A Spatially-Varying PSF Model
for Seidel Aberrations and Defocus

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ABSTRACT
Contrary to common assumptions in the literature, the blur kernel corresponding to lens-effect blur has been demonstrated to be spatially-varying across the image plane. Existing models for the corresponding point spread function (PSF) are either parameterized and spatially-invariant, or spatially-varying but ad-hoc and discretely-defined.

In this paper, we develop and present a novel, spatially-varying, parameterized PSF model that accounts for Seidel aberrations and defocus in an imaging system. We also demonstrate that the parameters of this model can easily be determined from a set of discretely-defined PSF observations, and that the model accurately describes the spatial variation of the PSF from a test camera.

Keywords: Point Spread Function, Seidel aberration model, Image Restoration

1. INTRODUCTION
The point-spread function (PSF) is a description of how a point of light is redistributed over a local area of the sensor in an imaging system. The formation of a blurred observation image \( O \) can be viewed as a linear function of the underlying sharp image \( I \) and the PSF \( P \) (Eqn. 1). Given a blurred observation \( O \), there are established methods for estimating the underlying sharp image \( I \), but these methods hinge on an accurate estimate of the PSF, and inaccurate PSF estimates have been demonstrated to result in inadequate image reconstructions

\[
O(x, y) = \sum_{i,j} I(x + i, y + j) P_{x,y}(i, j)
\]  

where \( (x, y) \) is in the coordinate-frame of the image, whereas \( (i, j) \) is in the coordinate-frame of the PSF, and the two coordinate frames differ only by translation of the origin. In the most general case, \( P \) is assumed to vary slowly as a function of the ideal image point \( (x, y) \), although in application \( P \) is commonly incorrectly assumed to be spatially-invariant.

Blur kernel models in the literature tend to fall into one of two categories: parameterized and spatially-invariant, or spatially-varying and discretely-defined. The downside of the spatially-invariant model is that both simulation\(^3\) and experiments\(^4, 5\) have demonstrated the need for \( P \) to have a spatial dependency. The downside of the discretely-defined model is that it has an unwieldily number of degrees of freedom, and requires a larger dataset for accurate kernel estimation.

In this paper, we develop and present a novel, spatially-varying PSF model which is parameterized, and which describes blur due to Seidel aberrations and defocus.

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2. IMAGING MODEL

The Seidel aberration model\textsuperscript{6} describes where a ray leaving a point $P_1'$ on the exit pupil should arrive on the focused-image plane (at aberrated arrival point $P_1^*$) (Fig. 1). The Seidel aberration model requires only 5 parameters, corresponding to the aberration coefficients representing spherical aberration ($B$), astigmatism ($C$), field curvature ($D$), radial distortion ($E$), and coma ($F$).

In Born's formulation of the aberration model, the actual optical system is abstracted away, leaving the more generally defined planes of the object, the entrance and exit pupils, and the focused-image plane.

Born’s approach, though, describes the arrival point of the aberrated ray on the focused-image plane (the conjugate plane of the object plane), which is not necessarily the plane of the image sensor. In order to take defocus into account, we use the Gaussian thick-lens geometric optics model\textsuperscript{7} to follow the path of a ray traveling from the object at point $P_0$ to the image sensor at point $P_{\text{Img}}$ (Fig. 2).

In the thick-lens model, there are three pairs of conjugate planes:

1. The focal plane and the image plane (at distances $Z_{\text{Foc}}$ and $Z_{\text{Img}}$), which represent the focus distance of the camera and the location of the image sensor, respectively.

2. The object plane and the focused-image plane (at distances $Z_{\text{Obj}}$ and $Z_{\text{Foc,Img}}$), which represent the object distance and the location where the image of the object would appear in focus in the absence of aberrations.

3. The entrance and exit pupils of the lens system, at distances $Z_{\text{Enter}}$ and $Z_{\text{Exit}}$.)
The thickness of the lens $T_{\text{Lens}}$ presents itself as an adjustment to the traditional thin-lens equation (Eqns 2-4). If the focal length of the lens is given as $f$, then the pairs of planes are related as:

\[
\begin{align*}
 f^{-1} &= (Z_{\text{Foc}} - T_{\text{Lens}})^{-1} + Z_{\text{Img}}^{-1} \\
 f^{-1} &= (Z_{\text{Obj}} - T_{\text{Lens}})^{-1} + Z_{\text{Foc,Img}}^{-1} \\
 f^{-1} &= (Z_{\text{Enter}} - T_{\text{Lens}})^{-1} + Z_{\text{Exit}}^{-1}
\end{align*}
\]  

(2)  
(3)  
(4)

In the common case of a lens containing many optical elements, the thick-lens model with offset apertures is an approximation to the overall effect of the lens, so $T_{\text{Lens}}$, $Z_{\text{Enter}}$, and $Z_{\text{Exit}}$ may be the effective thickness and offsets of the thick-lens approximation, and not the actual lens thickness or locations of the physical lens pupils.

By incorporating the Seidel aberration model into the thick-lens model, we get the aberrated point $P_1^*$ at which the ray is trying to intersect the focused-image plane, and we can therefore compute the point $P_{\text{Img}}^*$ at which the ray actually intersects the imaging sensor. Without the Seidel model, aberration would not be accounted for, and without the thick-lens model, defocus would not be accounted for, but with our combined model, the arrival point on the imaging plane at $(x_{\text{Img}}, y_{\text{Img}})$ of a ray arriving at the entrance pupil at $(x_0', y_0')$ is given by Eqns. 5-6:

\[
\begin{align*}
 x_{\text{Img}} &= x_1^* \frac{Z_2}{Z_1} \left( 1 - E \left( \frac{M^2 ((x_1^*)^2 + (y_1^*)^2)}{Z_1^2} \right) \right) + Mx_0' \left( 1 - \frac{Z_2}{Z_1} \right) \\
 &\quad + B \left( \frac{Z_2x_0'((x_0')^2 + (y_0')^2)}{M} \right) + D \left( \frac{Z_2Mx_0' ((x_1^*)^2 + (y_1^*)^2)}{Z_1^2} \right) \\
 &\quad + C \left( \frac{2Z_2Mx_1^* (x_1^*x_0' + y_1^*y_0')}{Z_1^2} \right) - F \left( \frac{Z_2 (3(x_0')^2x_1^* + (y_0')^2y_1^*)}{Z_1} \right) \\
 y_{\text{Img}} &= y_1^* \frac{Z_2}{Z_1} \left( 1 - E \left( \frac{M^2 ((y_1^*)^2 + (x_1^*)^2)}{Z_1^2} \right) \right) + My_0' \left( 1 - \frac{Z_2}{Z_1} \right) \\
 &\quad + B \left( \frac{Z_2y_0'((x_0')^2 + (y_0')^2)}{M} \right) + D \left( \frac{Z_2My_0' ((x_1^*)^2 + (y_1^*)^2)}{Z_1^2} \right) \\
 &\quad + C \left( \frac{2Z_2My_1^* (x_1^*y_0' + y_1^*y_0')}{Z_1^2} \right) - F \left( \frac{Z_2 (3(y_0')^2y_1^* + (x_0')^2x_1^* + 2x_0'y_0'x_1^*)}{Z_1} \right)
\end{align*}
\]  

(5)  
(6)

where $M$ is the magnification from entrance to exit pupil, where $Z_1$, $Z_2$ are the distances of the focused-image plane and the image plane, respectively, to the exit pupil of the system, and where $P_1^*$ is the non-aberrated arrival point of the ray on the focused-image plane. These simplifying parameters are computed from the full thick-lens model as:

\[
\begin{align*}
 M &= \frac{Z_{\text{Exit}}}{T_{\text{Lens}} - Z_{\text{Enter}}} \\
 Z_1 &= Z_{\text{Foc,Img}} - Z_{\text{Exit}} \\
 Z_2 &= Z_{\text{Img}} - Z_{\text{Exit}} \\
 P_1^* &= \left( 1 - \frac{Z_{\text{Foc,Img}}}{f} \right) P_0
\end{align*}
\]  

(7)  
(8)  
(9)  
(10)
3. PARAMETERIZED SPATIALLY-VARYING PSF MODEL

Suppose that we want to define \( P \) as a 2D Gaussian, with parameters that vary as a function of the ideal image point \((x_1^*, y_1^*)\). Given the forward imaging model, we can compute the 5 parameters of this Gaussian function, by integrating the incoming rays of light over the surface of the entrance pupil:

\[
P_{x_1^*, y_1^*}(i, j) = \frac{1}{2\pi(\sigma_x^2 \sigma_y^2 - \sigma_{xy}^2)^{1/2}} \exp \left( -\frac{1}{2} \left( \frac{(j - \mu_y)^2 \sigma_y^2 - 2(i - \mu_x)(j - \mu_y)\sigma_{xy} + (i - \mu_x)^2 \sigma_x^2}{\sigma_x^2 \sigma_y^2 - \sigma_{xy}^2} \right) \right)
\]

(11)

\[
\mu_x(x_1^*, y_1^*) = \frac{1}{\mathcal{A}} \int_{-\mathcal{R}_p}^{\mathcal{R}_p} \int_{-\mathcal{R}_p}^{\mathcal{R}_p} (x_{1mg}(x_1^*, y_1^*, x_0^*, y_0^*)) \, dx_0^* \, dy_0^*
\]

(12)

\[
\mu_y(x_1^*, y_1^*) = \frac{1}{\mathcal{A}} \int_{-\mathcal{R}_p}^{\mathcal{R}_p} \int_{-\mathcal{R}_p}^{\mathcal{R}_p} (y_{1mg}(x_1^*, y_1^*, x_0^*, y_0^*)) \, dx_0^* \, dy_0^*
\]

(13)

\[
\sigma_x^2(x_1^*, y_1^*) = \frac{1}{\mathcal{A}} \int_{-\mathcal{R}_p}^{\mathcal{R}_p} \int_{-\mathcal{R}_p}^{\mathcal{R}_p} (x_{1mg}(x_1^*, y_1^*, x_0^*, y_0^*) - \mu_x(x_1^*, y_1^*))^2 \, dx_0^* \, dy_0^*
\]

(14)

\[
\sigma_y^2(x_1^*, y_1^*) = \frac{1}{\mathcal{A}} \int_{-\mathcal{R}_p}^{\mathcal{R}_p} \int_{-\mathcal{R}_p}^{\mathcal{R}_p} (y_{1mg}(x_1^*, y_1^*, x_0^*, y_0^*) - \mu_y(x_1^*, y_1^*))^2 \, dx_0^* \, dy_0^*
\]

(15)

\[
\sigma_{xy}(x_1^*, y_1^*) = \frac{1}{\mathcal{A}} \int_{-\mathcal{R}_p}^{\mathcal{R}_p} \int_{-\mathcal{R}_p}^{\mathcal{R}_p} (x_{1mg}(x_1^*, y_1^*, x_0^*, y_0^*) - \mu_x(x_1^*, y_1^*)) (y_{1mg}(x_1^*, y_1^*, x_0^*, y_0^*) - \mu_y(x_1^*, y_1^*)) \, dx_0^* \, dy_0^*
\]

(16)

where \( \mathcal{A} = \pi \mathcal{R}_p^2 \), and \( \mathcal{R}_p \) is the radius of the entrance pupil. Eqns. 12-16 result in the necessary 5 parameters for the spatially-varying Gaussian PSF, as a function of the camera parameters, the Seidel aberration coefficients, and the location of the ideal image point:

\[
\mu_x(x_1^*, y_1^*) = x_1^* \left( \frac{(Z_2 - Z_1) - FZ_2 \mathcal{R}_p^2}{Z_1^3} - \frac{E Z_2 M^2 \left( x_1^{*2} + y_1^{*2} \right)}{Z_1^3} \right)
\]

(17)

\[
\mu_y(x_1^*, y_1^*) = y_1^* \left( \frac{(Z_2 - Z_1) - FZ_2 \mathcal{R}_p^2}{Z_1^3} - \frac{E Z_2 M^2 \left( x_1^{*2} + y_1^{*2} \right)}{Z_1^3} \right)
\]

(18)

\[
\sigma_x^2(x_1^*, y_1^*) = S_1 + S_2 \left( x_1^{*2} \right) + S_3 \left( x_1^{*2} + y_1^{*2} \right) + S_4 \left( x_1^{*4} + x_1^{*2} y_1^{*2} \right) + S_5 \left( x_1^{*2} + y_1^{*2} \right)^2
\]

(19)

\[
\sigma_y^2(x_1^*, y_1^*) = S_2 \left( x_1^{*2} \right) + S_4 \left( x_1^{*4} + x_1^{*2} y_1^{*2} \right) + S_5 \left( x_1^{*2} + y_1^{*2} \right)^2
\]

(20)

\[
\sigma_{xy}(x_1^*, y_1^*) = S_1 + S_2 \left( y_1^{*2} \right) + S_3 \left( x_1^{*2} + y_1^{*2} \right) + S_4 \left( y_1^{*4} + x_1^{*2} y_1^{*2} \right) + S_5 \left( x_1^{*2} + y_1^{*2} \right)^2
\]

(21)

The covariance matrix of \( P \) can therefore be summarized as a polynomial function of the location in the image plane, with only 5 unknown constants \( S_1, ..., S_5 \) (constant across the image plane), and where
\[
S_1 = \frac{(Z_1 - Z_2)^2 M^2 R_{Ap}^2}{4Z_1^2} + \frac{B(Z_1 - Z_2)Z_2 R_{Ap}^4}{3Z_1} + \frac{B^2 Z_2^2 R_{Ap}^6}{8M^2} \tag{22}
\]
\[
S_2 = \left(\frac{F^2 + 2BC}{3}\right) \left(\frac{Z_2 R_{Ap}^2}{Z_1}\right)^2 + C \left(\frac{Z_2(Z_1 - Z_2)M^2 R_{Ap}^2}{Z_1^3}\right) \tag{23}
\]
\[
S_3 = \left(\frac{F^2 + 2BD}{6}\right) \left(\frac{Z_2 R_{Ap}^2}{Z_1}\right)^2 + \frac{D}{2} \left(\frac{Z_2(Z_1 - Z_2)M^2 R_{Ap}^2}{Z_1^3}\right) \tag{24}
\]
\[
S_4 = \left(C^2 + CD\right) M^2 \left(\frac{Z_2 R_{Ap}}{Z_1^2}\right)^2 \tag{25}
\]
\[
S_5 = \left(\frac{DM}{2}\right)^2 \left(\frac{Z_2 R_{Ap}}{Z_1^2}\right)^2 \tag{26}
\]

These parameters are constant for a fixed \(Z_{Foc}\) and \(Z_{Obj}\), but the location \(P_0\) of the object on the object plane is allowed to vary freely in two dimensions. If the object-camera distance changed, the parameters \(S_1, ..., S_5\) would also change, through Eqns. 22–26 and the new value of \(Z_2\) as given by Eqn. 9. If the focal distance changed, however, the relationship is not so straightforward, and it remains for future work to see if the aberration coefficients \(B, C, D, E, F\) remain constant as the optical elements of the lens move to adjust the focal distance.

4. RESULTS

We verify our model with a set of observed PSFs from a Canon 550D, using a backlit pinhole setup as in Shih et. al.\(^{10}\) Given each PSF observation, it is trivial to compute the corresponding measures \(\sigma_x^2, \sigma_{xy}, \text{ and } \sigma_y^2\) numerically, and a set of \(N\) PSF observations yields a system of \(3N\) linear equations (built from Eqns. 19-21) that can be solved for \(S_1, ..., S_5\).

Our spatially-varying model consistently described the PSF observations as accurately as a 2D Gaussian fit to each individual PSF, but our approach requires only a fraction of the parameters (Fig. 3).

Figure 3. PSFs from the test camera as \(x_1^*\) spans the image plane, for a constant \(y_1^*\). Top row is the observed, discretely-defined PSF, middle row is the (spatially invariant) Gaussian fit to each independent observation, and the bottom row is our spatially-varying PSF model, evaluated at each corresponding value of \((x_1^*, y_1^*)\).

Over our full set of 128 PSF observations, the SNR of the Gaussian fit to each observation was on average only 0.29 dB better than the SNR of the spatially-varying model (average of 11.6197 dB versus 11.3307 dB), despite requiring more parameters and yielding less information about the underlying optical system.

5. CONCLUSIONS

We conclude that our novel spatially-varying PSF model, motivated from an established theoretical framework in physics, is able to accurately describe a set of PSF observations across the image plane. Although the model as presented here is based on a 2D Gaussian (described entirely by 1st and 2nd spatial moments), the framework
used to derive this model is easily extended to functions described by higher-order moments by continuing the progression laid out in Eqns. 12-16.

REFERENCES


