{\boldsymbol{i}} = \mathbf{S}^{\boldsymbol{n}}_{+}$$

2. $\mathbf{I}^{\boldsymbol{n}}_{\boldsymbol{r}} \subseteq \mathbf{I}^{\boldsymbol{n}}_{\boldsymbol{r}+\mathbf{1}} \subseteq \mathbf{S}^{\boldsymbol{n}}_{+}$ and

$$\mathrm{cl}\left(\bigcup_{\boldsymbol{i} \in \mathbb{N}} \mathbf{I}^{\boldsymbol{n}}_{\boldsymbol{i}}\right) = \mathbf{S}^{\boldsymbol{n}}_{+}$$

[Braun, Fiorini, Pokutta, Steurer '12] "Approximation limits of linear programs (beyond hierarchies)."

"It's not possible to approximate SDPs arbitrarily well using small LPs"

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2. Inner and Outer Polyhedral Hierarchies of the PSD Cone

3. Inner and Outer Hierarchies of Robust SDPs

4. Application : Robust Resistance Network Design Problem

Outer Approximations to Robust SDP

Recall the outer and inner polyhedral cones approximating $\mathbf{S}_+^{\boldsymbol{n}}$

$$\mathbf{I}_{\mathbf{0}}^{\boldsymbol{n}} \subseteq \mathbf{I}_{\mathbf{1}}^{\boldsymbol{n}} \subseteq \cdots \subseteq \mathbf{S}_+^n \subseteq \cdots \subseteq \mathbf{O}_{\mathbf{1}}^{\boldsymbol{n}} \subseteq \mathbf{O}_{\mathbf{0}}^{\boldsymbol{n}}.$$

Outer Approximations to Robust SDP

Recall the outer and inner polyhedral cones approximating $\mathbf{S}_+^{\boldsymbol{n}}$

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For any $r \in \mathbb{N}$, the robust LP:

- $$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & \displaystyle\sum_{i=1}^{k} \xi_i \mathcal{A}_i(x) \;\in\; \overset{\mathbf{O}_{\boldsymbol{r}}^{\boldsymbol{n}}}{\cancel{\mathbf{S}_+^n}} \quad \forall\, \xi \in \Xi, \end{array}$$

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Finite-Dimensional Outer Approximation

The **hyperplane representation** of the outer polyhedral cones $\mathbf{O}_{\boldsymbol{r}}^{\boldsymbol{n}}$

$$\mathbf{O}_{\boldsymbol{r}}^{\boldsymbol{n}} = \bigcap_{i=1}^{p_r} \text{half-space}\ _i$$

and strong duality gives a finite-dimensional representation of the robust LP.

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$$\mathbf{O}_{\boldsymbol{r}}^{\boldsymbol{n}} = \bigcap_{j=1}^{p_r} \{X \in \mathbf{S}^n \mid u_j^\top X u_j \geq 0,\ u_j \in \boldsymbol{\Delta}_{\boldsymbol{r}}\}$$

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Theorem : The robust LP over $\mathbf{O}_{\boldsymbol{r}}^{\boldsymbol{n}}$ admits an equivalent reformulation as a finite-dimensional conic linear program:

$$\begin{array}{lll} \text{minimize} & c^\top x & \\ \text{subject to} & x \in \mathbf{R}^{\boldsymbol{m}}, \mu_j \in \mathbf{R}_+, \lambda_j \in \mathbf{K}^*, & \forall\ j = 1, \ldots, p_r \\ & u_j^\top \mathcal{A}_i(x) u_j = \mu_j + e_i^\top B^\top \lambda_j, & \begin{array}{l} \forall\ i = 1, \ldots, k \\ \forall\ j = 1, \ldots, p_r \end{array} \end{array}$$

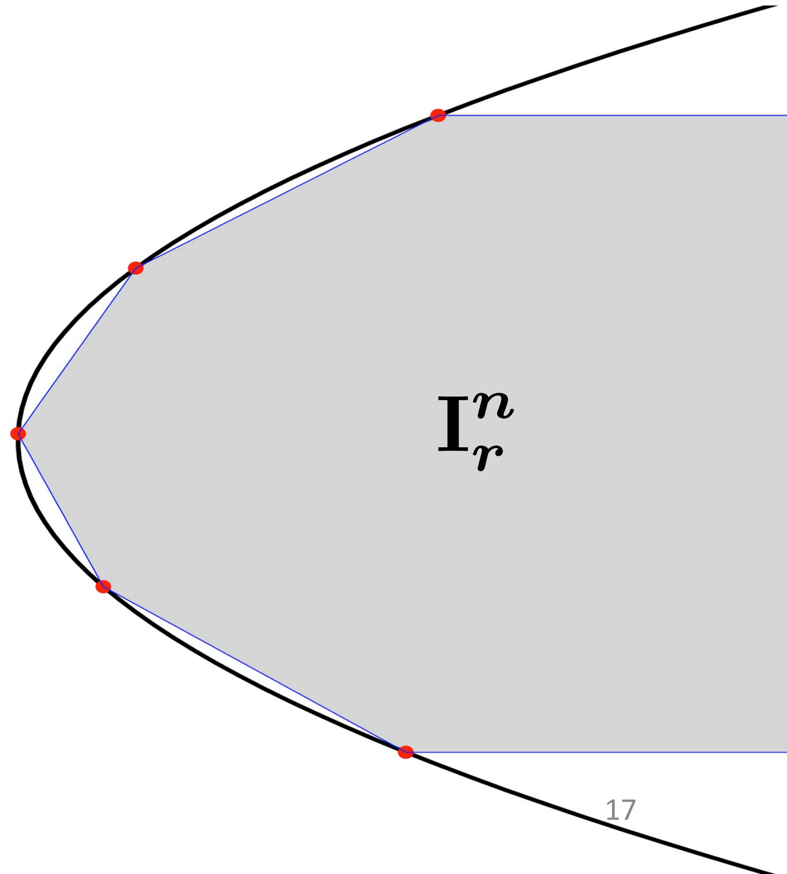
Its optimal value is a **lower bound** to the optimal value of the robust SDP.

A Challenge with the Inner Approximation

The **vertex representation** of the inner polyhedral cone $\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}} = (\mathbf{O}_{\boldsymbol{r}}^{\boldsymbol{n}})^*$

$$\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}} = \mathrm{cone}\left\{u_1 u_1^\top, \ldots, u_{p_r} u_{p_r}^\top\right\}$$

precludes a direct finite-dim. reformulation for the robust LP over $\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}}$



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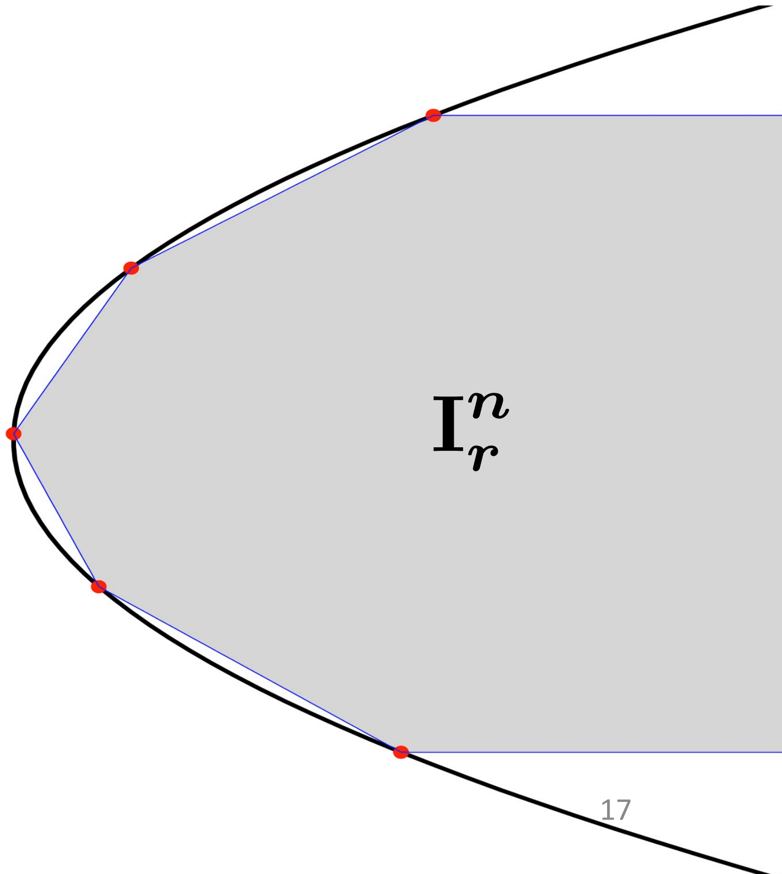

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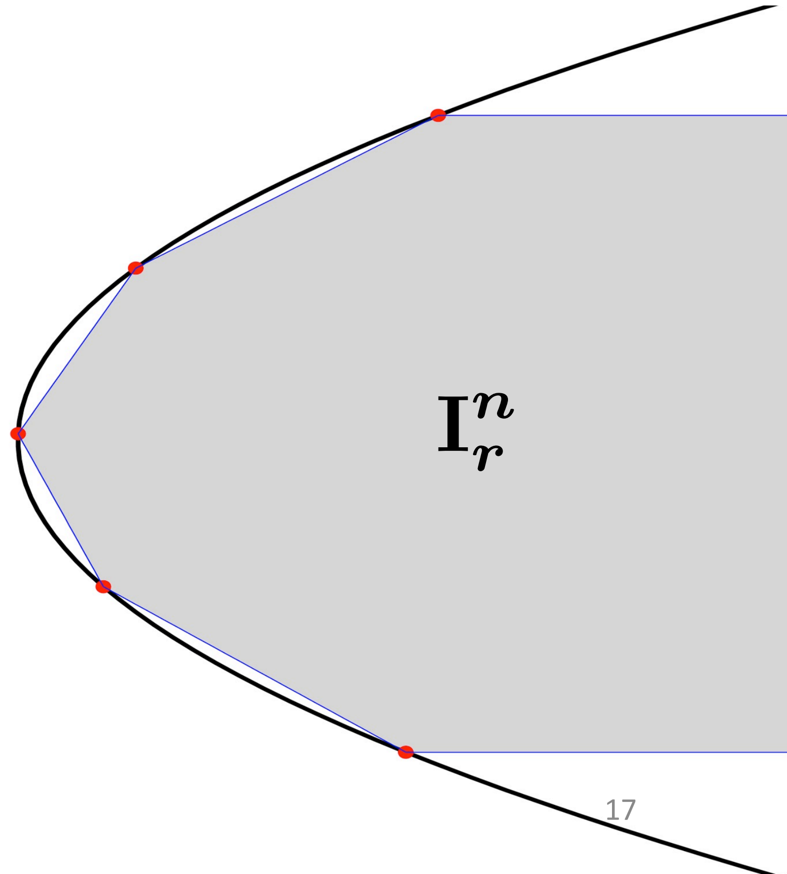

Finite-Dimensional Outer Approximation

Let

$$\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}} = \bigcap_{j=1}^{\boldsymbol{q}_r} \{X \in \mathbf{S}^n \mid \mathrm{tr}(H_i X) \geq 0\}$$

be the hyperplane representation of the inner polyhedral cone $\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}}$

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Theorem: The robust LP over $\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}}$ admits an equivalent reformulation as a finite-dimensional conic linear program:

$$\begin{array}{lll} \text{minimize} & c^\top x & \\ \text{subject to} & x \in \mathbf{R}^{\boldsymbol{m}},\ \mu_j \in \mathbf{R}_+,\ \lambda_j \in \mathbf{K}^*, & \forall\ j = 1, \ldots, q_r \\ & \mathrm{tr}(\mathcal{A}_i(x) H_j) = \mu_j + e_i^\top B^\top \lambda_j, & \forall\ i = 1, \ldots, k \\ & & \forall\ j = 1, \ldots, q_r \end{array}$$

Its optimal value is an **upper bound** to the optimal value of the robust SDP.

Inner Approximation to Robust LP over $\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}}$

A hyperplane representation of $\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}}$ can be

- **Computationally expensive** to compute
- **Impractical** : the number, q_r of hyperplanes can be rather large

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Question : Can we work directly with the vertex representation of $\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}}$?

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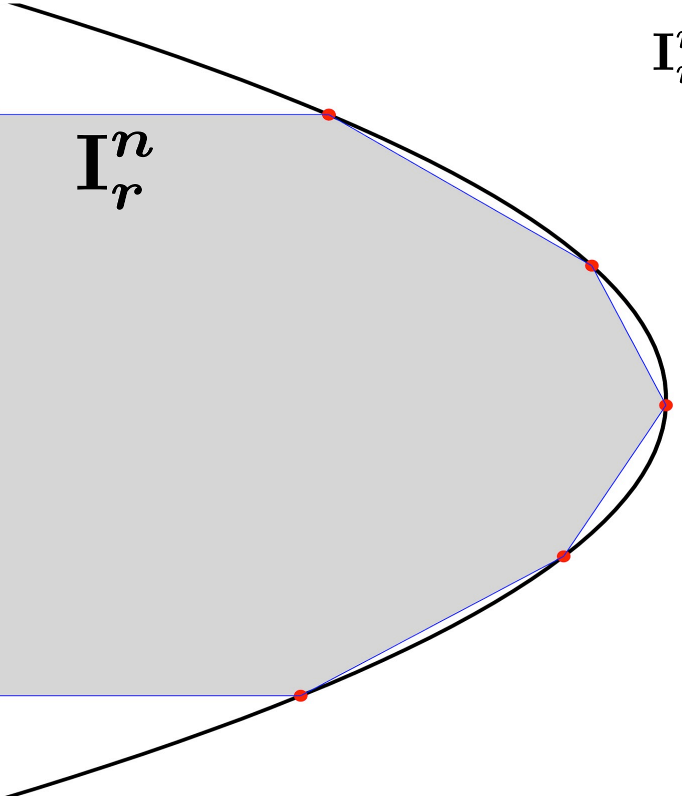

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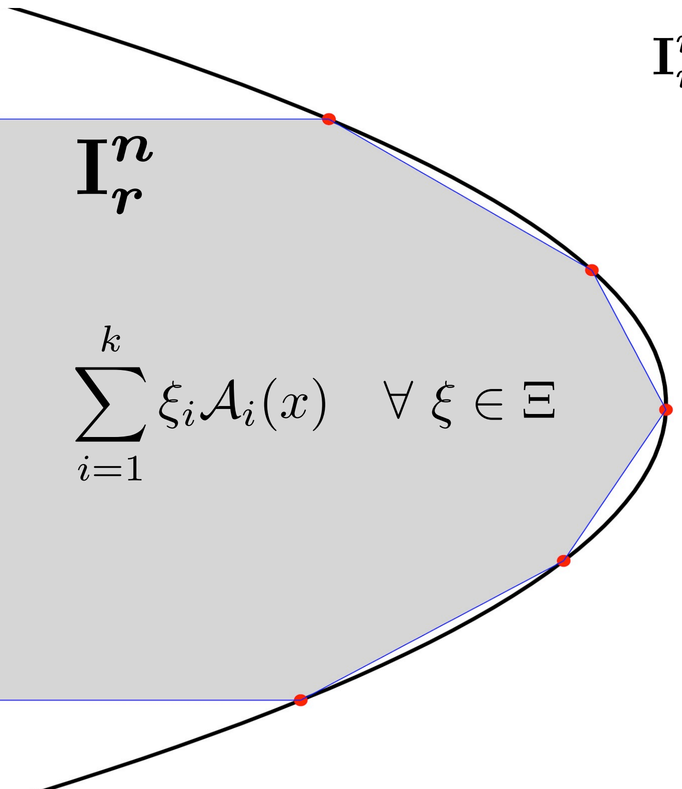

$$\sum_{i=1}^{k} \xi_i \mathcal{A}_i(x) \quad \forall\, \xi \in \Xi \quad \Longleftrightarrow$$

$\exists\ \boldsymbol{\phi_1, \ldots, \phi_{p_r}} : \mathbf{R}^{\boldsymbol{k}} \to \mathbf{R}_+$, such that

$$\sum_{i=1}^{k} \xi_i \mathcal{A}_i(x) = \sum_{j=1}^{p_r} \boldsymbol{\phi_j}(\xi) u_j u_j^\top, \quad \forall\, \xi \in \Xi.$$

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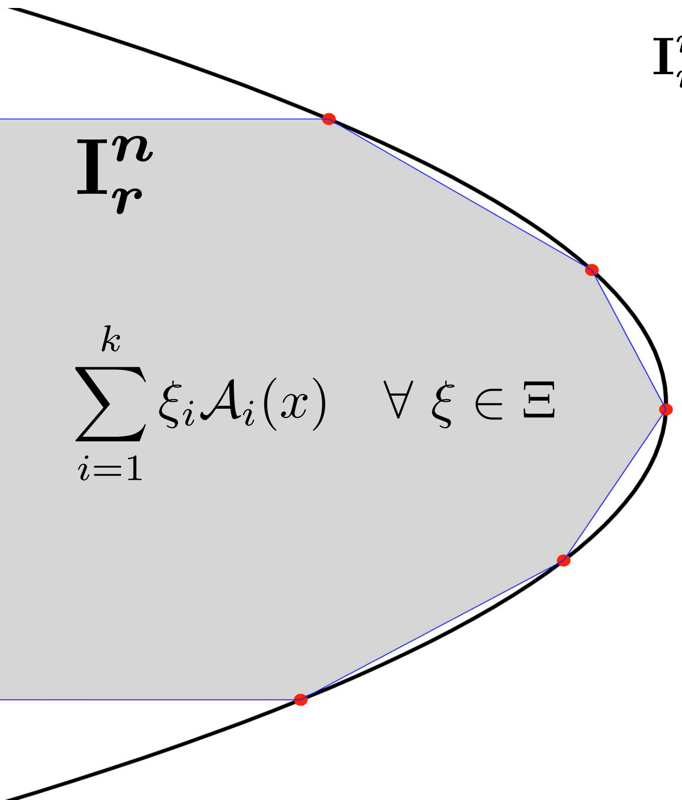

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$$\sum_{i=1}^{k} \xi_i \mathcal{A}_i(x) = \sum_{j=1}^{p_r} \overset{\varphi_j^\top \xi,\quad \varphi_j \in \mathbf{R}^k}{\cancel{\phi_j(\xi)}}\, u_j u_j^\top, \quad \forall\, \xi \in \Xi.$$

Restriction to affine functions yields a finite-dim. inner approximation to robust LP

Inner Approximation to Robust LP over $\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}}$

Let

$$\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}} = \text{cone}\left\{u_1 u_1^\top, \ldots, u_{p_r} u_{p_r}^\top\right\}$$

be the vertex representation of the inner polyhedral cone $\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}}$

Theorem: The robust LP over $\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}}$ admits an finite-dimensional inner approximation as a conic linear program:

$$\begin{array}{lll} \text{minimize} & c^\top x & \\ \text{subject to} & x \in \mathbf{R}^{\boldsymbol{m}},\ \mu_j \in \mathbf{R}_+,\ \lambda_j \in \mathbf{K}^*, & \forall\ j = 1, \ldots, p_r \\ & \mathcal{A}_i(x) = \displaystyle\sum_{j=1}^{\boldsymbol{p_r}} e_i^\top (\mu_j e_1 + B^\top \lambda_j) u_j u_j^\top, & \forall\ i = 1, \ldots, k \end{array}$$

Its optimal value is an **upper bound** to the optimal value of the robust SDP.

Talk Outline

1. Introduction

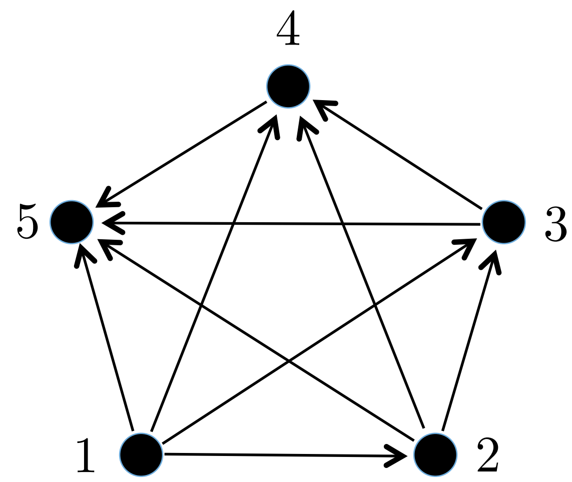
2. Inner and Outer Polyhedral Hierarchies of the PSD Cone

3. Inner and Outer Hierarchies of Robust SDPs

4. Application : Robust Resistance Network Design Problem

Robust Resistance Network Design Problem

Given a circuit topology and a set $\mathcal{I} = \{Q\xi \mid \xi \in \Xi\}$ of external currents



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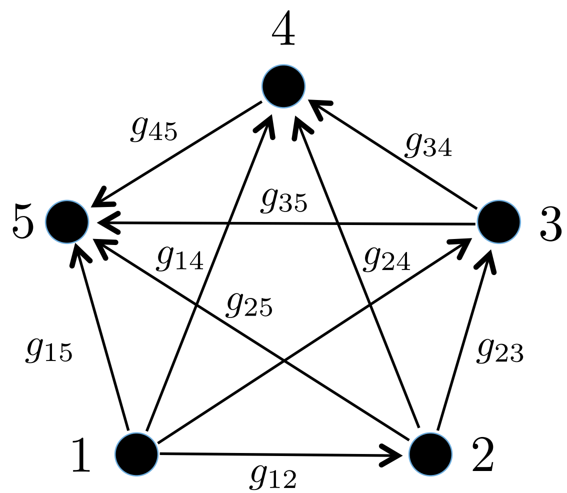
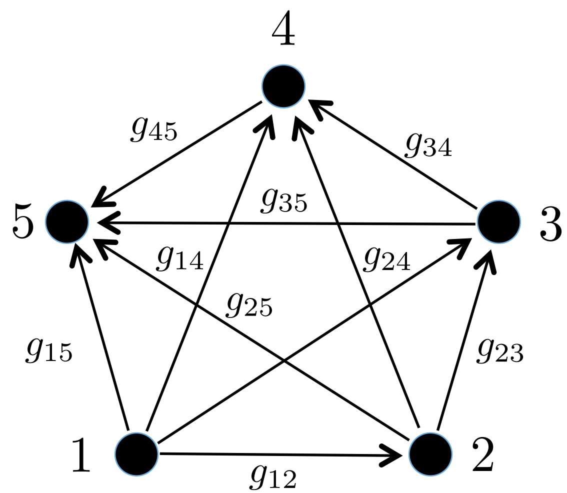


Objective: Choose a conductance g_{ij} for each line (i, j) such that:

$$
\begin{array}{lll}
\underset{g}{\text{minimize}} & \text{maximal dissipation over } \mathcal{I} & \\
\text{subject to} & \mathbf{1}^\top g \leq b & \text{budget constraint} \\
 & g \geq 0 & \text{physical constraints}
\end{array}
$$

Robust Resistance Network Design Problem

Given a circuit topology and a set $\mathcal{I} = \{Q\xi \mid \xi \in \Xi\}$ of external currents



Objective: Choose a conductance g_{ij} for each line (i, j) such that:

$$
\begin{array}{ll}
\underset{(\tau, g)}{\text{minimize}} & \tau \\
\text{subject to} & \mathbf{1}^\top g \leq \omega \\
& g \geq 0 \\
& \begin{bmatrix} \tau & Q\xi \\ Q\xi & \underbrace{M}_{\text{incidence matrix}} \text{diag}(g) M^\top \end{bmatrix} \succeq 0, \quad \forall\, \xi \in \Xi
\end{array}
$$

Unstructured Normed-Bounded Uncertainty

$$\Xi = \{\xi \in \mathbf{R}^6 \mid \|\xi\|_2 \leq 2,\ \xi_1 = 1\}$$

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Robust LP Hierarchies

	Level r in Hierarchy				
	0	**1**	**2**	**3**	**4**
Lower Bound ($\mathbf{O}_r^n$)					
Upper Bound I ($\mathbf{I}_r^n$)					
Upper Bound II ($\mathbf{I}_r^n$)					

Comparisons :

- **Ben-Tal et. al ['00]** – (Optimal Value to Robust SDP)

  2.37

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Lower Bound ($\mathbf{O}_r^n$)	0				
Upper Bound I ($\mathbf{I}_r^n$)	∞				
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Robust LP Hierarchies

	Level r in Hierarchy				
	0	**1**	**2**	**3**	**4**
Lower Bound ($\mathbf{O}_r^n$)	0	0			
Upper Bound I ($\mathbf{I}_r^n$)	∞	4.75			
Upper Bound II ($\mathbf{I}_r^n$)	∞	6.72			

Comparisons :

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Robust LP Hierarchies

	Level r in Hierarchy				
	0	**1**	**2**	**3**	**4**
Lower Bound $(\mathbf{O}_{\boldsymbol{r}}^{\boldsymbol{n}})$	0	0	2.25		
Upper Bound I $(\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}})$	∞	4.75	3.15		
Upper Bound II $(\mathbf{I}_{\boldsymbol{r}}^{\boldsymbol{n}})$	∞	6.72	4.94		

Comparisons :

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2.37

Unstructured Normed-Bounded Uncertainty

$$\Xi = \{\xi \in \mathbf{R}^6 \mid \|\xi\|_2 \leq 2,\ \xi_1 = 1\}$$

Robust LP Hierarchies

	Level r in Hierarchy				
	0	**1**	**2**	**3**	**4**
Lower Bound $(\mathbf{O}_r^n)$	0	0	2.25	2.34	2.36
Upper Bound I $(\mathbf{I}_r^n)$	∞	4.75	3.15	**comp. expensive**	
Upper Bound II $(\mathbf{I}_r^n)$	∞	6.72	4.94	4.56	4.55

Comparisons :

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2.37

Structured Normed-Bounded Uncertainty

$$\Xi = \{\xi \in \mathbf{R}^6 \mid \|(\xi_2, \xi_3)\|_2 \leq 1,\ \|(\xi_4, \xi_5, \xi_6)\|_2 \leq 1,\ \xi_1 = 1\}$$

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Robust LP Hierarchies

	Level r in Hierarchy				
	0	**1**	**2**	**3**	**4**
Lower Bound ($\mathbf{O}_r^n$)	0	1.65	3.66	4.19	4.24
Upper Bound I ($\mathbf{I}_r^n$)	∞	6.35	5.26	**comp. expensive**	
Upper Bound II ($\mathbf{I}_r^n$)	∞	8.02	6.80	6.61	6.51

Comparisons :

- **Ben-Tal et. al ['00]** – (Upper Bound to Robust SDP)

  ∞

- **Scherer, Hol ['06] –** (Upper Bound to Robust SDP)

  4.27

Polytopic Uncertainty

$$\Xi = \{\xi \in \mathbf{R}^6 \mid \|\xi\|_\infty \leq 1,\ L\xi \geq 0,\ \xi_1 = 1\}$$

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Robust LP Hierarchies

	Level r in Hierarchy				
	0	**1**	**2**	**3**	**4**
Lower Bound $(\mathbf{O}_r^n)$	0	3.40	8.17	8.17	8.17
Upper Bound I $(\mathbf{I}_r^n)$	∞	8.96	8.44	**comp. expensive**	
Upper Bound II $(\mathbf{I}_r^n)$	∞	8.96	8.44	8.34	8.26

Comparisons :

- **Nemirovski, El-Ghaoui ['00]** – (Not Applicable)
- **Scherer, Hol ['06] –** (Upper Bound to Robust SDP)

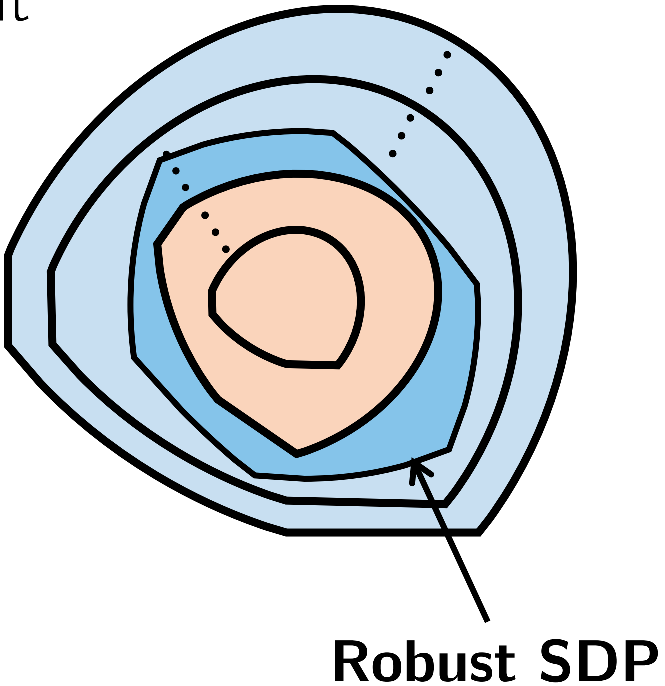
  8.22

Summary and Future Research

- Developed computationally tractable **inner and outer hierarchies to robust SDPs** that are exact in the limit

- **Approach**: Developed inner and outer polyhedral hierarchies to $\mathbf{S}_+^n$

- **Challenges:** Impractical for moderate levels in the hierarchy!



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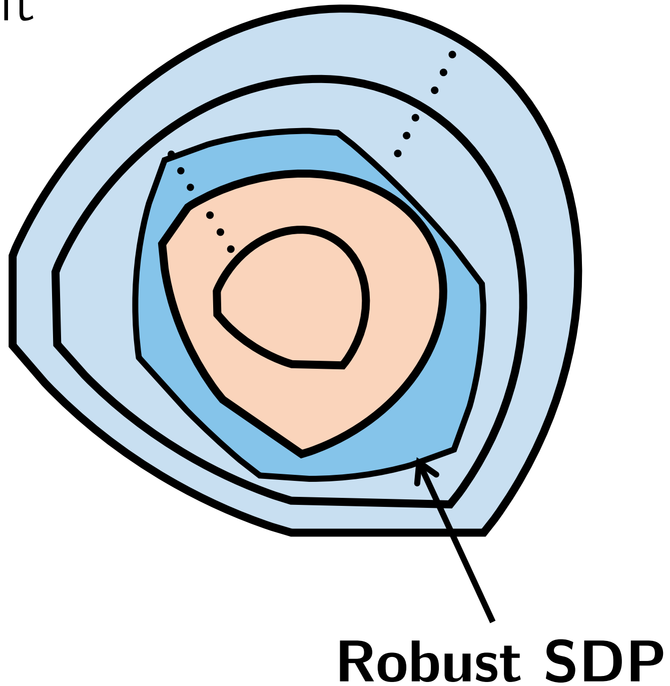

Future Research

- Adaptively improve the polyhedral approx. of $\mathbf{S}_+^n$ by using the guidance of the objective function!

Questions?

Thank you!

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