APPROSSIMATING PERMANENTS AND
HAFNIANS OF POSITIVE MATRICES

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Abstract. For any $0 < \delta \leq 1$, we present a deterministic algorithm, which, given an $n \times n$ real matrix $A$ with entries between $\delta$ and 1 and an $0 < \epsilon < 1$ approximates the permanent of $A$ within a relative error $\epsilon$ in $n^{O(\ln n - \ln \epsilon)}$ time. A similar algorithm is constructed to approximate the hafnian of a symmetric matrix with entries between $\delta$ and 1. In particular, we prove that $\ln \per A$ and $\ln \haf A$ are approximated within error $\epsilon$ by a polynomial of degree $O(\ln n - \ln \epsilon)$ in the entries of $A$.

1. Introduction and the main results

(1.1) Permanents. Let $A = (a_{ij})$ be an $n \times n$ real or complex matrix. The permanent of $A$ is defined as

$$
\per A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},
$$

where $S_n$ is the symmetric group of permutations of the set $\{1, \ldots, n\}$. It is a $\#P$-hard problem to compute the permanent of a given 0-1 matrix $A$ exactly [Va79], although a fully polynomial randomized approximation scheme is constructed for non-negative matrices [J+04]. In this paper, we are interested in deterministic algorithms to approximate per $A$ for a positive matrix $A$. The permanent of an $n \times n$ non-negative matrix $A$ can be approximated within a factor of $e^n$ in deterministic polynomial time [L+00] and the factor was improved to $2^n$ in [GS14] (with a conjectured improvement to $2^{n/2}$). If one assumes that the entries of $A$ are separated from 0, that is, if

$$\delta \leq a_{ij} \leq 1 \quad \text{for all} \quad i, j$$

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and some $0 < \delta \leq 1$ fixed in advance, then the polynomial algorithm of [L+00] actually results in an approximation factor of $n^{O(1)}$, where the implied constant in the “$O$” notation depends on $\delta$, see also [BS11]. Apart from that, deterministic polynomial time algorithms are known for special classes of matrices. For example, in [GK10], for any $\epsilon > 0$, fixed in advance, a polynomial time algorithm is constructed to approximate per $A$ within a factor of $(1 + \epsilon)^n$ if $A$ is the adjacency matrix of a constant degree expander.

In this paper, we present a quasi-polynomial deterministic algorithm, which, given an $n \times n$ matrix $A = (a_{ij})$ satisfying (1.1.1) with some $0 < \delta \leq 1$, fixed in advance, and an $\epsilon > 0$ approximates per $A$ within a relative error $\epsilon$ in $n^{O(\ln n - \ln \epsilon)}$ time. The implicit constant in the “$O$” notation depends on $\delta$.

More precisely, we prove the following result.

\textbf{(1.2) Theorem.} For any $0 < \delta \leq 1$ there exists $\gamma(\delta) > 0$ such that for any positive integer $n$ and any $0 < \epsilon < 1$ there exists a polynomial $p = p_{n,\delta,\epsilon}$ of degree $p \leq \gamma(\delta) (\ln n - \ln \epsilon)$ in the entries $a_{ij}$ of an $n \times n$ matrix $A$ such that

$$|\ln \text{per } A - p(A)| \leq \epsilon$$

for all $n \times n$ matrices $A = (a_{ij})$ satisfying

$$\delta \leq a_{ij} \leq 1 \text{ for all } i, j.$$

We show that the polynomial $p$ can be computed in quasi-polynomial time.

Our approach continues a line of work started in [B15+], see also [Mc14], and continued in [Ba15], [BS16] and [Re15]. The main idea is to relate approximability of a polynomial with its complex zeros. For a complex number $z = a + ib$, we denote by $\Re z = a$ and $\Im z = b$, the real and imaginary parts of $z$ correspondingly. We deduce Theorem 1.2 from the following result.

\textbf{(1.3) Theorem.} Let us fix a real $0 < \delta \leq 1$ and a real $\tau \geq 0$ such that

$$\tau \leq \delta \sin \left( \frac{\pi}{4} - \frac{1}{2} \arccos \delta^2 \right).$$

Let $Z = (z_{ij})$ be an $n \times n$ complex matrix such that

$$\delta \leq \Re z_{ij} \leq 1 \text{ and } |\Im z_{ij}| \leq \tau \text{ for all } 1 \leq i, j \leq n.$$

Then

$$\text{per } Z \neq 0.$$

In particular, the conclusion of the theorem holds if

$$\tau \leq \frac{1}{2} \delta^3.$$
since
\[
\frac{1}{2} \delta^3 \leq \delta \sin \left( \frac{\pi}{4} - \frac{1}{2} \arccos \delta^2 \right) \quad \text{for} \quad 0 \leq \delta \leq 1.
\]

In [B15+] we prove that for some absolute constant \(\delta_0 > 0\) we have \(\text{per} Z \neq 0\) for any \(n \times n\) complex matrix \(Z = (z_{ij})\) such that \(|1 - z_{ij}| \leq \delta_0\) for all \(i, j\). In [B15+] it is shown that one can choose \(\delta_0 = 0.195\), but the value can be improved to \(\delta_0 = 0.275\) using a slight modification of the argument, cf. [Ba15] and [BS16]. The best value of \(\delta_0\) is not known although one can show that \(\delta_0 < 0.708\). It is then shown in [B15+] that for any \(0 < \delta < \delta_0\), fixed in advance, there is a quasi-polynomial algorithm approximating the permanent of an \(n \times n\) real or complex matrix \(Z = (z_{ij})\) satisfying \(|1 - z_{ij}| \leq \delta\) for all \(i, j\) within a relative error \(\epsilon > 0\) in time \(n^{O(\ln n - \ln \epsilon)}\). This appears to be a fairly general phenomenon, which extends to other partition functions, including multi-dimensional permanents [B15+], partition functions enumerating dense subgraphs [Ba15], graph homomorphisms [BS16] and edge-coloring models [Re15] (for each partition function there is its own value of \(\delta_0\)).

Theorem 1.3 is of a somewhat different nature: here we allow the entries \(a_{ij}\) to be arbitrarily close to 0 but insist that the imaginary part of \(a_{ij}\) gets smaller as \(a_{ij}\) approaches 0. At the moment, it is not clear whether the approach can be generalized to other partition functions, and indeed there is a computational complexity obstacle of the P=NP type for extending it, for example, to the partition function of graph homomorphisms, cf. [BS16]. The only immediate extension appears to be from permanents to hafnians.

**1.4 Hafnian.** Let \(A = (a_{ij})\) be a \(2n \times 2n\) symmetric real or complex matrix. The hafnian of \(A\) is defined as
\[
\text{haf} A = \sum_{\{i_1, j_1\}, \ldots, \{i_n, j_n\}} a_{i_1,j_1} \cdots a_{i_n,j_n},
\]
where the sum is taken over \((2n)!/2^n n!\) unordered partitions of the set \(\{1, \ldots, 2n\}\) into \(n\) pairwise disjoint unordered pairs \(\{i_1, j_1\}, \ldots, \{i_n, j_n\}\), see for example, Section 8.2 of [Mi78]. Just as the permanent of the biadjacency matrix of a bipartite graph enumerates the perfect matchings in the graph, the hafnian of the adjacency matrix of a graph enumerates the perfect matchings in the graph. In fact, for any \(n \times n\) matrix \(A\) we have
\[
\text{haf} \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} = \text{per} A,
\]
and hence computing the permanent of an \(n \times n\) matrix reduces to computing the hafnian of a symmetric \(2n \times 2n\) matrix.

Computationally, the hafnian appears to be a more complicated object than the permanent. No fully polynomial (randomized or deterministic) approximation scheme is known to compute the hafnian of a non-negative symmetric matrix and
no deterministic polynomial time algorithm to approximate the hafnian of a $2n \times 2n$ non-negative matrix within an exponential factor of $c^n$ for some absolute constant $c > 1$ is known (though there is a randomized polynomial time algorithm achieving such an approximation [Ba99]). On the other hand, if the entries $a_{ij}$ of the matrix $A = (a_{ij})$ satisfy (1.1.1) for some $\delta > 0$, fixed in advance, there is a polynomial time algorithm approximating $\text{haf } A$ within a factor of $n^{O(1)}$, where the implicit constant in the “$O$” notation depends on $\delta$ [BS11].

In this paper, we prove the following versions of Theorem 1.2 and 1.3.

(1.5) Theorem. For any $0 < \delta \leq 1$ there exists $\gamma(\delta) > 0$ such that for any positive integer $n$ and any $0 < \epsilon < 1$ there exists a polynomial $p = p_{n, \delta, \epsilon}$ of $\text{deg } p \leq \gamma(\delta) (\ln n - \ln \epsilon)$ in the entries $a_{ij}$ of a $2n \times 2n$ symmetric matrix $A$ such that

$$|\ln \text{haf } A - p(A)| \leq \epsilon$$

for all $2n \times 2n$ symmetric matrices $A = (a_{ij})$ satisfying

$$\delta \leq a_{ij} \leq 1 \quad \text{for all } i, j.$$

The polynomial $p_{n, \delta, \epsilon}$ can be computed in $n^{O(\ln n - \ln \epsilon)}$ time, where the implicit constant in the “$O$” notation depends on $\delta$. Consequently, we obtain a deterministic quasi-polynomial algorithm to approximate the hafnian of a positive matrix $A = (a_{ij})$ satisfying (1.1.1) within any given relative error $\epsilon > 0$.

As is the case with permanents, we deduce Theorem 1.5 from the result on the complex zeros of the hafnian.

(1.6) Theorem. Let us fix a real $0 < \delta \leq 1$ and a real $\tau \geq 0$ such that

$$\tau \leq \delta \sin \left(\frac{\pi}{4} - \frac{1}{2} \arccos \delta^2\right).$$

Let $Z = (z_{ij})$ be an $2n \times 2n$ symmetric complex matrix such that

$$\delta \leq \Re z_{ij} \leq 1 \quad \text{and} \quad |\Im z_{ij}| \leq \tau \quad \text{for all } 1 \leq i, j \leq n.$$

Then

$$\text{haf } Z \neq 0.$$

We prove Theorem 1.3 and Theorem 1.6 in Section 2 and in Section 3 we prove Theorem 1.2 and Theorem 1.5.

2. Proofs of Theorems 1.3 and 1.6

We start with a couple of simple geometric arguments regarding angles between non-zero complex numbers. We identify $\mathbb{C} = \mathbb{R}^2$, thus identifying complex numbers with vectors in the plane. We denote by $\langle \cdot, \cdot \rangle$ the standard inner product in $\mathbb{R}^2$ and by $|\cdot|$ the corresponding Euclidean norm (the modulus of a complex number).
(2.1) Lemma.

(1) Let us fix a real $0 < \delta \leq 1$. Let $u_1, \ldots, u_n \in \mathbb{R}^2$ be non-zero vectors such that the angle between any two $u_i$ and $u_j$ does not exceed $\pi/2$, let 

$$\delta \leq \alpha_i, \beta_i \leq 1 \quad \text{for} \quad i = 1, \ldots, n$$

be reals and let

$$v = \sum_{i=1}^{n} \alpha_i u_i \quad \text{and} \quad w = \sum_{i=1}^{n} \beta_i u_i.$$ 

Then $v \neq 0$, $w \neq 0$ and the angle between $v$ and $w$ does not exceed $\arccos \delta^2$.

(2) Let $u, w \in \mathbb{R}^2$ be vectors such that $u \neq 0$ and $|w| < |u|$. Then $u + w \neq 0$ and the angle between $u + w$ and $u$ does not exceed

$$\arcsin \frac{|w|}{|u|}.$$ 

Proof. To prove Part (1), let 

$$u = \sum_{i=1}^{n} u_i.$$ 

Since 

$$\langle u_i, u_j \rangle \geq 0 \quad \text{for all} \quad i, j,$$

we have

$$\langle v, w \rangle = \sum_{1 \leq i, j \leq n} \alpha_i \beta_j \langle u_i, u_j \rangle \geq \delta^2 \sum_{1 \leq i, j \leq n} \langle u_i, u_j \rangle = \delta^2 \langle u, u \rangle = \delta^2 |u|^2$$

(hence, in particular, $v \neq 0$ and $w \neq 0$). Similarly, for any real $-1 \leq \gamma_1, \ldots, \gamma_n \leq 1$ and $x = \gamma_1 u_1 + \ldots + \gamma_n u_n$, we have

$$\langle x, x \rangle = \sum_{1 \leq i, j \leq n} \gamma_i \gamma_j \langle u_i, u_j \rangle \leq \sum_{1 \leq i, j \leq n} \langle u_i, u_j \rangle = \langle u, u \rangle,$$

so that $|x| \leq |u|$. In particular,

$$|v||w| \leq |u||u| = |u|^2.$$ 

Denoting by $\omega$ the angle between $u$ and $v$, we obtain

$$\cos \omega = \frac{\langle v, w \rangle}{|v||w|} \geq \delta^2,$$

as required.

To prove Part (2), we notice that clearly $u + w \neq 0$. If we fix $|u|$ and $|w|$, we observe that the angle between $u$ and $u + w$ is the largest when $w$ is orthogonal to $u + w$, in which case the angle is $\arcsin(|w|/|u|)$. \qed
(2.2) Lemma. Let us fix real $0 < \delta \leq 1$ and $0 < \tau \leq 1$. Let $u_1, \ldots, u_n \in \mathbb{C}$ be non-zero complex numbers such that the angle between any two $u_i, u_j$ does not exceed $\pi/2$. Let $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_n$ be complex numbers such that

$$\delta \leq \Re \xi_j, \Re \eta_j \leq 1$$

and let

$$v = \sum_{j=1}^{n} \xi_j u_j \quad \text{and} \quad w = \sum_{j=1}^{n} \eta_j u_j.$$

If

$$\tau \leq \delta \sin \left( \frac{\pi}{4} - \frac{1}{2} \arccos \delta^2 \right)$$

then $v \neq 0$, $w \neq 0$ and the angle between $v$ and $w$ does not exceed $\pi/2$.

Proof. Let us write

$$\xi_j = \alpha_j' + i\alpha_j'' \quad \text{and} \quad \eta_j = \beta_j' + i\beta_j''$$

where

$$\alpha_j' = \Re \xi_j, \quad \alpha_j'' = \Im \xi_j, \quad \beta_j' = \Re \eta_j \quad \text{and} \quad \beta_j'' = \Im \eta_j \quad \text{for} \quad j = 1, \ldots, n.$$ 

Let

$$v' = \sum_{j=1}^{n} \alpha_j' u_j, \quad v'' = i \sum_{j=1}^{n} \alpha_j'' u_j,$$

$$w' = \sum_{j=1}^{n} \beta_j' u_j, \quad w'' = i \sum_{j=1}^{n} \beta_j'' u_j,$$

so that $v = v' + v''$ and $w = w' + w''$. By Part (1) of Lemma 2.1, we have $v' \neq 0$, $w' \neq 0$ and the angle $\omega$ between $v'$ and $w'$ does not exceed $\arccos \delta^2$. As in the proof of Lemma 2.1, let

$$u = \sum_{j=1}^{n} u_j,$$

so that

$$|v'|, |v'| \geq \delta |u| \quad \text{and} \quad |v''|, |w''| \leq \tau |u|.$$ 

Since $\tau < \delta$, by Part (2) of Lemma 2.1, we have $v = v' + v'' \neq 0$, $w = w' + w'' \neq 0$ and the angle between $v$ and $v'$ as well as the angle between $w$ and $w'$ do not exceed

$$\theta = \arcsin \frac{\tau}{\delta}.$$ 

Hence the angle between $v$ and $w$ does not exceed

$$\omega + 2\theta = \arccos \delta^2 + 2 \arcsin \frac{\tau}{\delta} \leq \frac{\pi}{2}.$$ 

□

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Proof of Theorem 1.3. For a positive integer \( n \), let \( \mathcal{U}_n \) be the set of \( n \times n \) complex matrices \( Z = (z_{ij}) \) such that

\[
(2.3.1) \quad \delta \leq \Re z_{ij} \leq 1 \quad \text{and} \quad |\Im z_{ij}| \leq \tau \quad \text{for all} \quad i, j.
\]

We prove by induction on \( n \) a more general statement:

For any \( Z \in \mathcal{U}_n \) we have \( \text{per} \, Z \neq 0 \) and, moreover, if \( A, B \in \mathcal{U}_n \) are two matrices that differ in one row (or in one column) only, then the angle between non-zero complex numbers \( \text{per} \, A \) and \( \text{per} \, B \) does not exceed \( \pi/2 \).

Since \( \tau < \delta \), the statement holds for \( n = 1 \). Assuming that the statement holds for matrices in \( \mathcal{U}_{n-1} \) with \( n \geq 2 \), let us consider two matrices \( A, B \in \mathcal{U}_n \) that differ in one row or in one column only. Since the permanent of a matrix does not change when the rows or columns of the matrix are permuted or when the matrix is transposed, without loss of generality we assume that \( B \) is obtained from \( A \) by replacing the entries \( a_{1j} \) of the first row by complex numbers \( b_{1j} \) for \( j = 1, \ldots, n \).

Let \( A_j \) be the \((n-1) \times (n-1)\) matrix obtained from \( A \) by crossing out the first row and the \( j \)-th column. Then

\[
(2.3.2) \quad \text{per} \, A = \sum_{j=1}^{n} a_{1j} \text{per} \, A_j \quad \text{and} \quad \text{per} \, B = \sum_{j=1}^{n} b_{1j} \text{per} \, A_j.
\]

We observe that \( A_j \in \mathcal{U}_{n-1} \) for \( j = 1, \ldots, n \) and, moreover, any two matrices \( A_{j1}, A_{j2} \) after a suitable permutation of columns differ in one column only. Hence by the induction hypothesis, we have \( \text{per} \, A_j \neq 0 \) for \( j = 1, \ldots, n \) and the angle between any two non-zero complex numbers \( \text{per} \, A_{j1} \) and \( \text{per} \, A_{j2} \) does not exceed \( \pi/2 \). Applying Lemma 2.2 with \( u_j = \text{per} \, A_j, \quad \xi_j = a_{1j} \quad \text{and} \quad \eta_j = b_{1j} \quad \text{for} \quad j = 1, \ldots, n, \) we conclude that \( \text{per} \, A \neq 0, \, \text{per} \, B \neq 0 \) and that the angle between non-zero complex numbers \( \text{per} \, A \) and \( \text{per} \, B \) does not exceed \( \pi/2 \), which completes the induction. \( \square \)

Proof of Theorem 1.6. The proof is very similar to that of Section 2.3. For a positive integer \( n \), we define \( \mathcal{U}_n \) as the set of \( 2n \times 2n \) symmetric complex matrices \( Z = (z_{ij}) \) satisfying (2.3.1) and we prove by induction on \( n \) that for any \( Z \in \mathcal{U}_n \) we have \( \text{haf} \, Z \neq 0 \) and if \( A, B \in \mathcal{U}_n \) are two matrices that differ only in the \( k \)-th row and in the \( k \)-th column for some unique \( k \) then the angle between non-zero complex numbers \( \text{haf} \, A \) and \( \text{haf} \, B \) does not exceed \( \pi/2 \). Instead of the Laplace expansion (2.3.2) we use the recurrence

\[
\text{haf} \, A = \sum_{j=2}^{2n} a_{1j} \text{haf} \, A_j,
\]

where \( A_j \) is the \((2n-2) \times (2n-2)\) matrix obtained from \( A \) by crossing out the first row and the first column and the \( j \)-th row and the \( j \)-th column. We observe that, up to a simultaneous permutation of rows and columns (which does not change the hafnian), any two matrices \( A_{j1}, A_{j2} \) differ only in the \( k \)-th row and \( k \)-th column for some \( k \) and the induction proceeds as in Section 2.3. \( \square \)
3. Proofs of Theorems 1.2 and 1.5

The following simple result was obtained in [B15+], we give its proof here for completeness.

(3.1) Lemma. Let \( s : \mathbb{C} \rightarrow \mathbb{C} \) be a polynomial and let \( \beta > 1 \) be real such that \( s(z) \neq 0 \) for all \( |z| \leq \beta \). Let us choose a branch of

\[
    f(z) = \ln s(z) \quad \text{for} \quad |z| \leq 1
\]

and let

\[
    T_m(z) = f(0) + \sum_{k=1}^{m} \left( \frac{d^k}{dz^k} f(z) \bigg|_{z=0} \right) \frac{z^k}{k!}
\]

be the Taylor polynomial of \( f(z) \) of degree \( m \) computed at \( z = 0 \). Then

\[
    |f(1) - T_m(1)| \leq \frac{\deg s}{(m + 1)\beta^m(\beta - 1)}.
\]

Proof. Without loss of generality, we assume that \( n = \deg s > 0 \). Let \( z_1, \ldots, z_n \in \mathbb{C} \) be the roots of \( s \). Hence we can write

\[
    s(z) = s(0) \prod_{j=1}^{n} \left( 1 - \frac{z}{z_j} \right) \quad \text{where} \quad |z_j| > \beta \quad \text{for} \quad j = 1, \ldots, n
\]

and

\[
    f(z) = f(0) + \sum_{j=1}^{n} \ln \left( 1 - \frac{z}{z_j} \right) \quad \text{for all} \quad |z| \leq 1.
\]

Using the Taylor series expansion for the logarithm, we obtain

\[
    \ln \left( 1 - \frac{1}{z_j} \right) = -\sum_{k=1}^{m} \frac{1}{k z_j^k} + \xi_j
\]

where

\[
    |\xi_j| = \left| -\sum_{k=m+1}^{+\infty} \frac{1}{k z_j^k} \right| \leq \frac{1}{(m + 1)\beta^m(\beta - 1)}.
\]

Since

\[
    T_m(1) = f(0) - \sum_{j=1}^{n} \sum_{k=1}^{m} \frac{1}{k z_j^k},
\]

the proof follows. \( \square \)
Computing the derivatives. As is discussed in [B15+], the computation of the first $m$ derivatives $f^{(1)}(0), \ldots, f^{(m)}(0)$ of $f(z) = \ln s(z)$ reduces to the computation of the first $m$ derivatives $s^{(1)}(0), \ldots, s^{(m)}(0)$ of $s$. Indeed,

$$
f^{(1)}(z) = \frac{s^{(1)}(z)}{s(z)} \quad \text{and hence} \quad s^{(1)}(z) = f^{(1)}(z)s(z)
$$

and

(3.2.1) $$s^{(k)}(0) = \sum_{j=0}^{k-1} \binom{k-1}{j} s^{(j)}(0)f^{(k-j)}(0)$$

where $s^{(0)}(0) = s(0) \neq 0$. Writing equations (3.2.1) for $k = 1, \ldots, m$ we obtain a non-singular triangular system of linear equations (with numbers $s(0) \neq 0$ on the diagonal) from which the values of $f^{(1)}(0), \ldots, f^{(m)}(0)$ can be computed in $O(m^2)$ time from the values of $s(0), s^{(1)}(0), \ldots, s^{(m)}(0)$.

It follows from Lemma 3.1 that as long as the roots of a polynomial $s(z)$ stay at least distance $\beta$ away from 0 for some fixed $\beta > 1$ then to approximate $s(1)$ within a relative error $\epsilon$, we can use the Taylor polynomial of $f(z) = \ln s(z)$ at $s = 0$ of degree $m = O(\ln \deg s - \ln \epsilon)$, where the implicit constant in the “$O$” notation depends on $\beta$ only. In view of Theorems 1.3 and 1.6, we would like to approximate $s(1)$ under a weaker assumption that there are no roots of $s$ in an $\epsilon$-neighborhood of the interval $[0, 1] \subset \mathbb{C}$. We do that by constructing a polynomial $q : \mathbb{C} \rightarrow \mathbb{C}$ which maps the disk $\{z : |z| \leq \beta\}$ into an $\epsilon$-neighborhood of $[0, 1]$ while satisfying the constraints $q(0) = 0$ and $q(1) = 1$. We then apply Lemma 3.1 to the composition $s(q(z))$. We construct the polynomial $q$ in two steps, first constructing an analytic function $F$ with similar properties and then approximating $F$ by a polynomial.

(3.3) Lemma. For a $\rho > 0$, let us define $\mathbb{D}_\rho \subset \mathbb{C}$ by

$$\mathbb{D}_\rho = \left\{ z : |z| \leq 1 - \exp \left\{ -1 - \frac{1}{\rho} \right\} \right\}$$

and let $F_\rho : \mathbb{D}_\rho \rightarrow \mathbb{C}$ be the function

$$F_\rho(z) = \rho \ln \frac{1}{1 - z},$$

where we consider the branch of $\ln w$ in the halfplane $\Re w > 0$ that is 0 at $w = 0$. Then

1. We have

$$-\rho \ln 2 \leq \Re F_\rho(z) \leq 1 + \rho \quad \text{and} \quad |\Im F_\rho(z)| \leq \frac{\pi \rho}{2}.$$
for all \( z \in \mathbb{D}_\rho \);

(2) We have \( F_\rho(0) = 0 \);

(3) We have \( F_\rho(\alpha_\rho) = 1 \) where \( \alpha_\rho = 1 - \exp \left\{-\frac{1}{\rho} \right\}, \ z_\rho \in \mathbb{D}_\rho \).

Proof. Since \( \Re (1 - z) > 0 \) for all \( z \in \mathbb{D}_\rho \), we have

\[
\Re \frac{1}{1 - z} > 0 \quad \text{for all} \quad z \in \mathbb{D}_\rho
\]

and hence the branch of \( \ln \frac{1}{1 - z} \) is well-defined for all \( z \in \mathbb{D}_\rho \) and, moreover,

\[
-\frac{\pi}{2} \leq \Im \ln \frac{1}{1 - z} \leq \frac{\pi}{2} \quad \text{for all} \quad z \in \mathbb{D}_\rho.
\]

It follows that

\[
|\Im F_\rho(z)| \leq \frac{\pi \rho}{2} \quad \text{for all} \quad z \in \mathbb{D}_\rho.
\]

Since

\[
\exp \left\{-1 - \frac{1}{\rho} \right\} \leq |1 - z| < 2 \quad \text{for all} \quad z \in \mathbb{D}_\rho,
\]

we have

\[
-\ln 2 \leq \Re \ln \frac{1}{1 - z} = -\ln |1 - z| \leq 1 + \frac{1}{\rho}
\]

and the inequality

\[
-\rho \ln 2 \leq \Re F_\rho(z) \leq 1 + \rho \quad \text{for all} \quad z \in \mathbb{D}_\rho
\]

follows, which concludes the proof of Part (1).

Parts (2) and (3) are immediate. \( \square \)

(3.4) Lemma. For \( 0 < \rho \leq 1 \), let

\[
N = N_\rho \geq 50
\]

and let us define a polynomial \( q_\rho : \mathbb{C} \longrightarrow \mathbb{C} \) of degree \( N \) by

\[
q_\rho(z) = \frac{1}{\sigma_\rho} \sum_{m=1}^{N} \frac{\alpha_m}{m} z^m \quad \text{where}
\]

\[
\alpha_\rho = 1 - \exp \left\{-\frac{1}{\rho} \right\} < 1 \quad \text{and} \quad \sigma_\rho = \sum_{m=1}^{N} \frac{\alpha_m}{m}.
\]
Then

(1) 
\[ q_\rho(0) = 0 \] \quad \text{and} \quad \[ q_\rho(1) = 1; \]

(2) 
\[ |\Im q_\rho(z)| \leq 1.7\rho \quad \text{and} \quad -0.75\rho \leq \Re q_\rho(z) \leq 1 + 1.05\rho \]

provided

\[ |z| \leq \beta_\rho \quad \text{where} \quad \beta_\rho = \frac{1 - \exp\left\{-1 - \frac{1}{\rho}\right\}}{1 - \exp\left\{-\frac{1}{\rho}\right\}} > 1. \]

**Proof.** Part (1) is immediate, it remains to prove Part (2). Let

\[ P_n(z) = \sum_{m=1}^{n} \frac{z^m}{m}. \]

Then

\[
\left| \ln \frac{1}{1-z} - P_n(z) \right| = \left| \sum_{m=n+1}^{\infty} \frac{z^m}{m} \right| \leq \frac{|z|^{n+1}}{(n+1)(1-|z|)} \quad \text{provided} \quad |z| < 1.
\]

In particular,

\[
\left| \ln \frac{1}{1-\alpha_\rho z} - P_N(\alpha_\rho z) \right|
\]

\[
\leq \frac{1}{N+1} \exp\left\{1 + \frac{1}{\rho}\right\} \left(1 - \exp\left\{-1 - \frac{1}{\rho}\right\}\right)^{N+1}
\]

\[
\leq \frac{1}{N+1} \quad \text{provided} \quad |z| \leq \beta_\rho.
\]

Let

\[ F_\rho(z) = \rho \ln \frac{1}{1-z} \]

be the function of Lemma 3.3. From (3.4.1), we obtain

\[
|F_\rho(\alpha_\rho z) - \rho P_N(\alpha_\rho z)| \leq \frac{\rho}{N+1} \leq \frac{\rho}{50} \quad \text{provided} \quad |z| \leq \beta_\rho.
\]

Since \( F_\rho(\alpha_\rho) = 1 \) and \( \sigma_\rho = P_N(1) \), from (3.4.2) we conclude that

\[
|1 - \rho \sigma_\rho| \leq 0.02\rho.
\]
From Part (1) of Lemma 3.3 and (3.4.2), we conclude that

\begin{equation}
-0.72\rho \leq \Re \rho P_N(z) \leq 1 + 1.02\rho \quad \text{and} \quad |\Im \rho P_N(z)| \leq 1.6\rho \quad \text{provided} \quad |z| \leq \beta \rho.
\end{equation}

Now,

\[ q_\rho(z) = \frac{1}{\sigma_\rho} P_N(\alpha_\rho z). \]

Combining (3.4.3) and (3.4.4), we conclude that

\[ -0.75\rho \leq \Re q_\rho(z) \leq 1 + 1.05\rho \]

and

\[ |\Im q_\rho(z)| \leq 1.7\rho, \]

as required. \(\square\)

(3.5) Proof of Theorem 1.2. Let \( A = (a_{ij}) \) be an \( n \times n \) matrix satisfying

\[ \delta \leq a_{ij} \leq 1 \quad \text{for all} \quad i, j, \]

let \( J = J_n \) be the \( n \times n \) matrix filled with 1s and let us define a univariate polynomial

\[ r(z) = \per (J + z(A - J)) \quad \text{for} \quad z \in \mathbb{C}, \]

so that \( r(0) = \per J = n! \), \( r(1) = \per A \) and \( \deg r \leq n \). For the \( k \)-th derivative of \( r(z) \) at \( z = 0 \) we have

\[ r^{(k)}(0) = \frac{d^k}{dz^k} \sum_{\sigma \in S_n} \prod_{i=1}^n (1 + z(a_{i\sigma(i)} - 1)) \bigg|_{z=0} = \sum_{\sigma \in S_n} \sum_{1 \leq i_1, \ldots, i_k \leq n} (a_{i_1\sigma(i_1)} - 1) \cdots (a_{i_k\sigma(i_k)} - 1) = (n - k)! \sum_{1 \leq i_1, \ldots, i_k \leq n} (a_{i_1j_1} - 1) \cdots (a_{i_kj_k} - 1), \]

where the last sum is taken over all pairs of ordered \( k \)-subsets \( (i_1, \ldots, i_k) \) and \( (j_1, \ldots, j_k) \) of the set \( \{1, \ldots, n\} \). Hence \( r^{(k)}(0) \) is a polynomial of degree at most \( k \) in the entries \( a_{ij} \) of the matrix \( A \) computable in \( n^{O(k)} \) time (where the implied constant in the “\( O \)” notation is absolute).

If \( \Re z \) lies in the interval \([-\delta, 1 + \delta]\), the real parts of the entries of the matrix \( J + z(A - J) \) lie in the interval \([\delta^2, 1 + \delta - \delta^2]\). Let

\[ \tau = \frac{1}{2} \left( \frac{\delta^2}{1 + \delta - \delta^2} \right)^3. \]
If $|\Im z| \leq \tau$, the imaginary parts of the entries of the matrix $J + z(A - J)$ do not exceed $\tau(1 - \delta)$ in the absolute value. Using that the permanent is a homogeneous polynomial in the matrix entries, we conclude from Theorem 1.3 that

$$r(z) \neq 0 \text{ if } -\delta \leq \Re z \leq 1 + \delta \text{ and } |\Im z| \leq \tau.$$  

(3.5.1)

Let us choose

$$\rho = \min \left\{ \frac{\delta}{1.05}, \frac{\tau}{1.7} \right\},$$

let $q(z) = q_\rho(z)$ be the polynomial of Lemma 3.4 and let

$$\beta = \beta_\rho = \frac{1 - \exp \left\{ -1 - \frac{1}{\rho} \right\}}{1 - \exp \left\{ -\frac{1}{\rho} \right\}} > 1.$$  

Hence $q(z)$ is a univariate polynomial of some degree $N = N(\delta)$ such that $q(0) = 0$, $q(1) = 1$,

$$-\delta \leq \Re q(z) \leq 1 + \delta \text{ and } |\Im q(z)| \leq \tau \text{ provided } |z| \leq \beta.$$  

(3.5.2)

Let

$$s(z) = r(q(z)) \text{ for } z \in \mathbb{C}.$$  

Then $s(z)$ is a univariate polynomial such that $\deg s \leq Nn$,

$$s(0) = r(0) = \per J = n!, \quad s(1) = r(1) = \per A.$$  

(3.5.3)

Since $q(0) = 0$, only monomials of degree not exceeding $k$ in $r(z)$ and $q(z)$ contribute to the coefficient of $z^k$ in the composition $s(z) = r(q(z))$. Consequently, the derivative $s^{(k)}(0)$ is a polynomial of degree at most $k$ in the entries $a_{ij}$ of $A$ computable in $n^{O(k)}$ time (the implied constant in the “$O$” notation is absolute).

Combining (3.5.1) and (3.5.2) we conclude

$$s(z) \neq 0 \text{ for all } |z| \leq \beta.$$  

(3.5.4)

Let $T_m(z)$ be the Taylor polynomial of $f(z) = \ln s(z)$ of degree $m$ computed at $z = 0$. Applying Lemma 3.1, from (3.5.3) and (3.5.4) we conclude that

$$|\ln \per A - T_m(1)| \leq \frac{Nn}{(m + 1)\beta^m(\beta - 1)}.$$  

In particular, for some

$$m = O \left( \ln n - \ln \epsilon \right)$$  

(3.5.5)
we obtain

\[
(3.5.6) \quad |\ln \per A - T_m(1)| \leq \epsilon,
\]

where the implicit constant in the “O” notation depends on \( \delta \) alone.

We have

\[
(3.5.7) \quad T_m(1) = f(0) + \sum_{k=1}^{m} \frac{f^{(k)}(0)}{k!}.
\]

From the triangular system of linear equations (3.2.1) with \( s(0) = n! \), we conclude that \( f^{(k)}(0) \) is a polynomial of degree at most \( k \) in the matrix entries \( a_{ij} \), computable in \( n^{O(k)} \) time. Combining that with (3.5.5)–(3.5.7), we complete the proof. \( \square \)

(3.6) Proof of Theorem 1.5. We define the univariate polynomial \( r(z) \) by

\[
\frac{d^{k}}{dz^{k}} r(z) \bigg|_{z=0} = \frac{d^{k}}{dz^{k}} \sum_{\{i_1,j_1\}, \ldots, \{i_n,j_n\}} (1 + z(a_{i_1,j_1} - 1)) \cdots (1 + z(a_{i_1,j_1} - 1)) \bigg|_{z=0} = k! \frac{(2n-2k)!}{2^{n-k}(n-k)!} \sum_{\{i_1,j_1\}, \ldots, \{i_k,j_k\}} (a_{i_1,j_1} - 1) \cdots (a_{i_k,j_k} - 1),
\]

where the last sum is taken over all unordered collection of pairwise disjoint unordered pairs of indices \( 1 \leq i_1, j_1, \ldots, i_k, j_k \leq 2n \). The proof then proceeds as in Section 3.5. \( \square \)

REFERENCES


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