A Completeness Property of Wilke's Tree Algebras

Saeed Salehi

Turku Center for Computer Science
Lemminkäisenkatu 14 A
FIN-20520 Turku
saeed@cs.utu.fi

Abstract. Wilke's tree algebra formalism for characterizing families of tree languages is based on six operations involving letters, binary trees and binary contexts. In this paper a completeness property of these operations is studied. It is claimed that all functions involving letters, binary trees and binary contexts which preserve all syntactic tree algebra congruence relations of tree languages are generated by Wilke's functions, if the alphabet contains at least seven letters. The long proof is omitted due to page limit. Instead, a corresponding theorem for term algebras, which yields a special case of the above mentioned theorem, is proved: in every term algebra whose signature contains at least seven constant symbols, all congruence preserving functions are term functions.

1 Introduction

A new formalism for characterizing families of tree languages was introduced by Wilke [13], which can be regarded as a combination of *universal algebraic* framework of Steinby [11] and Almeida [1], in the case of binary trees, based on *syntactic algebras*, and *syntactic monoid/semigroup* framework of Thomas [12] and of Nivat and Podelski [8],[9]. It is based on three-sorted algebras, whose signature Σ consists of six operation symbols involving the sorts ALPHABET, TREE and CONTEXT. Binary trees over an alphabet are represented by terms over Σ, namely as Σ-terms of sort TREE. A *tree algebra* is a Σ-algebra satisfying certain identities which identify (some) pairs of Σ-terms representing the same tree. The *syntactic tree algebra congruence relation* of a tree language is defined in a natural way (see Definition 1 below.) The TREE-sort component of the *syntactic tree algebra* of a tree language is the syntactic algebra of the language in the sense of [11], while its CONTEXT-component is the semigroup part of the syntactic monoid of the tree language, as in [12].

A tree language is regular iff its syntactic tree algebra is finite ([13], Proposition 2.) A special sub-class of regular tree languages, that of k-frontier testable tree languages, is characterized in [13] by a set of identities satisfied by the corresponding syntactic tree algebra. For characterizing this sub-class, three-sorted tree algebra framework appears to be more suitable, since "frontier testable tree languages cannot be characterized by syntactic semigroups and there is no known

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finite characterization of frontier testability (for an arbitrary k) in the universal algebra framework" [7].

This paper concerns Wilke's functions (Definition 2) by which tree algebra formalism is established for characterizing families of tree languages ([13]). We claim that Wilke's functions generate all congruence preserving operations on the term algebra of trees, when the alphabet contains at least seven labels. For the sake of brevity, we do not treat tree languages and Wilke's functions in many-sorted algebra framework as done in [13], our approach is rather a continuation of the lines of the traditional framework, as of e.g. [11]. A more comprehensive general study of tree algebras and Wilke's formalism (independent from this work) has been initiated by Steinby and Salehi [10].

2 Tree Algebraic Functions

For an alphabet A, let Σ^{A} be the signature which contains a constant symbol c_a and a binary function symbol f_a for every $a \in A$, that is $\Sigma^{\mathrm{A}} = \{c_a \mid a \in A\} \cup \{f_a \mid a \in A\}$.

The set of binary trees over A, denoted by T_{A}, is defined inductively by:

- $c_a \in T_{\mathrm{A}}$ for every $a \in A$; and
- $f_a(t_1, t_2) \in T_{\mathrm{A}}$ whenever $t_1, t_2 \in T_{\mathrm{A}}$ and $a \in A$.

Fix a new symbol ξ which does not appear in A. Binary contexts over A are binary trees over $A \cup \{\xi\}$ in which ξ appears exactly once. The set of binary contexts over A, denoted by C_{A}, can be defined inductively by:

- $\xi, f_a(t,\xi), f_a(\xi,t) \in C_{\mathrm{A}}$ for every $a \in A$ and every $t \in T_{\mathrm{A}}$; and
- $f_a(t,p), f_a(p,t) \in C_{\mathrm{A}}$ for every $a \in A$, every $t \in T_{\mathrm{A}}$, and every $p \in C_{\mathrm{A}}$.

For $p, q \in C_{\mathrm{A}}$ and $t \in T_{\mathrm{A}}$, $p(q) \in C_{\mathrm{A}}$ and $p(t) \in T_{\mathrm{A}}$ are obtained from p by replacing the single occurrence of ξ by q or by t, respectively.

Definition 1. For a tree language $L \subseteq T_{\mathrm{A}}$ we define the *syntactic tree algebra congruence* relation of L, denoted by $\approx^L = (\approx^L_{\mathrm{A}}, \approx^L_{\mathrm{T}}, \approx^L_{\mathrm{C}})$, as follows:

1. For any $a, b \in A$, $a \approx^L_{\mathrm{A}} b \equiv \forall p \in C_{\mathrm{A}}\{p(c_a) \in L \leftrightarrow p(c_b) \in L\}$ & $\forall p \in C_{\mathrm{A}} \forall t_1, t_2 \in T_{\mathrm{A}}\{p(f_a(t_1,t_2)) \in L \leftrightarrow p(f_b(t_1,t_2)) \in L\}$.
2. For any $t, s \in T_A$, $t \approx^L_{\mathrm{T}} s \equiv \forall p \in C_{\mathrm{A}}\{p(t) \in L \leftrightarrow p(s) \in L\}$.
3. For any $p, q \in C_{\mathrm{A}}$, $p \approx^L_{\mathrm{C}} q \equiv \forall r \in C_{\mathrm{A}} \forall t \in T_{\mathrm{A}}\{r(p(t)) \in L \leftrightarrow r(q(t)) \in L\}$.

Remark 1. Our definition of syntactic tree algebra congruence relation of a tree language is that of [13], but we have corrected a mistake in Wilke's definition of $\approx^L_{\mathrm{A}}$; it is easy to see that the original definition (page 72 of [13]) does not yield a congruence relation. Another difference is that ξ is not a context in [13].

Definition 2. ([13], page 88) For an alphabet A, *Wilke's functions over A* are defined by:

$$\begin{array}{ll}
\iota^{\mathrm{A}} : A \to T_{\mathrm{A}} & \iota^{\mathrm{A}}(a) = c_a \\
\kappa^{\mathrm{A}} : A \times T_{\mathrm{A}}^2 \to T_{\mathrm{A}} & \kappa^{\mathrm{A}}(a, t_1, t_2) = f_a(t_1, t_2) \\
\lambda^{\mathrm{A}} : A \times T_{\mathrm{A}} \to C_{\mathrm{A}} & \lambda^{\mathrm{A}}(a, t) = f_a(\xi, t) \\
\rho^{\mathrm{A}} : A \times T_{\mathrm{A}} \to C_{\mathrm{A}} & \rho^{\mathrm{A}}(a, t) = f_a(t, \xi) \\
\sigma^{\mathrm{A}} : C_{\mathrm{A}}^2 \to C_{\mathrm{A}} & \sigma^{\mathrm{A}}(p_1, p_2) = p_1(p_2) \\
\eta^{\mathrm{A}} : C_{\mathrm{A}} \times T_{\mathrm{A}} \to T_{\mathrm{A}} & \eta^{\mathrm{A}}(p, t) = p(t)
\end{array}$$

Recall that projection functions $\pi_j^n : B_1 \times \cdots \times B_n \to B_j$ (for sets $B_1, \cdots, B_n$) are defined by $\pi_j^n(b_1, \cdots, b_n) = b_j$. For a $b \in B_j$, the constant function from $B_1 \times \cdots \times B_n$ to B_j, determined by b, is defined by $(b_1, \cdots, b_n) \mapsto b$.

Definition 3. For an alphabet A, a function $F : A^n \times T_{\mathrm{A}}^m \times C_{\mathrm{A}}^k \to X$ where $X \in \{A, T_{\mathrm{A}}, C_{\mathrm{A}}\}$ is called *tree-algebraic* over A, if it is a composition of Wilke's functions over A, constant functions and projection function.

Example 1. Let $A = \{a, b\}$. The function $F : A \times T_A \times C_A \to C_A$ defined by

$$F(x, t, p) = f_a\Big(f_x\big(f_b(c_a, c_a), \xi\big), p\big(f_b(t, c_x)\big)\Big),$$

for $x \in A$, $t \in T_A$ and $p \in C_A$, is tree-algebraic over A. Indeed

$$F(x, t, p) = \sigma^A\Big(\lambda^A\Big(a, \eta^A\big(p, \kappa^A(b, t, \iota^A(x))\big)\Big), \rho^A\big(x, f_b(c_a, c_a)\big)\Big).$$

Definition 4. A function $F : A^n \times T_{\mathrm{A}}^m \times C_{\mathrm{A}}^k \to X$ where $X \in \{A, T_{\mathrm{A}}, C_{\mathrm{A}}\}$ is called *congruence preserving* over A, if for every tree language $L \subseteq T_{\mathrm{A}}$ and for all $a_1, b_1, \cdots, a_n, b_n \in A$, $t_1, s_1, \cdots, t_m, s_m \in T_{\mathrm{A}}$, $p_1, q_1, \cdots, p_k, q_k \in C_{\mathrm{A}}$,

if $a_1 \approx_{\mathrm{A}}^L b_1, \cdots, a_n \approx_{\mathrm{A}}^L b_n,\ t_1 \approx_{\mathrm{T}}^L s_1, \cdots, t_m \approx_{\mathrm{T}}^L s_m,$
and $p_1 \approx_{\mathrm{C}}^L q_1, \cdots, p_k \approx_{\mathrm{C}}^L q_k,$
then $F(a_1, \cdots, a_n, t_1, \cdots, t_m, p_1, \cdots, p_k)$
$\approx_x^L F(b_1, \cdots, b_n, s_1, \cdots, s_m, q_1, \cdots, q_k),$

where x is A, T, or C, if $X = A$, $X = T_{\mathrm{A}}$, or $X = C_{\mathrm{A}}$, respectively.

Remark 2. In universal algebra, the functions which preserve congruence relations of an algebra, are called *congruence preserving* functions. On the other hand it is known that every congruence relation over an algebra is the intersection of some syntactic congruence relations (see Remark 2.12 of [1] or Lemma 6.2 of [11].) So, a function preserve all congruence relations of an algebra iff it preserves the syntactic congruence relations of all subsets of the algebra. This justifies the notion of congruence preserving function in our Definition 4, even though we require that the function preserves only all the syntactic tree algebra congruence relations of tree languages.

Example 2. For $A = \{a, b\}$, the root function root: $T_A \to A$, which maps a tree to its root label, is not congruence preserving: Let $L = \{f_a(c_b, c_b)\}$, then

$$f_a(c_a, c_a) \approx_{\mathrm{T}}^{L} f_b(c_a, c_a),$$

but since $f_a(c_b, c_b) \in L$, and $f_b(c_b, c_b) \notin L$, then

$$\mathrm{root}\big(f_a(c_a, c_a)\big) = a \not\approx_{\mathrm{A}}^{L} b = \mathrm{root}\big(f_b(c_a, c_a)\big).$$

Lemma 1. All tree-algebraic functions are congruence preserving.

The easy proof is omitted. We claim the converse for alphabets containing at least seven labels:

Theorem 1. For an alphabet A which contains at least seven labels, every congruence preserving function over A is tree-algebraic over A.

Remark 3. The condition $|A| \geq 7$ in Theorem 1 may seem odd at the first glance, but the theorem does not hold for $|A| = 2$: let $A = \{a, b\}$ and define $F : A \to T_{\mathrm{A}}$ by $F(a) = f_a(c_b, c_b),\ F(b) = f_b(c_a, c_a)$. It can be easily seen that F is congruence preserving but is not tree-algebraic over A. It is not clear at the moment whether Theorem 1 holds for $3 \leq |A| \leq 6$.

The long detailed proof of Theorem 1 will not be given in this paper because of space shortage. Instead, in the next section, a corresponding theorem for term algebras, which immediately yields Theorem 1 for congruence preserving functions of the form $F : T_{\mathrm{A}}^{m} \to T_{\mathrm{A}}$, is proved.

3 Congruence Preserving Functions in Term Algebras

Our notation follows mainly [2], [3], [5], [6], and [11]. A *ranked alphabet* is a finite nonempty set of symbols each of which has a unique non-negative *arity* (or *rank*). The set of m-ary symbols in a ranked alphabet Σ is denoted by Σ_m (for each $m \geq 0$). $T_\Sigma(X)$ is the set of Σ-terms with variables in X. For empty X it is simply denoted by T_Σ. Note that $(T_\Sigma(X), \Sigma)$ is a Σ-algebra, and (T_Σ, Σ) is called the *term algebra* over Σ. For $L \subseteq T_\Sigma$, let $\approx^L$ be the syntactic congruence relation of L ([11]), i.e., the greatest congruence on the term algebra T_Σ saturating L.

Let Σ denote a signature with the property that $\Sigma \neq \Sigma_0$. Throughout X is always a set of variables.

Definition 5. A function $F : (T_\Sigma)^n \to T_\Sigma$ is *congruence preserving* if for every congruence relation Θ over T_Σ and all $t_1, \cdots, t_n, s_1, \cdots, s_n \in T_\Sigma$, if $t_1 \Theta s_1, \cdots, t_n \Theta s_n$, then $F(t_1, \cdots, t_n) \Theta F(s_1, \cdots, s_n)$.

Remark 4. A congruence preserving function $F : (T_\Sigma)^n \to T_\Sigma$ induces a well-defined function $F_\Theta : (T_\Sigma/\Theta)^n \to T_\Sigma/\Theta$ on any *quotient algebra*, for any congruence Θ on T_Σ, defined by $F_\Theta([t_1]_\Theta, \cdots, [t_n]_\Theta) = [F(t_1, \cdots, t_n)]_\Theta$.

For terms $u_1, \cdots, u_n \in T_\Sigma(X)$ and $t \in T_\Sigma(X \cup \{x_1, \cdots, x_n\})$ with $x_1, \cdots, x_n \notin X$, the term $t[x_1/u_1, \cdots, x_n/u_n]$ [1] $\in T_\Sigma(X)$ is resulted from t by replacing all the occurrences of x_i by u_i for all $i \leq n$. The function $(T_\Sigma)^n \to T_\Sigma(X)$ defined by $(u_1, \cdots, u_n) \mapsto t[x_1/u_1, \cdots, x_n/u_n]$ for all $u_1, \cdots, u_n \in T_\Sigma$, is called the *term function* [2] defined by t.

The rest of the paper is devoted to the proof of the following Theorem:

Theorem 2. If $|\Sigma_0| \geq 7$, then every congruence preserving $F : (T_\Sigma)^n \to T_\Sigma$, for every $n \in \mathbb{N}$, is a term function (i.e., there is a term $t \in T_\Sigma(\{x_1, \cdots, x_n\})$, where $x_1, \cdots, x_n$ are variables, such that $F(u_1, \cdots, u_n) = t[x_1/u_1, \cdots, x_n/u_n]$ for all $u_1, \cdots, u_n \in T_\Sigma$.)

Remark 5. Theorem 2 dose not hold for $|\Sigma_0| = 1$: Let $\Sigma = \Sigma_0 \cup \Sigma_1$ be a signature with $\Sigma_1 = \{\alpha\}$ and $\Sigma_0 = \{\zeta_0\}$. The term algebra (T_Σ, Σ) is isomorphic to $(\mathbb{N}, \mathbf{0}, \mathbf{S})$, where $\mathbf{0}$ is the constant zero and $\mathbf{S}$ is the successor function. Let $F : \mathbb{N} \to \mathbb{N}$ be defined by $F(n) = 2n$. It is easy to see that F is congruence preserving: for every congruence relation Θ, if $n\Theta m$ then $\mathbf{S}n\Theta\mathbf{S}m$ and by repeating the same argument for n times we get $\mathbf{S}^n n\Theta\mathbf{S}^n m$ or $2n\Theta n + m$. Similarly $\mathbf{S}^m n\Theta\mathbf{S}^m m$, so $m + n\Theta 2m$, hence $2m\Theta 2n$ that is $F(n)\Theta F(m)$. But F is not a term function, since all term functions are of the form $n \mapsto \mathbf{S}^{\mathbf{k}} n = \mathbf{k} + n$ for a fixed $\mathbf{k} \in \mathbb{N}$. It is not clear at the moment whether Theorem 2 holds for $2 \leq |\Sigma_0| \leq 6$.

Remark 6. Finite algebras having the property that all congruence preserving functions are term functions are called *hemi-primal* in universal algebra (see e.g. [3]). Our assumption $\Sigma \neq \Sigma_0$ in Theorem 2 implies that T_Σ is infinite.

Remark 7. Theorem 2 yields Theorem 1 for congruence preserving functions of the form $F : T_{\mathrm{A}}^n \to T_{\mathrm{A}}$, since $(T_{\mathrm{A}}, \Sigma^{\mathrm{A}})$ is the term algebra over the signature Σ^{A}, and its every term function can be represented by ι^{A} and κ^{A} (recall that $c_a = \iota^{\mathrm{A}}(a)$, and $f_a(t_1, t_2) = \kappa^{\mathrm{A}}(a, t_1, t_2)$, for every $a \in A$, and $t_1, t_2 \in T_{\mathrm{A}}$).

Proof of Theorem 2

Definition 6.
- An *interpretation* of X in T_Σ is a function $\varepsilon : X \to T_\Sigma$. Its unique extension to the Σ-homomorphism $T_\Sigma(X) \to T_\Sigma$ is denoted by ε^*.
- Any congruence relation Θ on T_Σ is extended to a congruence relation Θ^* on $T_\Sigma(X)$ defined by the following relation for any $p, q \in T_\Sigma(X)$:

$$p\,\Theta^* q, \text{ if for every interpretation } \varepsilon : X \to T_\Sigma,\ \varepsilon^*(p)\ \Theta\ \varepsilon^*(q) \text{ holds.}$$

- A function $G : T_\Sigma \to T_\Sigma(X)$ is *congruence preserving* if for every congruence relation Θ on T_Σ and $t, s \in T_\Sigma$, if $t\Theta s$, then $G(t)\Theta^* G(s)$.

The classical proof of the following lemma is not presented here.

[1] Denoted by $t[u_1, \cdots, u_n]$ in [4].
[2] It is also called *tree substitution operation*, see e.g. [4].

Lemma 2. The term function $T_\Sigma \to T_\Sigma(X)$, $u \mapsto t[x/u]$ defined by any term $t \in T_\Sigma(X \cup \{x\})$ $(x \notin X)$, is congruence preserving.

Definition 7. Let t be a term in $T_\Sigma(X)$, and $C \subseteq T_\Sigma(X)$, then t is called *independent* from C, if it is not a subterm of any member of C and no member of C is a subterm of t.

For a term rewriting system $\mathcal{R}$, and a term u, let $\Delta^*_{\mathcal{R}}(u)$ be the set of $\mathcal{R}$-descendants of $\{u\}$ (cf. [6]) and for a set of terms C, let $\Delta^*_{\mathcal{R}}(C) = \bigcup_{u \in C} \Delta^*_{\mathcal{R}}(u)$. A useful property of the notion of independence is the following:

Lemma 3. Let $u \in T_\Sigma(X)$ be independent from $C \subseteq T_\Sigma(X)$ and $\mathcal{R}$ be the single-ruled (ground-)term rewriting system $\{w \to u\}$ where w is any term in $T_\Sigma(X)$. Then $L = \Delta^*_{\mathcal{R}}(C)$ is closed under the rewriting rule $u \to w$, and also $u \approx^L w$. Moreover, every member of L results from a member of C by replacing some w subterms of it by u.

Proof. Straightforward, once we note that any application of the rule $w \to u$ to a member of C, does not result in a new subterm of the form w, and all u's appearing in the members of L (as subterms) are obtained by applying the (ground-term) rewriting rule $w \to u$. □

Proposition 1. For any $C \subset T_\Sigma(X)$ such that $|C| < |\Sigma_0|$, there is a term in T_Σ which is independent from C.

Proof. For each $c \in \Sigma_0$ choose a $t_c \in T_\Sigma$ that is higher (has longer height) than all terms in C and contains no other constant symbol than this c. Then, no t_c is a subterm of any member of C. On the other hand, no term in C may appear as a subterm in more than one of the terms t_c (for any $c \in \Sigma_0$). Since the number of t_c's for $c \in \Sigma_0$ are more than the number of elements of C, then by the Pigeon Hole Principle, there must exist a t_c that is independent from C. □

Lemma 4. Let $G : T_\Sigma \to T_\Sigma(X)$ be congruence preserving, $\varepsilon : X \to T_\Sigma$ be an interpretation, and $u, v \in T_\Sigma$. If v is independent from $\{u, \varepsilon^*(G(u))\}$, then

$$\varepsilon^*(G(v)) \in \Delta^*_{\{u \to v\}}\big(\varepsilon^*(G(u))\big).$$

Moreover, $\varepsilon^*(G(v))$ results from $\varepsilon^*(G(u))$ by replacing some u subterms by v.

Proof. Let $L = \Delta^*_{\{u \to v\}}\big(\varepsilon^*(G(u))\big)$. By Lemma 3, $u \approx^L v$. The function G is congruence preserving, so $\varepsilon^*(G(u)) \approx^L \varepsilon^*(G(v))$, and since $\varepsilon^*(G(u)) \in L$, then $\varepsilon^*(G(v)) \in L$. The second claim follows from the independence of v from $\{u, \varepsilon^*(G(u))\}$. □

Recall that for a position p of the term t, $t|_p$ is the subterm of t at the position p (cf. [2]).

Lemma 5. Suppose $|\Sigma_0| \geq 7$, and let $G : T_\Sigma \to T_\Sigma(X)$ be congruence preserving. If v is independent from $\{u, G(u)\}$, for $u, v \in T_\Sigma$, then

$$G(v) \text{ results from } G(u) \text{ by replacing some of its } u \text{ subterms by } v.$$

Proof. By Proposition 1, there are w, w_1, w_2 such that w is independent from $\{u, G(u), v, G(v)\}$, w_1 is independent from $\{w, u, G(u), v, G(v)\}$, and w_2 is independent from $\{w, w_1, u, G(u), v, G(v)\}$.

Define the interpretation $\varepsilon : X \to T_\Sigma$ by setting $\varepsilon(x) = w$ for all $x \in X$. By the choice of w, v is independent from $\{u, \varepsilon^*(G(u))\}$. So we can apply Lemma 4 to infer that $\varepsilon^*(G(v))$ results from $\varepsilon^*(G(u))$ by replacing some u subterms by v. Note that $G(v)$ is obtained by substituting all w's in $\varepsilon^*(G(v))$ by members of X. The same is true about $G(u)$ and $\varepsilon^*(G(u))$.

The positions of $\varepsilon^*(G(v))$ in which w appear are exactly the same positions of $\varepsilon^*(G(u))$ in which w appear (by the choice of w). So, positions of $G(v)$ in which a member of X appear are exactly the same positions of $G(u)$ in which a member of X appear. We claim that identical members of X appear in those identical positions of $G(u)$ and $G(v)$: if not, there are $x_1, x_2 \in X$ such that $G(v)|_p = x_1$ and $G(u)|_p = x_2$ for some position p of $G(u)$ (and of $G(v)$).

Define the interpretation $\delta : X \to T_\Sigma$ by $\delta(x_1) = w_1$, $\delta(x_2) = w_2$, and $\delta(x) = w$ for all $x \neq x_1, x_2$. Then $\delta^*(G(v))|_p = w_1$ and $\delta^*(G(u))|_p = w_2$. On the other hand by Lemma 4, $\delta^*(G(v))$ results from $\delta^*(G(u))$ by replacing some u subterms by v. By the choice of w_1 and w_2, such a replacement can not affect the appearance of w_1 or w_2, and hence the subterms of $\delta^*(G(v))$ and $\delta^*(G(u))$ in the position p must be identical, a contradiction. This proves the claim which implies that $G(v)$ results from $G(u)$ by replacing some u subterms by v. □

Lemma 6. Suppose $|\Sigma_0| \geq 7$, and let $G : T_\Sigma \to T_\Sigma(X)$ be congruence preserving. Then for any $u, v \in T_\Sigma$,

$$G(v) \text{ results from } G(u) \text{ by replacing some } u \text{ subterms by } v.$$

Proof. By Proposition 1, there is a $w \in T_\Sigma$ independent from $\{u, G(u), v, G(v)\}$. By Lemma 5, $G(w)$ is obtained from $G(u)$ by replacing some u subterms by w, and also results from $G(v)$ by replacing some v subterms by w. By the choice of w, all w's appearing in $G(w)$ have been obtained either by replacing u by w in $G(u)$ or by replacing v by w in $G(v)$. Since the only difference between $G(v)$ and $G(w)$ is in the positions of $G(w)$ where w appears, and the same is true for the difference between $G(u)$ and $G(w)$, then $G(v)$ can be obtained from $G(u)$ by replacing some u subterms of it, the same u subterms which have been replaced by w to get $G(w)$, by v. □

Lemma 7. If $|\Sigma_0| \geq 7$, then every congruence preserving function $G : T_\Sigma \to T_\Sigma(X)$ is a term function (i.e., there is a term $t \in T_\Sigma(X \cup \{x\})$, where $x \notin X$, such that $G(u) = t[x/u]$ for all $u \in T_\Sigma$.)

Proof. Fix a $u_0 \in T_\Sigma$, and choose a $v \in T_\Sigma$ such that v is independent from $\{u_0, G(u_0)\}$. (By Proposition 1 such a v exists.) Then by Lemma 6, $G(v)$ results from $G(u_0)$ by replacing some u_0 subterms by v. Let y be a new variable ($y \notin X$) and let $t \in T_\Sigma(X \cup \{y\})$ result from $G(u_0)$ by putting y exactly in the same positions that u_0's are replaced by v's to get $G(v)$. So, $G(u_0) = t[y/u_0]$ and $G(v) = t[y/v]$, moreover all v's in $G(v)$ are obtained from t by substituting all y's by v. We show that for any arbitrary $u \in T_\Sigma$, $G(u) = t[y/u]$ holds:

Take a $u \in T_\Sigma$. By Proposition 1, there is a w independent from the set $\{u_0, G(u_0), v, G(v), u, G(u)\}$. By Lemma 6, $G(w)$ results from $G(v)$ by replacing some v subterms by w. We claim that all v's are replaced by w's in $G(v)$ to get $G(w)$. If not, then v must be a subterm of $G(w)$. From the fact (Lemma 6) that $G(u_0)$ results from $G(w)$ by replacing some w subterms by u_0 (and the choice of w) we can infer that v is a subterm of $G(u_0)$ which is in contradiction with the choice of v. So the claim is proved and then we can write $G(w) = t[y/w]$, moreover all w's in $G(w)$ are obtained from t by substituting y by w. Again by Lemma 6, $G(u)$ results from $G(w)$ by replacing some w subterms by u. We can claim that all w's appearing in $G(w)$ are replaced by u to get $G(u)$. Since otherwise w would have been a subterm of $G(u)$ which is in contradiction with the choice of w. This shows that $G(u) = t[y/u]$. □

Theorem 2. If $|\Sigma_0| \geq 7$, then every congruence preserving $F : (T_\Sigma)^n \to T_\Sigma$, for every $n \in \mathbb{N}$, is a term function.

Proof. We proceed by induction on n: For $n = 1$ it is Lemma 7 with $X = \emptyset$.

For the induction step let $F : (T_\Sigma)^{n+1} \to T_\Sigma$ be a congruence preserving function. For any $u \in T_\Sigma$ define $F_u : (T_\Sigma)^n \to T_\Sigma$ by $F_u(u_1, \cdots, u_n) = F(u_1, \cdots, u_n, u)$. By the induction hypothesis every F_u is a term function, i.e., there is a $s \in T_\Sigma(\{x_1, \cdots, x_n\})$ such that $F_u(u_1, \cdots, u_n) = s[x_1/u_1, \cdots, x_n/u_n]$ for all $u_1, \cdots, u_n \in T_\Sigma$. Denote the corresponding term for u by t_u (it is straightforward to see that such a term s is unique for every u). The mapping $T_\Sigma \to T_\Sigma(\{x_1, \cdots, x_n\})$ defined by $u \mapsto t_u$ is also congruence preserving. Hence by Lemma 7, it is a term function. So there is a $t \in T_\Sigma(\{x_1, \cdots, x_n, x_{n+1}\})$ such that $t_u = t[x_{n+1}/u]$, hence $F(u_1, \cdots, u_n, u_{n+1}) = F_{u_{n+1}}(u_1, \cdots, u_n) = t_{u_{n+1}}[x_1/u_1, \cdots, x_n/u_n] = t[x_{n+1}/u_{n+1}][x_1/u_1, \cdots, x_n/u_n]$.
So $F(u_1, \cdots, u_n, u_{n+1}) = t[x_1/u_1, \cdots, x_n/u_n, x_{n+1}/u_{n+1}]$ is a term function. □

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References

1. Almeida J., "On pseudovarieties, varieties of languages, fiters of congruences, pseudoidentities and related topics", *Algebra Universalis*, Vol. 27 (1990) pp. 333-350.
2. Bachmair L., "Canonical equational proofs", Progress in Theoretical Computer Science, Birkhäuser, Boston Inc., Boston MA, 1991.

3. Denecke K. & Wismath S. L., "Universal algebra and applications in theoretical computer science", Chapman & Hall/CRC, Boca Raton FL, 2002.
4. Fülöp Z. & Vágvölgyi S. "Minimal equational representations of recognizable tree languages" *Acta Informatica* Vol. 34, No. 1 (1997) pp. 59-84.
5. Gécseg F. & Steinby M., "Tree languages", in: Rozenberg G.& Salomaa A. (ed.) *Handbook of formal languages*, Vol. 3, Springer, Berlin (1997) pp. 1-68.
6. Jantzen M., "Confluent string rewriting", EATCS Monographs on Theoretical Computer Science 14, Springer-Verlag, Berlin 1988.
7. Salomaa K., Review of [13] in AMS-MathSciNet, MR-97f:68134.
8. Nivat M. & Podelski A., "Tree monoids and recognizability of sets of finite trees", *Resolution of Equations in Algebraic Structures*, Vol. 1, Academic Press, Boston MA (1989) pp. 351-367.
9. Podelski A., "A monoid approach to tree languages", in: Nivat M. & Podelski A. (ed.) *Tree Automata and Languages*, Elsevier-Amsterdam (1992) pp. 41-56.
10. Salehi S. & Steinby M., "Tree algebras and regular tree languages" in preparation.
11. Steinby M., "A theory of tree language varieties", in: Nivat M. & Podelski A. (ed.) *Tree Automata and Languages*, Elsvier-Amsterdam (1992) pp. 57-81.
12. Thomas W., "Logical aspects in the study of tree languages", *Ninth Colloquium on Trees in Algebra and in Programming* (Proc. CAAP'84), Cambridge University Press (1984) pp. 31-51.
13. Wilke T., "An algebraic characterization of frontier testable tree languages", *Theoretical Computer Science*, Vol. 154, N. 1 (1996) pp. 85-106.