

MULTISTEP METHODS FOR COUPLED SECOND ORDER INTEGRO-DIFFERENTIAL EQUATIONS: STABILITY, CONVERGENCE AND ERROR BOUNDS

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ABSTRACT. In this paper multistep methods for systems of coupled second order Volterra integro-differential equations are proposed. Stability and convergence properties are studied and an error bound for the discretization error is given.

KEY WORDS AND PHRASES: Multistep methods, Convergence, Stability, Error bounds.

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1. INTRODUCTION

Systems of coupled second order integral equations and integro-differential equations have been used to model problems from a number of application areas including heat transfer solids and gases, superfluidity theory, mechanical systems, optics, physics of atoms, scattering theory, etc. A few references are included [4], [5], [7], [10], [16], [19]. Such systems also appear using semidiscretization techniques for solving scalar partial integro-differential equations [6], [18], [21]. Second order integro-differential systems can be transformed into an extended system of first order integro-differential equations, [14, p. 188]. Collocation methods for second order Volterra integro-differential equations are proposed in [1]. However, there are still advantages in studying methods for particular classes of second order systems of integro-differential equations for several reasons:

- (a) the transformation of a second order system into an extended first order system increases the computational cost,
- (b) the physical meaning of the original magnitudes is lost with the transformation of the system,
- (c) by requiring less generality we may be able to produce more efficient algorithms,
- (d) useful concepts may be identified, leading to a better understanding of what we require of a numerical method for problems in our chosen class

In this paper we consider multistep methods for matrix coefficients for systems of coupled second order Volterra integro-differential equations of the form

$$Y''(x) = F(x, Y(x), Z(x)), \quad 0 \leq x \leq a, \quad (1.1)$$

$$Z(x) = \int_0^x K(x, t, Y(t)) dt, \quad Y(0) = \Omega_0, \quad Y'(0) = \Omega_1 \quad (1.2)$$

which is to be solved for $Y(x)$ in $0 \leq x \leq a$, where $F : [0, a] \times \mathbb{R}^r \times \mathbb{R}^r \rightarrow \mathbb{R}^r$, $K : [0, a] \times [0, a] \times \mathbb{R}^r \rightarrow \mathbb{R}^r$ are uniformly continuous in all variables and satisfy the following Lipschitz conditions:

$$\|F(x, y_1, z) - F(x, y_2, z)\| \leq L_1 \|y_1 - y_2\| \tag{1.3}$$

$$\|F(x, y, z_1) - F(x, y, z_2)\| \leq L_2 \|z_1 - z_2\| \tag{1.4}$$

$$\|K(x, t, y_1) - K(x, t, y_2)\| \leq L_3 \|y_1 - y_2\| \tag{1.5}$$

Under these hypotheses the problem (1.1)-(1.2) has a unique solution in $[0, a]$, [14, chapter 11]

The aim of this paper is to provide error bounds for coupled integro-differential systems using a matrix approach that avoids the increase of the computational cost and preserves the meaning of the original magnitudes of the problem.

This paper is organized as follows. In section 2 we introduce the concept of a linear multistep matrix method for the numerical solution of problem (1.1)-(1.2). Consistency and the concept of zero-stability intrinsically related to the method, and not expressed in terms of its behavior with respect to any test equation are also defined in section 2. In section 3 we provide error bounds for the introduced multistep matrix methods and it is proven that consistent and zero-stable methods are convergent.

If A is a matrix with complex entries, element of $\mathbb{C}^{r \times r}$, we denote by $\|A\|$ its 2-norm, defined in [8, p. 15]. The set of all eigenvalues of A is denoted by $\sigma(A)$ and the spectral radius of A , denoted by $\rho(A)$ is the maximum of the set $\{|z|; z \in \sigma(A)\}$. In accordance with the definition given in [12], we say that a matrix $A \in \mathbb{C}^{r \times r}$ is of class N if for every eigenvalue $z \in \sigma(A)$ such that $|z| = \rho(A)$ the corresponding Jordan blocks of A associated with z have size 1×1 or 2×2 .

2. MULTISTEP MATRIX METHODS

A way to solve (1.1)-(1.2) numerically consists in the application of linear multistep methods for ordinary differential equations to equation (1.1) and in the approximation of $Z(x)$ by a quadrature formula (see [3, p. 151]). To solve (1.1) we use linear multistep matrix methods recently introduced in [12]. Multistep methods with matrix coefficients have also been studied in [11], [13] to solve numerically first order matrix ordinary differential equations.

DEFINITION 2.1. A linear k -step matrix method for the Volterra integro-differential system (1.1)-(1.2) is a relationship of the form

$$Y_{n+k} + A_{k-1}Y_{n+k-1} + \dots + A_0Y_n = h^2\{B_kF_{n+k} + \dots + B_0F_n\}, \quad n \geq p \geq 0, \quad k \geq 2, \tag{2.1}$$

where $A_i \in \mathbb{C}^{r \times r}$ for $0 \leq i \leq k - 1$, $B_q \in \mathbb{C}^{r \times r}$ for $0 \leq q \leq k$, $h > 0$, $\|A_0\| + \|B_0\| > 0$,

$$F_n = F(x_n, Y_n, Z_n), \quad Z_n = h \sum_{i=0}^n w_{n,i} K(x_n, x_i, Y_i), \quad n \geq p \tag{2.2}$$

and $w_{n,i}$ is a real number for $0 \leq i \leq n$.

The method (2.1)-(2.2) is said to be consistent if

$$\left. \begin{aligned} A_0 + A_1 + \dots + A_{k-1} + I &= 0, \\ A_1 + 2A_2 + \dots + (k-1)A_{k-1} + kI &= 0, \\ 2A_2 + \dots + (k-1)(k-2)A_{k-1} + k(k-1)I &= 2(B_0 + \dots + B_k) \end{aligned} \right\} \tag{2.3}$$

and the weights $w_{n,i}$, are bounded for all n and $i \leq n$, $|w_{n,i}| < W$, and are such that

$$\int_0^x f(t)dt - h \sum_{i=0}^n w_{n,i} f(x_i) = \theta(h), \tag{2.4}$$

for any continuous function $f(x)$ where $\theta(h) \rightarrow 0$ as $h \rightarrow 0$, $n \rightarrow \infty$, $nh = x$

The method (2.1)-(2.2) is said to be zero-stable if the matrix

$$C = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & I \\ -A_0 & -A_1 & -A_2 & \dots & -A_{k-1} \end{bmatrix} \tag{2.5}$$

is of class N and $\rho(C) = 1$.

REMARK 1. The concept of zero-stability introduced here for multistep matrix methods extends the one of zero-stability for scalar multistep methods given in [2], [3]. To our knowledge the only discussion of the stability in the case of systems is given in [17]. However, in [17] Matthys uses the concept of A -stability, that is not intrinsically related to the method but, it depends on a particular test equation. As we show in the following, the concept of zero-stability given in Definition 2.1 permits us to obtain error bounds of consistent and zero-stable matrix methods for systems of Volterra integro-differential equations.

The next example provides a family of 3-step methods depending on a matrix parameter

EXAMPLE 1. Let A be a matrix in $\mathbb{C}^{r \times r}$ of class N such that

$$\rho(A) \leq 1 \quad \text{and} \quad A + I \quad \text{is invertible} \tag{2.6}$$

and let us consider the method defined by

$$Y_{n+3} + (A - 2I)Y_{n+2} + (I - 2A)Y_{n+1} + AY_n = h^2\{B_3F_{n+3} + B_2F_{n+2} + B_1F_{n+1} + B_0F_n\} \tag{2.7}$$

where matrices B_q for $0 \leq q \leq 3$ are matrices in $\mathbb{C}^{r \times r}$ such that

$$B_0 + B_1 + B_2 + B_3 = I + A. \tag{2.8}$$

F_m is defined by (2.2), where $\{w_{n,i}\}_{0 \leq i \leq n}$ is bounded and the condition (2.4) is satisfied. From Theorem 1 of [12] the method defined by (2.6)-(2.8) is zero-stable and consistent.

DEFINITION 2.2. The method (2.1)-(2.2) is said to be convergent if, for all initial value problem (1.1)-(1.2) subject to hypotheses (1.3)-(1.5), we have that

$$\lim_{\substack{h \rightarrow 0 \\ nh=x}} Y_n = Y(x)$$

holds for all $x \in [0, a]$, and for all solutions $\{Y_n\}$ of the difference system (2.1) satisfying starting conditions $Y_s = \Omega_s(h)$ for which

$$\|Y_s - Y(sh)\| \leq h\delta, \quad 0 \leq s \leq p + k, \tag{2.9}$$

for some positive number δ .

For the sake of clarity we state a result whose proof is given in [12]

THEOREM 1. [12] Let $A_j \in \mathbb{C}^{r \times r}$ for $0 \leq j \leq k - 1$, $k \geq 2$, and let us suppose that matrix C defined by (2.5) is of class N and $\rho(C) = 1$. Let the matrix coefficients $\gamma_n \in \mathbb{C}^{m \times m}$ be defined by

$$[I + A_{k-1}z + \dots + A_0z^{k-1}]^{-1} = \sum_{n \geq 0} \gamma_n z^n, \quad |z| < 1.$$

Then there exist two positive constants Γ and γ such that

$$\|\gamma_n\| \leq n\Gamma + \gamma, \quad n = 0, 1, 2, \dots \tag{2.10}$$

$$\gamma_m + \gamma_{m-1}A_{k-1} + \dots + \gamma_{m-k}A_0 = \begin{cases} I, & m = 0 \\ 0, & m > 0 \end{cases} \tag{2.11}$$

where it is assumed that $\gamma_m = 0$ for $m < 0$

We conclude this section with a result that will be used in the next section to study the discretization error of methods of the type (2.1)-(2.2)

THEOREM 2. Let us consider the difference equation

$$Z_{m+k} + A_{k-1}Z_{m+k-1} + \dots + A_0Z_m = h^2\{B_{k,m}\|Z_{m+k}\| + \dots + B_{0,m}\|Z_m\|\} + h^3\left\{C_{k,m}\sum_{i=0}^{m+k}\|Z_i\| + \dots + C_{0,m}\sum_{i=0}^m\|Z_i\|\right\} + \Lambda_m, \quad m \geq p \tag{2.12}$$

where $A_i \in \mathbb{C}^{r \times r}$ for $0 \leq i \leq k-1$, $C_{j,m}, B_{j,m} \in \mathbb{C}^r$ for $0 \leq j \leq k$, $\Lambda_m \in \mathbb{C}^r$ and $h > 0$ with $Nh = b$, N integer. Let us assume that method (2.1)-(2.2) is zero-stable and let B, C , and Λ be positive constants such that

$$\|B_{j,j}\| \leq B, \quad \|C_{j,m}\| \leq C, \quad \|\Lambda_m\| \leq \Lambda, \quad p \leq m \leq N. \tag{2.13}$$

If $\{Z_m\}$ is a solution of (2.12) such that

$$\|Z_m\| \leq Z, \quad p \leq m \leq N \tag{2.14}$$

and

$$B_* = (k+1)B, \quad C_* = (k+1)C, \quad h < [(N\Gamma + \gamma)(B_* + bC_*)]^{-1/2}, \tag{2.15}$$

then

$$\|Z_m\| \leq K_* \exp(mh^2L_*), \quad N \geq m \geq p \tag{2.16}$$

where

$$K_* = \frac{(N\Gamma + \gamma)(N\Lambda + AZk)}{1 - h^2(N\Gamma + \gamma)(B_* + bC_*)} = \frac{1}{h^2} \frac{(b\Gamma + h\gamma)(b\Lambda + AZhk)}{1 - h(b\Gamma + h\gamma)(B_* + bC_*)} \tag{2.17}$$

$$L_* = \frac{(N\Gamma + \gamma)(B_* + bC_*)}{1 - h^2(N\Gamma + \gamma)(B_* + bC_*)} = \frac{h}{h^2} \frac{(b\Gamma + h\gamma)(B_* + bC_*)}{1 - h(b\Gamma + h\gamma)(B_* + bC_*)} \tag{2.18}$$

$$A = \|A_0\| + \|A_1\| + \dots + \|A_{k-1}\| + 1, \tag{2.19}$$

and Γ, γ are defined by Theorem 1.

PROOF. Let us write equation (2.12) for $m = n - k, n - k - 1, \dots, p$ and let us premultiply the resulting equation by $\gamma_0, \gamma_1, \dots, \gamma_{n-k-p}$, respectively, obtaining

$$\begin{aligned} \gamma_0 Z_n + \gamma_0 A_{k-1} Z_{n-1} + \dots + \gamma_0 A_0 Z_{n-k} &= h^2 \gamma_0 \{B_{k,n-k} \|Z_n\| + \dots + B_{0,n-k} \|Z_{n-k}\|\} \\ &+ h^3 \gamma_0 \left\{ C_{k,n-k} \sum_{i=0}^n \|Z_i\| + \dots + C_{0,n-k} \sum_{i=0}^{n-k} \|Z_i\| \right\} + \gamma_0 \Lambda_{n-k} \\ \gamma_1 Z_{n-1} + \gamma_1 A_{k-1} Z_{n-2} + \dots + \gamma_1 A_0 Z_{n-k-1} &= h^2 \gamma_1 \{B_{k,n-k-1} \|Z_{n-1}\| + \dots + B_{0,n-k-1} \|Z_{n-k-1}\|\} \\ &+ h^3 \gamma_1 \left\{ C_{k,n-k-1} \sum_{i=0}^{n-1} \|Z_i\| + \dots + C_{0,n-k-1} \sum_{i=0}^{n-k-1} \|Z_i\| \right\} + \gamma_1 \Lambda_{n-k-1} \end{aligned} \tag{2.20}$$

$$\begin{aligned} \dots \dots \dots \\ \gamma_{n-k-p} Z_{p+k} + \gamma_{n-k-p} A_{k-1} Z_{p+k-1} + \dots + \gamma_{n-k-p} A_0 Z_p &= h^2 \gamma_{n-k-p} \{B_{k,p} \|Z_{p+k}\| + \dots + B_{0,p} \|Z_p\|\} \\ &+ h^3 \gamma_{n-k-p} \left\{ C_{k,p} \sum_{i=0}^{p+k} \|Z_i\| + \dots + C_{0,p} \sum_{i=0}^p \|Z_i\| \right\} + \gamma_{n-k-p} \Lambda_p. \end{aligned}$$

Adding the left hand side of the above equations (2.20) one gets

$$\begin{aligned}
 S_n = & \gamma_0 Z_n + (\gamma_0 A_{k-1} + \gamma_1) Z_{n-1} + (\gamma_0 A_{k-2} + \gamma_1 A_{k-1} + \gamma_2) Z_{n-2} + \dots \\
 & + (\gamma_0 A_0 + \gamma_1 A_1 + \dots + \gamma_{k-1} A_{k-1} + \gamma_k) Z_{n-k} \\
 & + (\gamma_1 A_0 + \dots + \gamma_{k+1}) Z_{n-k-1} + \dots + (\gamma_{n-2k-p} A_0 + \dots + \gamma_{n-k-p}) Z_{p+k} \\
 & + (\gamma_{n-k-p} A_{k-1} + \dots + \gamma_{n-2k-p+1} A_0) Z_{p+k-1} + \dots + \gamma_{n-k-p} A_0 Z_p .
 \end{aligned}$$

Taking into account (2.11) we have

$$S_n = Z_n + (\gamma_{n-k-p} A_{k-1} + \dots + \gamma_{n-2k-p+1} A_0) Z_{p+k-1} + \dots + \gamma_{n-k-p} A_0 Z_p \tag{2.21}$$

and adding the right hand side it follows that

$$\begin{aligned}
 S_n = & h^2 \{ \gamma_0 B_{k,n-k} \|Z_n\| + (\gamma_0 B_{k-1,n-k} + \gamma_1 B_{k,n-k-1}) \|Z_{n-1}\| + \dots \\
 & + (\gamma_0 B_{0,n-k} + \dots + \gamma_k B_{k,n-2k}) \|Z_{n-k}\| + \dots + \gamma_{n-k-p} B_{0,p} \|Z_p\| \} \\
 & + h^3 \left\{ \gamma_0 C_{k,n-k} \sum_{i=0}^n \|Z_i\| + (\gamma_0 C_{k-1,n-k} + \gamma_1 C_{k,n-k-1}) \sum_{i=0}^{n-1} \|Z_i\| + \dots \right. \\
 & + (\gamma_0 C_{0,n-k} + \dots + \gamma_k C_{k,n-2k}) \sum_{i=0}^{n-k} \|Z_i\| + \dots + \gamma_{n-k-p} C_{0,p} \sum_{i=0}^p \|Z_i\| \} \\
 & + \gamma_0 \Lambda_{n-k} + \dots + \gamma_{n-k-p} \Lambda_p .
 \end{aligned} \tag{2.22}$$

From (2.10) and (2.13) it follows that

$$\|\gamma_0 \Lambda_{n-k} + \dots + \gamma_{n-k-p} \Lambda_p\| \leq \Lambda \sum_{j=0}^{n-k} (j\Gamma + \gamma) \leq \Lambda(N\Gamma + \gamma)N . \tag{2.23}$$

Equating the right hand sides of (2.21) and (2.22) one gets

$$\begin{aligned}
 Z_n = & -(\gamma_{n-k-p} A_{k-1} + \dots + \gamma_{n-2k-p+1} A_0) Z_{p+k-1} - \dots - \gamma_{n-k-p} A_0 Z_p \\
 & + h^2 \{ \gamma_0 B_{k,n-k} \|Z_n\| + (\gamma_0 B_{k-1,n-k} + \gamma_1 B_{k,n-k-1}) \|Z_{n-1}\| + \dots \\
 & + (\gamma_0 B_{0,n-k} + \dots + \gamma_k B_{k,n-2k}) \|Z_{n-k}\| + \dots + \gamma_{n-k-p} B_{0,p} \|Z_p\| \} \\
 & + h^3 \left\{ \gamma_0 C_{k,n-k} \sum_{i=0}^n \|Z_i\| + (\gamma_0 C_{k-1,n-k} + \gamma_1 C_{k,n-k-1}) \sum_{i=0}^{n-1} \|Z_i\| + \dots \right. \\
 & + (\gamma_0 C_{0,n-k} + \dots + \gamma_k C_{k,n-2k}) \sum_{i=0}^{n-k} \|Z_i\| + \dots + \gamma_{n-k-p} C_{0,p} \sum_{i=0}^p \|Z_i\| \} \\
 & + \gamma_0 \Lambda_{n-k} + \dots + \gamma_{n-k-p} \Lambda_p .
 \end{aligned} \tag{2.24}$$

Taking into account that from Theorem 1, $\gamma_0 = I$, and from (2.10), (2.14), (2.15), (2.19), (2.21), (2.23) and (2.24) it follows that

$$\begin{aligned}
 \|Z_n\| \leq & h^2(N\Gamma + \gamma) B_* \sum_{i=p}^n \|Z_i\| + h^3(N\Gamma + \gamma) C_* \sum_{j=0}^{n-p} \sum_{i=0}^{p+j} \|Z_i\| \\
 & + N(N\Gamma + \gamma) \Lambda + kAZ(N\Gamma + \gamma) \\
 \leq & h^2(N\Gamma + \gamma) B_* \sum_{i=0}^n \|Z_i\| + h^3(N\Gamma + \gamma) C_* N \sum_{i=0}^n \|Z_i\| + N(N\Gamma + \gamma) \Lambda + kAZ(N\Gamma + \gamma) \\
 = & h^2(N\Gamma + \gamma) B_* \|Z_n\| + h^2(N\Gamma + \gamma) B_* \sum_{i=0}^{n-1} \|Z_i\| + h^3(N\Gamma + \gamma) C_* N \sum_{i=0}^{n-1} \|Z_i\| \\
 & + h^3(N\Gamma + \gamma) C_* N \|Z_n\| + N(N\Gamma + \gamma) \Lambda + kAZ(N\Gamma + \gamma) .
 \end{aligned}$$

From the last inequality and from (2.17)-(2.18) we can write

$$\|Z_n\| \leq h^2 L_* \sum_{i=0}^{n-1} \|Z_i\| + K_* . \tag{2.25}$$

Note that $A \geq 1$, and $N\Gamma + \gamma \geq 1$. Then from (2.17) we have that $K_* \geq Z \geq \|Z_0\|$. Thus for $m = 0$ one verifies

$$\|Z_m\| \leq K_*(1 + h^2 L_*)^m. \tag{2.26}$$

Let us assume that (2.26) holds for $m = 0, 1, \dots, n - 1$. Substituting (2.26) for $0 \leq m \leq n - 1$ into (2.25) it follows that

$$\begin{aligned} \|Z_n\| &\leq h^2 L_* \sum_{i=0}^{n-1} K_*(1 + h^2 L_*)^i + K_* = h^2 L_* K_* \frac{K_*(1 + h^2 L_*)^n - 1}{h^2 L_*} + K_* \\ &= K_*(1 + h^2 L_*)^n. \end{aligned}$$

Using the inequality $1 + h^2 L_* \leq \exp(h^2 L_*)$ from the last expression one gets

$$\|Z_n\| \leq K_* \exp(nh^2 L_*), \quad p \leq n \leq N.$$

Thus the result is established.

3. CONVERGENCE AND ERROR BOUNDS

The global truncation error of the method (2.1)-(2.2) is defined by

$$e_m = Y_m - Y(x_m), \quad x_m = mh, \tag{3.1}$$

where $Y(x_m)$ is the value of the theoretical solution $Y(x)$ of problem (1.1)-(1.2) at x_m , and Y_m is the solution of the difference equation (2.1).

Let us introduce the operator L_{nh} defined by

$$\begin{aligned} L_{nh} &= L[Y(x_n); h] = \\ &= Y(x_{n+k}) + A_{k-1}Y(x_{n+k-1}) + \dots + A_0Y(x_n) - h^2[B_k \bar{F}_{n+k} + \dots + B_0 \bar{F}_n] \end{aligned} \tag{3.2}$$

where

$$\bar{F}_n = F\left(x_n, Y(x_n), h \sum_{i=0}^n w_{n,i} K(x_n, x_i, Y(x_i))\right). \tag{3.3}$$

THEOREM 3. Let us suppose that the method (2.1)-(2.2) is consistent and let $L_{mh} = L[Y(x_m); h]$ be the corresponding operator defined by (3.2). Then

$$\|L[Y(x_m); h]\| \leq h^2(k + 1)\bar{B}L_2\|\theta(h)\| \tag{3.4}$$

where $\theta(h)$ is a C^r valued vector function such that $\theta(h) \rightarrow 0$ as $h \rightarrow 0$, and

$$\bar{B} = \max\{\|B_i\|; 0 \leq i \leq k\}. \tag{3.5}$$

PROOF. From (3.2)-(3.3) we have

$$\begin{aligned} L[Y(x_m); h] &= Y(x_{m+k}) + A_{k-1}Y(x_{m+k-1}) + \dots + A_0Y(x_m) - h^2[B_k \bar{F}_{m+k} + \dots + B_0 \bar{F}_m] \\ &= [Y(x_{m+k}) + \dots + A_0Y(x_m) - h^2 B_k Y''(x_{m+k}) - \dots - h^2 B_0 Y''(x_m)] \\ &\quad + h^2 \left[B_k F(x_{m+k}, Y(x_{m+k}), \int_0^{x_{m+k}} K(x_{m+k}, t, Y(t)) dt) + \dots \right. \\ &\quad \left. + B_0 F(x_m, Y(x_m), \int_0^{x_m} K(x_m, t, Y(t)) dt) - B_k \bar{F}_{m+k} - \dots - B_0 \bar{F}_m \right]. \end{aligned} \tag{3.6}$$

From expressions (3.12)-(3.13) of [10] and from the consistency conditions (2.3) it follows that expression (3.6) is of the form $h^2 \kappa(h)$ where $\kappa(h)$ is a vector function such that $\kappa(h) \rightarrow 0$ as $h \rightarrow 0$. From the consistency condition (2.4), the Lipschitz condition (1.4), and from (3.6) it follows that

$$\begin{aligned} \|L[Y(x_m); h]\| \leq h^2 \left\{ L_2 \|B_k\| \left\| \int_0^{x_{m+k}} K(x_{m+k}, t, Y(t)) dt - h \sum_{i=0}^{m+k} w_{m+k,i} K(x_{m+k}, x_i, Y(x_i)) \right\| \right. \\ \left. + \dots + L_2 \|B_0\| \left\| \int_0^{x_m} K(x_m, t, Y(t)) dt - h \sum_{i=0}^m w_{m,i} K(x_m, x_i, Y(x_i)) \right\| \right\} \\ \leq (k+1) \bar{B} L_2 h^2 \|\theta(h)\| \end{aligned} \tag{3.7}$$

where \bar{B} is defined by (3.5).

Thus the result is established.

If e_n is defined by (3.1), subtracting equation (3.2) from (2.1) it follows that

$$e_{n+k} + A_{k-1} e_{n+k-1} + \dots + A_0 e_n - h^2 \{ B_k (F_{n+k} - \bar{F}_{n+k}) + \dots + B_0 (F_n - \bar{F}_n) \} = -L_n h. \tag{3.8}$$

Let us introduce the vector sequences $\{G_n\}$, $\{g_n\}$, $\{d_n\}$, $\{H_n\}$ defined by

$$G_n = F \left(x_n, Y_n, h \sum_{i=0}^n w_{n,i} K(x_n, x_i, Y_i) \right) - F \left(x_n, Y(x_n), h \sum_{i=0}^n w_{n,i} K(x_n, x_i, Y_i) \right) \tag{3.9}$$

$$H_n = F \left(x_n, Y(x_n), h \sum_{i=0}^n w_{n,i} K(x_n, x_i, Y_i) \right) - F \left(x_n, Y(x_n), h \sum_{i=0}^n w_{n,i} K(x_n, x_i, Y(x_i)) \right) \tag{3.10}$$

$$g_n = \begin{cases} G_n \|e_n\|^{-1} & \text{if } e_n \neq 0 \\ 0 & \text{if } e_n = 0 \end{cases}, \quad d_n = \begin{cases} \left[h \sum_{i=0}^n \|e_i\| \right]^{-1} H_n & \text{if } \sum_{i=0}^n \|e_i\| > 0 \\ 0 & \text{if } \sum_{i=0}^n \|e_i\| = 0 \end{cases}. \tag{3.11}$$

From (1.3)-(1.5) and (3.11) it follows that

$$\|g_n\| \leq L_1 \quad \text{and} \quad \|d_n\| \leq L_2 L_3 W, \tag{3.12}$$

where $|w_{n,i}| \leq W$ for $0 \leq i \leq n$.

From (3.11), equation (3.8) can be written in the form

$$\begin{aligned} e_{n+k} + A_{k-1} e_{n+k-1} + \dots + A_0 e_n = h^2 \{ B_k g_{n+k} \|e_{n+k}\| + \dots + B_0 g_n \|e_n\| \} \\ + h^3 \left\{ B_k d_{n+k} \sum_{i=0}^{n+k} \|e_i\| + \dots + B_0 d_n \sum_{i=0}^n \|e_i\| \right\} - L_n h. \end{aligned} \tag{3.13}$$

From Theorem 3 we have $\|L_n h\| \leq h^2 (k+1) \bar{B} L_2 \|\theta(h)\|$, where \bar{B} is given by (3.5) and $\|\theta(h)\| \rightarrow 0$ as $h \rightarrow 0$. Taking into account this bound of $\|L_n h\|$ and by application of Theorem 2 to equation (3.13) it follows that

$$\|e_n\| \leq K_* \exp(h^2 x_n L_*)$$

where

$$B_* = (k+1) L_1 \bar{B}, \quad C_* = (k+1) \bar{B} L_2 L_3 W, \quad N = \frac{x_n}{h} \text{ integer} \tag{3.14}$$

$$Z = h\delta(h), \quad \delta(h) = \max\{\|Y_s - Y(sh)\|; 0 \leq s \leq p+k-1\} \tag{3.15}$$

$$\begin{aligned} K_* &= \frac{(N\Gamma + \gamma)(Nh^2(k+1)\bar{B}L_2\|\theta(h)\| + Akh\delta(h))}{1 - h^2(k+1)(N\Gamma + \gamma)\bar{B}(L_1 + aWL_2L_3)} \\ L_* &= \frac{(k+1)(N\Gamma + \gamma)\bar{B}(L_1 + aWL_2L_3)}{1 - h^2(k+1)(N\Gamma + \gamma)\bar{B}(L_1 + aWL_2L_3)} \end{aligned}$$

Taking into account that $N = \frac{x_n}{h}$ we can write

$$N\Gamma + \gamma = \Gamma h^{-1}x_n + \gamma,$$

$$L_* = \frac{(k+1)h^{-1}(\gamma h + \Gamma x_n)\overline{B}(L_1 + aWL_2L_3)\overline{B}(L_1 + aWL_2L_3)}{1 - h(k+1)(\gamma h + x_n\Gamma)\overline{B}(L_1 + aWL_2L_3)}, \tag{3 16}$$

$$nh^2L_* = \frac{x_n(k+1)\overline{B}(\gamma h + x_n\Gamma)(L_1 + aWL_2L_3)}{1 - h(k+1)(\gamma h + x_n\Gamma)(L_1 + aWL_2L_3)},$$

$$K_* = \frac{(\gamma h + x_n\Gamma)[x_n\overline{B}L_2\|\theta(h)\| + kA\delta(h)]}{1 - h(k+1)(\gamma h + x_n\Gamma)\overline{B}(L_1 + aWL_2L_3)}. \tag{3 17}$$

Hence the following result has been established.

THEOREM 4. Let us consider a consistent and stable method of the form (2.1)-(2.2) and let W be an upper bound of the weights $w_{n,i}$ appearing in (2.4). Let L_1, L_2 and L_3 be positive constants satisfying (1.3)-(1.5), let A and \overline{B} be defined by (2.19) and (3.5) respectively, and let $N = \frac{x_n}{h}$ integer such that

$$\gamma h^2 + a\Gamma h < [\overline{B}(k+1)L_1 + aWL_2L_3]^{-1}, \quad h > 0.$$

If K_* is defined by (3.17), where $\theta(h)$ satisfies (2.4), then the discretization error $e_n = Y(x_n) - Y_n$ at $x_n \in [0, a]$ satisfies

$$\|e_n\| \leq K_* \exp \left[\frac{x_n(k+1)\overline{B}(\gamma h + x_n\Gamma)(L_1 + aWL_2L_3)}{1 - h(k+1)(\gamma h + x_n\Gamma)(L_1 + aWL_2L_3)} \right], \tag{3 18}$$

where Γ and γ are defined by Theorem 1.

REMARK 2. A scalar version of the results of sections 2 and 3 are given in the recent Ph.D Thesis [20]. The starting values $Y_0, Y_1, \dots, Y_{k+p-1}$ of the method (2.1)-(2.2) can be obtained by transforming the problem (1.1)-(1.2) into the first order system

$$V = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}; \quad V' = \begin{bmatrix} Y_2 \\ F(x, Y_1(x), \int_0^x K(x, t, Y_1(t))dt) \end{bmatrix}; \quad V(0) = \begin{bmatrix} \Omega_0 \\ \Omega_1 \end{bmatrix}.$$

Then using Simpson's rule and quadratic interpolation like in section 3 of [15] for first order scalar Volterra integro-differential systems, starting values Y_0, \dots, Y_{k+p-1} satisfying condition (2.9) can be obtained

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REFERENCES

[1]AGUILAR, M. and BRUNNER, H., Collocation methods for second order Volterra integro-differential equations, *Appl. Numer. Math.*, 4 (1988), 455-470.
 [2]BRUNNER, H. and LAMBERT, J.D., Stability of numerical methods for Volterra integro-differential equations, *Computing* 12 (1974), 75-89.
 [3]BRUNNER, H. and VAN DER HOUWEN, P.J., *The Numerical Solution of Volterra Equations* (North-Holland, Amsterdam, 1986).
 [4]CHAWLA, M.M., SUBRANAURIAN, R. and SHIVAKUMAR, P.N., Numerov's method for nonlinear two-point boundary value problems II, Monotone approximations, *Intern. J. Computer Math.*, 26 (1989), 219-227.
 [5]FADDEYEV, L.D., The inverse problem in the quantum theory of scattering, *J. Math. Phys.* 4 No 1 (1963), 72-104

- [6]FORD, W T., Mathematical programming and integrodifferential equations, *SIAM J. Numer. Anal.* 2 (1965), 171-202.
- [7]GAREY, L.E. and GLADWIN, C.J., Numerical methods of second order Volterra integro-differential equations with two boundary value conditions, *Utilitas Mathematica* 35 (1989), 107-113
- [8]GOLUB, G and VAN LOAN, C F., *Matrix Computations* (John Hopkins Univ. Press, Baltimore, Maryland, 1985).
- [9]HENRICI, P., *Discrete Variable Methods in Ordinary Differential Equations* (John Wiley, New York, 1962).
- [10]HORN, S. and FRASER, P.A., Low-energy ortho-positronium scattering by hydrogen atoms, *J. Physics B* (1975), 2472-2475.
- [11]JÓDAR, L , MORERA, J.L. and NAVARRO, E., On convergent linear multistep matrix methods, *Int. J. Computer. Math.* 40 (1991), 211-219.
- [12]JÓDAR, L, MORERA, J.L. and VILLANUEVA, R.J., Numerical multistep matrix methods for $Y'' = f(t, Y)$, *Applied Math. & Comput.* 59 (1993), 257-274.
- [13]LAMBERT, J.D and SIGURDSSON, S.T., Multistep methods with variable matrix coefficients, *SIAM J. Numer. Anal.* 9 No. 4 (1972), 715-733.
- [14]LINZ, P., *Analytic and Numerical Methods for Volterra Integral Equations* (SIAM Studies in Appl Maths., Philadelphia, 1985).
- [15]LINZ, P., Linear multistep methods for Volterra integro-differential equations, *J. of the Ass. Comp. Mach.*, 16 No 2 (1969), 295-301.
- [16]MANN, W.R. and WOLF, F., Heat transfer between solids and gases under nonlinear boundary conditions, *Quart. J. Appl. Math.* 9 (1951), 163-184.
- [17]MATTHYS, J. A-stable linear multistep methods for Volterra integro-differential equations, *Numer. Math.* 27 (1976), 85-94.
- [18]NETA, B , Numerical solution of nonlinear integro-differential equation, *J. Math. Anal. Appl.* 89 (1982), 598-611.
- [19]ROBERTS, J.H and MANN, W.R., A nonlinear integral equation of Volterra type, *Pacific J. Math.* (1951), 431-445.
- [20]SHAW, R.E., Numerical Solutions for Two Point Boundary Value Problems, Ph.D Thesis, University of New Brunswick, Canada, 1994.
- [21]VEMURI, V. and KARPLUS, W.J., *Digital Computer Treatment of Partial Differential Equations* (Prentice Hall, Englewood Cliffs, New Jersey, 1981).

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