OPTIMAL REINSURANCE AND INVESTMENT STRATEGIES FOR INSURER UNDER INTEREST RATE AND INFLATION RISKS

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Abstract. In this paper, we investigate an optimal reinsurance and investment problem for an insurer whose surplus process is approximated by a Brownian motion with drift. Interest rate risk and inflation risk are considered. We assume that the instantaneous nominal interest rate follows an Ornstein-Uhlenbeck process, and the inflation index is given by a generalized Fisher equation. To make the market complete, zero-coupon bonds and Treasury Inflation Protected Securities (abbr. TIPS) are included in the market. The insurer can invest in the nominal bond, zero-coupon bond, TIPS and stock to hedge the risk of market. We finally derive closed-forms of the optimal reinsurance and investment strategies as well as optimal utility function under the constant relative risk aversion (abbr. CRRA) utility maximization. Sensitivity analysis is given to show the economic behavior of the optimal strategies and optimal utility.

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1. Introduction

Optimal investment strategy for insurer has recently become an important subject. The insurer can participate in the financial market to avoid risk. More recently, many works have considered the maximizing the utility of terminal value or minimizing probability of ruin for the insurer. Browne[9](1995) firstly derived the closed solution of reinsurer to maximize the exponential utility of terminal wealth and minimize the probability of ruin, the surplus process in this model is given by the

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Lundberg risk model. Hipp and Plum[21] (2000) minimized the ruin probability later when the claim was exponential distributed and was Pareto with the densities. Wang, Xia and Zhang[32](2007) solved the same problem of mean-variance efficient investment and expected constant absolute risk aversion(abbr.CARA) utility maximization by the martingale method. Many literatures can refer to, for example, Yang and Zhang[33](2005), Wang[31](2007), Liu and Yang[27](2004), Bai and Guo[2](2008) and references therein.

Besides the risk of market, the insurer also takes into account the risk of insurance. The risk of insurance can not be avoided by singly investing in the bond and other assets in the market. The business of reinsurance provides a way for insurer to hedge the risk, and this way has also recently attracted a lot of attention. Promislow and Young[29] (2005) obtained an analytic expression for the minimum probability of ruin when purchasing quota-share reinsurance was available. Bäuerle[6](2005) initially minimized the expected quadratic distance of the risk reserve to a given benchmark and successfully solved the mean-variance problem in proportional reinsurance. Zeng and Li[34](2011) also essentially gets the mean-variance efficient frontier of the diffusion model with multiple risky assets. The stock price in the above models generally follows the geometric Brownian motion and the market is often complete. But in the real market, the stock price may have other features, for example, stochastic volatility. Liang,Yuen and Guo[26](2011) maximizes the exponential utility where the instantaneous rate of the stock return follows an Ornstein-Uhlenbeck process so that the stock price can have features of bull and bear markets. Gu, Guo, Li, Zeng[17](2012) firstly model the stock price in a constant elasticity of variance (abbr.CEV) framework and then derive closed-form of the optimal reinsurance and investment policies. In Bäuerle and Blatter[7] (2011)’s work, the surplus of the insurer and the stock index in the market followed the Lévy process and optimal investment and reinsurance policies were gotten. Further more, optimal investment strategy with maximal risk aversion and its ruin probability in the presence of stochastic volatility on investments was beautifully solved by Badaoui and Fernández[1] (2013).

Based on the investment and reinsurance strategy, the insurer can successfully avoid its risk. However, the investment strategy may take a long time for the insurer, so we can not ignore the risk of interest
rate in our model. So far, few literature exists for insurer under stochastic interest rate. Elliott and Siu[12](2011) uses a game theory to solve the optimal investment for insurer when the interest rate depends on the state of an economy described by a continuous-time, finite-state, Markov chain. Most of the work of investment under stochastic interest focus on portfolio selection. In the case of stochastic interest rate, zero coupon bonds, which deliver a fixed return of $1 at maturity, are issued in the market to hedge the risk of interest rate. With the help of zero coupon bonds, we can change our market into a complete market. Bajeux-Besnainou and Portait[5](1998) firstly solved portfolio selection problem when the instantaneous interest rate was described by stochastic model. They introduced pricing kernel in their work and derived the mean-variance efficient frontier under the generalized Vasicek model. Bajeux-Besnainou, Jordan and Portait[4](2003)considered a case when the interest rate followed an Ornstein-Uhlenbeck process and got the optimal investment strategies to maximize CRRA and hyperbolic absolute risk aversion(abbr.HARA) utility for investors by martingale methods. Mean-variance problem with extended Cox-Ingersoll-Ross (abbr.CIR) stochastic interest rate model was studied by Ferland and Waiter[13](2010). Boulier, Huang and Taillard[10](2001), Josa-Fombellida and Rincón-Zapatero[23](2011) solved the optimal investment problem under stochastic interest rate in defined contribution(abbr.DC) and defined benefit(abbr.DB) pension plans, respectively.

On the other hand, the risk of inflation risk is also an important factor in financial market. To hedge the risk of inflation, in the case of optimal asset allocation with inflation, Treasury Inflation Protected Securities(abbr.TIPS) are needed. There are many TIPS in practice, in which people often use inflation-indexed zero coupon bond in the market. The model of inflation often includes nominal interest rate, real interest rate and the inflation index. The inflation index is a factor to characterize the connection between the nominal market and the real market. The most famous equation between them is given by the famous Fisher equation. Jarrow and Yildirim[22](2003) made a breakthrough in establishing the Jarow-Yildirim (abbr.JY) model to characterize the forward nominal interest rate, forward real interest rate and the inflation index. Brennan and Xia[8](2002) considered dynamic asset allocation under inflation when only nominal assets were available. Zhang, Korn and
Ewald[36](2007) extended the Fisher’s equation under the risk-neutral measure. They used the martingale method to solve the optimal investment problem in DC pension plan with inflation protection, and in their model, the nominal and real interest rate are deterministic and TIPS are included. Later, Han and Hung[18](2012) considered optimal asset allocation problem in DC pension plans with downside protection under inflation, and nominal interest rate is the CIR model.

Unfortunately, as far as we are concerned, no literature of insurer cares about the above two important risks of market at the same time. But when we concerns the optimal reinsurance and investment strategies for a long time, the both risks of interest rate and inflation should be included. More precisely, in this paper, we will concentrate on studying the optimal reinsurance and investment problem for an insurer whose surplus process is assumed to be approximated by a Brownian motion with drift under risks of interest rate and inflation. The objective of the insurer is to maximize the expected CRRA utility of the terminal real wealth, where we assume that the nominal interest rate follows an Ornstein-Uhlenbeck process, the connection among real stochastic interest rate, nominal stochastic interest rate and the stochastic inflation index is given by the famous Fisher equation. To make the market complete and hedge the risk of market, zero-coupon bonds, TIPS and stock are also included in the market. Moreover, we also assume that proportional reinsurance is allowed. By using stochastic dynamic programming method, we first derive the Hamilton-Jacobi-Bellman (abbr.HJB) equations for the problem, and then solve it by employing a variable change technique, finally derive closed-form of the optimal reinsurance and investment strategies in the dynamic optimization problem. However, since the existence of insurance, we will not get a self-financing wealth process and this makes the problem difficult. To handle this situation, auxiliary process will be introduced to make the market also be self-financing, and the auxiliary process will play an important role in solving the optimal reinsurance and investment problem for insurer.

The paper is organized as follows. The model of proportional reinsurance with stochastic nominal interest rate and inflation index is presented in Section 2, and the dynamics of zero coupon bonds and TIPS are also given. Section 3 introduces an auxiliary problem and derive the
optimal reinsurance and investment strategies by stochastic dynamic programming. Section 4 shows the numerical analysis of optimal strategies and optimal utility. Section 5 is a conclusion.

2. The model

In this section we consider an insurer for whom reinsurance is available. \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})\) is a filtered complete probability space and \(\mathcal{F}_t\) is the information available up to time \(t\). \([0,T]\) is a fixed time horizon. We assume that all the processes introduced below are adapted to \(\mathcal{F}_t\).

2.1. Market. We consider an insurer whose surplus process is described by the classical Lundberg model

\[
dX(t) = cd t - \sum_{i=1}^{N_t} Y_i,
\]

where \(N_t\) is a homogeneous Poisson process with intensity \(\lambda > 0\). \(Y_i\) is the size of the \(i\)th claim and \(Y_i, i = 1, 2, 3, \ldots\) are assumed to be independent and identically distributed (i.i.d) with finite first moment \(\mu_1 > 0\) and second moment \(\mu_2\). \(c > 0\) represents the premium intensity of the insurer. In the case with safety loading \(\eta\), we may set \(c = \lambda \mu_1 (1 + \eta)\) with \(\eta > 0\).

In addition, proportional reinsurance is allowed for the insurer. The insurer can divide \(100(1 - a(t))\%\) of its insurance risk to a reinsurer. For the \(i\)th claim \(Y_i\), the insurer pays \(a(t)Y_i\) while the reinsurer pays \((1 - a(t))Y_i\). To achieve this, the insurer has to pay a premium at the rate of \((1 + \theta)\lambda \mu_1 (1 - a(t))(\theta > 0)\) to the reinsurer based on the expectation principle. In general, \(\theta > \eta\), otherwise arbitrage will exist.

The insurer can hedge its insurance risk by the reinsurance strategy \(a(t)\). If \(a(t)\) is small, the insurer takes a little risk of insurance by himself and divides most of the risk to the reinsurer. \(a(t) > 1\) means taking new reinsurance business. In this case, the surplus process \(X(t)\) takes the following form:

\[
dX(t) = \lambda \mu_1[a(t)(1 + \theta) - (\theta - \eta)]dt - a(t)\sum_{i=1}^{N_t} Y_i. \quad (2.1)
\]

Following the same procedure as in Grandll[16] (1991), the above process can be approximated by the following diffusion process:

\[
dX(t) = \lambda \mu_1(\eta - \theta)dt + \lambda \mu_1 \theta a(t)dt + \sqrt{\lambda \mu_2} a(t)dW_0(t), \quad (2.2)
\]

where \(W_0(t)\) is a standard Brownian motion on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})\).
In the market, we assume that the instantaneous nominal rate $r_n(t)$ and the inflation index $I(t)$ are stochastic processes while the instantaneous real rate $r_r(t)$ is deterministic function of $t$ to simplify the model.

The stochastic nominal rate $r_n(t)$ follows the following Ornstein-Uhlenbeck equation of the form:

$$d r_n(t) = a(b - r_n(t))dt - \sigma_{r_n}dW_{r_n}(t),$$

where $a, b, \sigma_{r_n}$ are positive constants and $W_{r_n}(t)$ is a standard Brownian motion, and it is independent of $W_0(t)$.

The relationship between real interest rate, nominal interest rate and the inflation risk is given by the Fisher equation. We can derive the model of the inflation index based on the extended Fisher’s equation given by Zhang [36] (2007):

$$r_n(t) - r_r(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E}[i(t, t + \Delta t) | \mathcal{F}_t]$$

$$i(t, t + \Delta t) = \frac{I(t + \Delta t) - I(t)}{I(t)}$$

where $\mathbb{E}$ is the expectation with respect to the risk neutral measure $\mathbb{P}$. $i(t, t + \Delta t)$ is the inflation rate within time horizon $[t, t + \Delta t]$, and the stochastic inflation index $I(t)$ is given by the following stochastic differential equation:

$$\frac{dI(t)}{I(t)} = (r_n(t) - r_r(t))dt + \sigma_{I_r}d\tilde{W}_{r_n}(t) + \sigma_{I_I}d\tilde{W}_I(t).$$

where $\tilde{W}_{r_n}(t)$ and $\tilde{W}_I(t)$ are standard Brownian motions under the risk-neutral measure $\tilde{\mathbb{P}}$.

Assume that the market price of risk of $W_I(t)$ is $\lambda_I$, then by the Girsanov theorem we know that the model of the stochastic inflation index $I(t)$ under the original measure $P$ is also as follows:

$$\frac{dI(t)}{I(t)} = (r_n(t) - r_r(t))dt + \sigma_{I_r}[\lambda_{r_n}dt + dW_{r_n}(t)] + \sigma_{I_I}[\lambda_Idt + dW_I(t)].$$

The riskless asset price $S_0(t)$ evolves according to

$$\frac{dS_0(t)}{S_0(t)} = r_n(t)dt, \quad S_0(0) = 1.$$
differential equation:
\[
\begin{aligned}
\frac{\partial B_n(t,T)}{\partial t} &+ [a(b - r_n) + \lambda_r \sigma_{r_n}] \frac{\partial B_n(t,T)}{\partial r_n} + \frac{1}{2} \sigma_{r_n}^2 \frac{\partial^2 B_n(t,T)}{\partial r_n^2} = r_n B(t,T), \\
B(T, T) &= 1,
\end{aligned}
\]  
(2.7)

where \( \lambda_{r_n} \) is the market price of risk of \( W_{r_n}(t) \). Then \( B_n(t,T) \) has the following closed-form:
\[
B_n(t,T) = \exp[r_n(t)C(t,T) - A(t,T)],
\]  
(2.8)

where \( C(t,T) = \frac{e^{-a(T-t)}}{a}, A(t,T) = - \int_t^T [(ab + \lambda_{r_n} \sigma_{r_n})C(s,T) + \frac{1}{2} \sigma_{r_n}^2 C(s,T)^2] ds \).

In addition, \( B_n(t,T) \) also satisfies the following backward stochastic differential equation (abbr. BSDE):
\[
\begin{aligned}
\frac{dB_{K_1}(t)}{B_{K_1}(t)} &= r_n(t) dt + \sigma_{B_1}(T-t) [\lambda_{r_n} dt + dW_{r_n}(t)], \\
B_n(T, T) &= 1,
\end{aligned}
\]  
(2.9)

where \( \sigma_{B_1}(t) = \frac{1-e^{-at}}{a} \sigma_{r_n} \).

Boulier, Huang and Taillard [10] (2001) stated that there may not exist zero-coupon bonds with any maturity \( t > 0 \) in the market, so we need to introduce a rolling bond with a constant maturity \( K_1 \) in the market.

The rolling bond \( B_{K_1}(t) \) follows the stochastic differential equation:
\[
\frac{dB_{K_1}(t)}{B_{K_1}(t)} = r_n(t) dt + \sigma_{B_1}(K_1) [\lambda_{r_n} dt + dW_{r_n}(t)].
\]  
(2.10)

The relationship between \( B_{K_1} \) and \( B_n(t,T) \) is given by
\[
\frac{dB_n(t,T)}{B_n(t,T)} = (1 - \frac{\sigma_{B_1}(T-t)}{\sigma_{B_1}(K_1)}) \frac{dS_0(t)}{S_0(t)} + \frac{\sigma_{B_1}(T-t)}{\sigma_{B_1}(K_1)} \frac{dB_{K_1}(t)}{B_{K_1}(t)}. 
\]  
(2.11)

We can see that the stochastic inflation index model in this paper is a particular case of the JY model in Jarrow and Yildirim [22] (2003) when the nominal interest rate is an Ornstein-Uhlenbeck process and the real interest rate is deterministic. To reduce the risk of inflation, TIPS are issued in the market. We consider a particular TIPS named indexed zero coupon bond \( P(t,T) \) with a final payment of \( I(t) \) at maturity \( T \).

The bond \( P(t,T) \) satisfies the following partial differential equation:
\[
\begin{aligned}
\frac{\partial P}{\partial t} + \frac{P}{\sigma_{r_n}} [a(b - r_n) + \lambda_{r_n} \sigma_{r_n}] + P T (r_n - r_r) + \frac{1}{2} P r_{r_n} \sigma_{r_n}^2 \\
+ \frac{1}{2} P T r_r^2 (\sigma_{I_1}^2 + \sigma_{I_2}^2) - P r_{r_n} I \sigma_{r_n} \sigma_{I_1} = r_n P,
\end{aligned}
\]  
(2.12)

The closed-form of \( P(t,T) \) is given by
\[
P(t,T) = I(t) \exp[- \int_t^T r_r(s) ds].
\]  
(2.13)
In addition, \( P(t, T) \) satisfies the following BSDE:
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dP(t, T)}{P(t, T)} = r_n(t)dt + \sigma_{I_1}[\lambda_{r_n}dt + dW_{r_n}(t)] + \sigma_{I_2}[\lambda_Idt + dW_I(t)], \\
\end{array}
\right.
\quad P(T, T) = 1.
\end{align*}
\] (2.14)

We also consider a rolling indexed bond with constant maturity \( K_2 \) satisfying
\[
\frac{dP_{K_2}(t)}{P_{K_2}(t)} = r_n(t)dt + \sigma_{I_1}[\lambda_{r_n}dt + dW_{r_n}(t)] + \sigma_{I_2}[\lambda_Idt + dW_I(t)].
\] (2.15)

The relationship between \( P_{K_2}(t) \) and \( P(t, T) \) is
\[
dP(t, T) = \frac{dP_{K_2}(t)}{P_{K_2}(t)} P(t, T).
\] (2.16)

We can see that the differential of TIPS \( P(t, T) \) is not correlated with its maturity \( T \).

Besides, there is a stock in the market, and we assume that the price of the stock follows the following stochastic differential equation:
\[
\frac{dS_1(t)}{S_1(t)} = r_n(t)dt + \sigma_{S_1}[\lambda_{r_n}dt + dW_{r_n}(t)] + \sigma_{S_2}[\lambda_Idt + dW_I(t)] \\
+ \sigma_{S_3}[\lambda_Sdt + dW_S(t)].
\] (2.17)

where \( \lambda_S \) is the market price of risk of standard Brownian motion \( W_S(t) \) on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})\), and \( W_0(t), W_{r_n}(t), W_I(t), W_S(t) \) are independent.

In the above market, the wealth \( X(t) \) of the insurer must satisfy the following SDE:
\[
dX(t) = \lambda_{I_1}(\eta - \theta)dt + \lambda_{I_2}a(t)dt + \sqrt{\lambda_{I_2}a(t)}dW_0(t) \\
+ \theta_0(t) \frac{dS_0(t)}{S_0(t)} + \theta_B(t) \frac{dB_{K_1}(t)}{B_{K_1}(t)} + \theta_P(t) \frac{dP_{K_2}(t)}{P_{K_2}(t)} + \theta_S(t) \frac{dS_1(t)}{S_1(t)},
\] (2.18)

where \( \theta_0(t), \theta_B(t), \theta_P(t), \theta_S(t) \) are the money invested in the risk-free bond, zero coupon bond, TIPS and the stock, respectively. The wealth of our model also satisfies \( X(t) = \theta_0(t) + \theta_B(t) + \theta_P(t) + \theta_S(t) \). We call \( \pi(t) = (a(t), \theta_B(t), \theta_P(t), \theta_S(t))^T \) a strategy. \( \pi(t) \) is a combination of the reinsurance strategy and the investment strategy. We say \( \pi(t) \) an admissible strategy if \( \pi(t) \) is adapted to the filtration \( \mathcal{F} = \{\mathcal{F}_t\}_{t\in[0,T]} \) and the reinsurance strategy \( a(t) \) in \( \pi(t) \) is not less than zero. Besides, the wealth process \( X(t) \) corresponding to \( \pi(t) \) should satisfy \( X(t) \geq 0 \). Substituting (2.6), (2.10), (2.15) and (2.17) into the last equation above, we can rewrite \( X(t) \) in compact form:
\[
dX(t) = \lambda_{I_1}(\eta - \theta)dt + \pi(t)^T \sigma[Adt + dW(t)],
\] (2.19)
where
\[
\Lambda = \begin{pmatrix}
\lambda \mu_1 \\
\lambda \mu_2 \\
\lambda_1 \\
\lambda_2
\end{pmatrix}, \quad \sigma = \begin{pmatrix}
\sqrt{\lambda} & 0 & 0 & 0 \\
0 & \sigma_{B_1}(K_1) & 0 & 0 \\
0 & \sigma_{I_1} & \sigma_{I_2} & 0 \\
0 & \sigma_{S_1} & \sigma_{S_2} & \sigma_{S_3}
\end{pmatrix}, \quad dW(t) = \begin{pmatrix}
dW_0(t) \\
dW_{r_n}(t) \\
dW_I(t) \\
dW_S(t)
\end{pmatrix}.
\]

2.2. The optimization problem. In this paper, our goal is to maximize the expected utility of the terminal wealth by continuously adjust reinsurance and investment strategies within time horizon \([0, T]\). Because inflation risk exists in the market, we need to maximize the expected utility of the real value of the terminal wealth \(X(T)\). So the optimization problem can be written as follows:
\[
\begin{aligned}
\max & \mathbb{E}[U(\frac{X(T)}{H(T)})] \\
\text{subject to:} & \\
X(0) &= x \\
\pi(t) & \text{admissible}
\end{aligned}
\tag{2.20}
\]

There are many utility functions in practice. In this paper we consider the CRRA utility function:
\[
U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma > 0 \quad \text{and} \quad \gamma \neq 1,
\tag{2.21}
\]
where \(\gamma\) is the relative risk aversion.

3. Solution of the optimization problem

The optimization problem (2.20) is not a classical self-financing problem. The insurer has a continuous income of the premium. So maybe we can not solve the problem via the traditional methods. Besides the problem involves reinsurance strategy and investment strategies, it is not a single investment problem. The existence of reinsurance can effect the solution of the optimal strategy. But similar to the single-agent consumption and investment problem in Karatzas and Shreve[24](1998), we have the following lemma about \(X(t)\):

**Lemma 3.1.** Let \(H(t) = \exp\{\int_0^t (r_n(s) + \frac{1}{2} \| \Lambda \|^2)ds + \int_0^t \Lambda^T dW(s)\}\) and \(H(t)\) satisfy the following SDE:
\[
\frac{dH(t)}{H(t)} = [r_n(t) + \Lambda^T \Lambda]dt + \Lambda^T dW(t), \quad H(0) = 1,
\]
then the \(X(t)\) must be the following form:
\[
X(t) = \mathbb{E}[ - \int_t^T \frac{\lambda \mu_1(\eta - \theta)H(t)}{H(s)} ds + \frac{X(T)H(T)}{H(T)} | \mathcal{F}_t], \quad t \in [0, T].
\]
Proof. Applying Itô formula to the process \( \frac{X(t)}{H(t)} \), we have

\[
\begin{align*}
\frac{X(t)}{H(t)} = & \lambda \mu_1(\eta - \theta) H^{-1}(t) dt + [H^{-1}(t)t^T(t) - X(t)H^{-1}t] dW(t). \\
\end{align*}
\] (3.1)

Then the differential of \( \frac{X(t)}{H(t)} + \int_t^T \frac{\lambda \mu_1(\eta - \theta)}{H(s)} ds \) is

\[
\begin{align*}
\frac{X(t)}{H(t)} + \int_t^T \frac{\lambda \mu_1(\eta - \theta)}{H(s)} ds = & \frac{H^{-1}(t) u^T(t) - X(t)H^{-1}t}{H(t)} dW(t). \\
\end{align*}
\] (3.2)

So we see that \( \{ \frac{X(t)}{H(t)} + \int_t^T \frac{\lambda \mu_1(\eta - \theta)}{H(s)} ds, 0 \leq t \leq T \} \) is a martingale. Thus this lemma follows. \( \square \)

Similar to the general investment problem, \( H(t) \) may act as the pricing kernel of the financial market. But because of the existence of reinsurance risk, \( H(t) \) is in fact the combination of risk of reinsurance and the financial market. Besides, in the general case of self-financing problem, we simply have \( X(t) = E[\frac{X(T)}{H(T)} | \mathcal{F}_t] \), which means that \( X(t) \) is a martingale under the risk neutral measure. The term \( \lambda \mu_1(\eta - \theta) dt \) in(2.19) acts as a continuously outcome of the wealth. When the insurer chooses the reinsurance and investment strategies, the effect of the outcome should also be considered. We denote \( F(t) = E[\int_t^T \frac{\lambda \mu_1(\eta - \theta) H(t)}{H(s)} ds | \mathcal{F}_t] \). It can be seen as the discounted expected value of the continuous outcome \( \lambda \mu_1(\eta - \theta) dt \) of the wealth \( X(t) \), and we can calculate \( F(t) \) via the following lemma:

**Lemma 3.2.** The discounted value \( F(t) \) can be written as \( F(t) = \lambda \mu_1(\eta - \theta) \int_t^T B_n(t,s) ds \), and \( F(t) \) satisfies the following BSDE:

\[
\begin{align*}
\{dF(t) = & -\lambda \mu_1(\eta - \theta) dt + F(t)[r_n(t) + \lambda \sigma_F(t,T)] dt + F(t) \sigma_F(t,T) dW_n(t), \\
F(T) = & 0, \\
\end{align*}
\] (3.3)

where \( \sigma_F(t,T) = \int_t^T \frac{\lambda \mu_1(\eta - \theta) \sigma_F(s-T) B_n(s,t)}{F(t)} ds \).

Proof. Rewriting \( F(t) \) as the following form:

\[
F(t) = \lambda \mu_1(\eta - \theta) \int_t^T E[\frac{H(t)}{H(s)} | \mathcal{F}_t] ds, 
\] (3.4)
we see that it suffices to calculate $\mathbb{E}\left[ \frac{H(t)}{H(s)} \mid \mathcal{F}_t \right], s \geq t$. By the independence of $W_0(t)$, $W_{rn}(t)$, $W_I(t)$ and $W_S(t)$, it easily follows

$$
\mathbb{E}\left[ \frac{H(t)}{H(s)} \mid \mathcal{F}_t \right] = \mathbb{E}\left[ -\int_t^s (r_n(u) + \frac{1}{2} \| \Lambda \|^2) du - \int_t^s \Lambda^T dW(u) \mid \mathcal{F}_t \right] \\
= \mathbb{E}\left[ -\int_t^s (r_n(u) + \frac{1}{2} \lambda^2) du - \int_t^s \lambda_n dW_r(u) \mid \mathcal{F}_t \right] = \tilde{\mathbb{E}}\left[ -\int_t^s (r_n(u) \mid \mathcal{F}_t) \right] \\
= B_n(t, s).
$$

(3.5)

So $F(t) = \lambda \mu_1 (\eta - \theta) \int_t^T B_n(t, s) ds$. Differentiating it directly, we obtain the second equation. \qed

3.1. **Construction of an auxiliary problem.** In this paper we introduce an auxiliary process $Y(t)$ defined by $Y(t) = X(t) + F(t)$ with the initial value $F(0) = f$. Substituting (2.19) and (3.3) into the above equation, we have

$$
dY(t) = dX(t) + dF(t) \\
= r_n(t) Y(t) dt + \begin{pmatrix} a(t) \\
\theta_B(t) \\
\frac{F(t) \sigma_F(t, T)}{\sigma_B(t, K_1)} \\
\theta_P(t) \\
\theta_S(t) \\
\end{pmatrix}^T \sigma [dt + dW(t)] \\
= r_n(t) Y(t) dt + u(t)^T \sigma [dt + dW(t)],
$$

(3.6)

where $u(t) = \bar{a}(t) + \begin{pmatrix} F(t) \sigma_F(t, T) \\
\sigma_B(t, K_1) \\
0, 0 \end{pmatrix}^T$.

Because $F(T) = 0$ and what we are cared about is the terminal value at time $T$, we can change the original problem (2.20) into the following auxiliary self-financing problem:

$$
\left\{ \begin{array}{l}
\max \mathbb{E}[U(Y(T))] \\
\text{subject to :} \\
Y(0) = x + f \\
u(t) \text{ admissible}
\end{array} \right. 
$$

(3.7)

We can see that compared with the self-financial problem, the insurer should buy more zero-coupon bond to hedge the risk of market due to the outcome of the wealth. In addition, $Y(0) \geq 0$ should be satisfied in our model, otherwise bankruptcy may take place within $[0, T]$.

3.2. **Solution to the auxiliary problem.** As the problem introduced above is a self-financing problem, it is solvable. There are mainly two methods to solve it, one is the stochastic dynamic programming method,
another one is martingale method. In this paper, we will solve it by the former. Define

\[ V(t, r_n, I, y) = \max_{u(t)} \mathbb{E}\{ U(Y(T)) | r_n(t) = r_n \text{ and } I(t) = I, Y(t) = y \}, \]

we have the following.

**Theorem 3.3.** The associated HJB equation of the auxiliary problem (3.7) is

\[
\begin{aligned}
& \sup \left\{ \frac{V_t + V_y [r_n y + u^* T(t)] \sigma \Lambda + V_{r_n} a(b - r_n) + V_I I(r_n - r_r + \sigma_{I^2}) + \gamma \sigma_I \gamma_I + \frac{1}{2} V_{yy} u^* T(t) \sigma \sigma^T u^* (t) + \frac{1}{2} V_{r_n r_n} \sigma_I^2 \sigma_r + \frac{1}{2} V_{I I} I^2 \sigma_I^2 \right\}, \\
& \quad \left\{ V_{y r_n} u^* T(t) \sigma \sigma_r + V_{y I} I \sigma_I + V_{r_n} I \sigma_r \right\} = 0, 
\end{aligned}
\]

where \( \sigma_r = (0, -\sigma_{r_n}, 0, 0)^T \) and \( \sigma_I = (0, \sigma_{I_1}, \sigma_{I_2}, 0) \).

**Proof.** The proof is very standard, see Merton [28](1969), Fleming and Soner [14](1993), Vigna and Haberman [30](2001), He and Liang [19](2009) and references therein, we omit it here. \( \square \)

We can get the optimal feedback function \( u^*(t, y) \) by its first order condition. The expression of \( u^*(t, y) \) is:

\[
u^*(t, y) = -\frac{V_y \Sigma^{-1} \sigma \Lambda}{V_{yy}} - \frac{V_{y I} \Sigma^{-1} \sigma \sigma_I}{V_{yy}} - \frac{V_{y r_n} \Sigma^{-1} \sigma \sigma_r}{V_{yy}}, \]

where \( \Sigma = \sigma \sigma^T \). Substituting \( u^*(t, y) \) into the HJB equation, we can get the closed-form of \( V(t, r_n, I, y) \) and thus the optimal strategy \( u^*(t) = u^*(t, Y^*(t)) \), where \( Y^*(t) \) is uniquely determined by SDE (3.6) with replacing the coefficient \( u^*(t) \) there by \( u^*(t, Y^*(t)) \). So we have the following proposition:

**Proposition 3.4.** The optimal reinsurance-investment strategy \( u^*(t) \) is

\[
\frac{u^*(t)}{\gamma} = \frac{Y^*(t)}{\gamma} \Sigma^{-1} \sigma \Lambda + \frac{1}{\gamma} Y^*(t) \Sigma^{-1} \sigma \sigma_I \\
= \frac{X^*(t) + F(t)}{\gamma} \left( \begin{array}{c} \frac{\lambda_r}{\sigma_{B_1(K_1)}} - \frac{\mu_1 \theta}{\mu_2} \\ \frac{\lambda_I \sigma_{P_1}}{\sigma_{B_1(K_1)} \sigma_{P_2}} + \frac{\lambda_r \sigma_{P_1}}{\sigma_{B_1(K_1)} \sigma_{P_2} \sigma_{S_3}} \\ \frac{\lambda_r \sigma_{S_1}}{\sigma_{B_1(K_1)} \sigma_{S_3}} \\ \frac{\lambda_r \sigma_{S_2}}{\sigma_{B_1(K_1)} \sigma_{S_3}} \end{array} \right) \\
+ \left( 1 - \frac{1}{\gamma} \right) (X^*(t) + F(t)) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},
\]

(3.10)
The closed-form of $V(t,r_n,I,y)$ is

$$V(t,r_n,I,y) = \frac{1}{1-\gamma}(\frac{y}{I})^{1-\gamma}h(t),$$

(3.11)

where

$$h(t)=\exp\left[\int_t^T (\gamma-1)[-r_s+\lambda_s+\sigma_s T \frac{1}{2\gamma}\Lambda T \Lambda-(1-\frac{1}{\gamma})\Lambda T \sigma_T \frac{1}{2\gamma} \sigma_T \sigma_I]ds\right].$$

Proof. See Appendix.

The first term in $u^*(t)$ is the general form in self-financing framework. It maximize the utility in the self-financing case with initial value $x + f$. The second term is to invest only in the TIPS. The role of it is to hedge the risk of inflation. In our model, the nominal interest rate and the inflation are closely correlated. Besides, what we concerned about is the real value. In our model the real interest rate is deterministic. So we only need to hedge the risk of inflation and the risk of interest rate can be ignored. We can see that the optimal utility $V(t,r_n,I,y)$ is also not correlated with the nominal interest rate.

3.3. Solution to the original problem. Once we get the solution of the auxiliary problem, we can easily derive the solution to the original problem. The optimal reinsurance-investment strategy of the original problem is

$$u^*(t) = u(t) - (0, \frac{F(t)\sigma_F(t,T)}{\sigma_B(K_1)}, 0, 0)^T,$$

(3.12)

i.e., because there is a continuously outcome in our model, we have to borrow $\frac{F(t)\sigma_F(t,T)}{\sigma_B(K_1)}$ zero coupon bond to get the optimal utility.

3.4. Transform the optimal strategies. We can observe that $Y^*(t)$ exists in the optimal strategies $\pi^*(t)$. But in the market $Y^*(t)$ is not observable. In the form $Y^*(t) = X^*(t)+F(t), F(t) = \lambda\mu_1(\eta-\theta) \int_t^T B_n(t,s)ds$, we can approximate $F(t)$ by zero coupon bonds with different maturities in the market and thus obtain $Y^*(t)$. But we may not have so many zero coupon bonds in the market and the method is not such accurate. We can transform $Y^*(t)$ in terms of the assets and indexes in the market. By observing the value of the assets and indexes, we can easily get $Y^*(t)$ and thus the optimal strategies.

Substituting (3.10) into (3.6), we have

$$dY^*(t) = Y^*(t)\{[r_n(t)+\frac{1}{\gamma}\Lambda_T \Lambda+(1-\frac{1}{\gamma})\sigma_T^T \Lambda]dt+[\frac{1}{\gamma}\Lambda_T + (1-\frac{1}{\gamma})\sigma_T^T]dW(t)\}.$$

(3.13)
The above formula indicates that $Y^*(t)$ follows a geometric Brownian motion. Because in the market we do not have any assets to represent the insurance risk. In order to represent $Y^*(t)$, we firstly introduce a fictitious asset $Z(t)$ defined by

$$\frac{dZ(t)}{Z(t)} = \frac{\lambda \mu_1 \theta}{\gamma \sqrt{\lambda \mu_2}} dW_0(t). \quad (3.14)$$

By conjugation, we rewrite $Y^*(t)$ as the following:

$$Y^*(t) = (x + f) e^{\mu t} \left( \frac{S_0(t)}{S_0(0)} \right)^{\alpha_1 (B_{K_1}(t) - B_{K_1}(0))}\alpha_2 \left( \frac{P_{K_2}(t)}{P_{K_2}(0)} \right)^{\alpha_3} \left( \frac{S_1(t)}{S_1(0)} \right)^{\alpha_4} \frac{Z(t)}{Z(0)}. \quad (3.15)$$

Differentiating $Y^*(t)$, and then comparing it with (3.13), the parameters should satisfy the following equations:

$$\begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} \sigma_{B_1(K_1)} & \sigma_{I_1} & \sigma_{S_1} \\ 0 & \sigma_{I_2} & \sigma_{S_2} \end{pmatrix}^{-1} \begin{pmatrix} \Lambda_m \\ \lambda_1 \\ \lambda_2 \end{pmatrix} + \frac{1}{\gamma} \begin{pmatrix} \sigma_{B_1(K_1)} & \sigma_{I_1} & \sigma_{S_1} \\ 0 & \sigma_{I_2} & \sigma_{S_2} \end{pmatrix}^{-1} \begin{pmatrix} \sigma_{I_1} \\ \sigma_{I_2} \end{pmatrix}. \quad (3.16)$$

Moreover,

$$\begin{align*}
\alpha_1 &= 1 - \alpha_2 - \alpha_3 - \alpha_4, \\
m &= \frac{1}{\gamma} \lambda^T \Lambda + \frac{1}{\gamma} \sigma_{I_1}^T \Lambda - \alpha_2 \lambda_r \sigma_{B_1(K_1)} - \frac{1}{2} \alpha_2 (\alpha_2 - 1) \sigma_{B_1(K_1)}^2 \\
&\quad - \alpha_3 (\sigma_{I_1} \lambda_r + \sigma_{I_2} \lambda_t) - \frac{1}{2} \alpha_3 (\alpha_3 - 1) (\sigma_{I_1}^2 + \sigma_{I_2}^2) - \alpha_4 (\sigma_{S_1} \lambda_{r_1}) \\
&\quad + \sigma_{S_2} \lambda_t + \sigma_{S_1} \lambda_r - \frac{1}{2} \alpha_4 (\alpha_4 - 1) (\sigma_{S_1}^2 + \sigma_{S_2}^2 + \sigma_{S_3}^2) \\
&\quad - \alpha_2 \alpha_3 \sigma_{B_1(K_1)} \sigma_{I_1} - \alpha_2 \alpha_4 \sigma_{B_1(K_1)} \sigma_{S_1} - \alpha_3 \alpha_4 (\sigma_{I_1} \sigma_{S_1} + \sigma_{I_2} \sigma_{S_2}).
\end{align*}$$

By these parameters, we can express $Y^*(t)$ in terms of $S_0(t)$, $B_{K_1}(t)$, $P_{K_2}(t)$ and $Z(t)$. Because $Z(t)$ is a fictitious asset, it does not exist in the market. However, we can observe it by the claims in the insurance market. In fact, we have a closed-form of $Z(t)$:

$$Z(t) = Z(0) \exp \left[ -\frac{\lambda \mu_1 \theta^2}{2 \gamma^2 \mu_2} t + \frac{\lambda \mu_1 \theta}{\gamma \sqrt{\lambda \mu_2}} W_0(t) \right]. \quad (3.17)$$

Since the initial value of $Z(t)$ can be arbitrarily chosen, so for simplicity we set $Z(0) = 1$. As we approximate the claims by $W_0(t)$ in the preceding research, we can also approximate $Z(t)$ by the claims and thus $Z(t)$ is observable in the market. Also following Grandll[16](1991), we calculate $Z(t)$ by

$$Z(t) = \exp \left[ -\frac{\lambda \mu_1 \theta^2}{2 \gamma^2 \mu_2} t + \frac{\lambda \mu_1 \theta}{\gamma \mu_2} t + \frac{\mu_1 \theta}{\gamma \mu_2} \sum_{i=1}^{N_t} Y_i \right]. \quad (3.18)$$
4. Sensitivity analysis

In this section, we give some numerical examples to show how the optimal strategies and optimal utility vary. In contrast to the case of self-financial problem, neither the optimal invest amounts nor the optimal proportions are deterministic, so we can only study the exact amounts of mean allocations or mean proportions. Firstly, we have the following:

**Proposition 4.1.** The expectation of $Y(t)$ is

$$E[Y^*(t)] = Y_0 \exp \left\{ (r_0 - b) \frac{1 - \exp(-at)}{a} + bt + \frac{1}{\gamma} \Lambda^T \Lambda t \right\} + \left( 1 - \frac{1}{\gamma} \right) \sigma_f^2 T \Lambda t + \frac{\sigma_r^2}{2a^2} \left[ t + \frac{2 \exp(-at)}{a} - \frac{\exp(-2at)}{2a} - \frac{3}{2a} \right] - \sigma_r \left[ \frac{1}{\gamma} \lambda_r + (1 - \frac{1}{\gamma}) \sigma_f \left[ \frac{t}{a} - \frac{1}{a^2} + \frac{e^{-at}}{a^2} \right] \right].$$

**Proof.** See Appendix.

Next, we investigate the sensitivity analysis of the optimal investment and reinsurance strategies. Unless otherwise stated, the data we adopt for the model is listed in the following table.

**Table 1.** values of the parameters in the financial market

<table>
<thead>
<tr>
<th>text interpretation</th>
<th>symbol</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>risk aversion</td>
<td>$\gamma$</td>
<td>2</td>
</tr>
<tr>
<td>proportional reinsurance</td>
<td></td>
<td></td>
</tr>
<tr>
<td>insurance intensity</td>
<td>$\lambda$</td>
<td>3</td>
</tr>
<tr>
<td>mean of the claim</td>
<td>$\mu_1$</td>
<td>0.08</td>
</tr>
<tr>
<td>variance of the claim</td>
<td>$\mu_2$</td>
<td>0.05</td>
</tr>
<tr>
<td>safety loading of insurer</td>
<td>$\eta$</td>
<td>0.05</td>
</tr>
<tr>
<td>safety loading of reinsurer</td>
<td>$\theta$</td>
<td>0.1</td>
</tr>
<tr>
<td>nominal interest rate</td>
<td>$r_0$</td>
<td>0.05</td>
</tr>
<tr>
<td>initial value</td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean reversion</td>
<td>$a$</td>
<td>0.1</td>
</tr>
<tr>
<td>mean rate</td>
<td>$b$</td>
<td>0.03</td>
</tr>
<tr>
<td>volatility of nominal interest rate</td>
<td>$\sigma_{rn}$</td>
<td>0.01</td>
</tr>
<tr>
<td>real interest rate</td>
<td>$r_r$</td>
<td>0.045</td>
</tr>
<tr>
<td>volatility of inflation index</td>
<td>$(\sigma_{I_1}, \sigma_{I_2})$</td>
<td>(0.08,0.05)</td>
</tr>
<tr>
<td>maturity of rolling zero coupon bond</td>
<td>$K_1$</td>
<td>10</td>
</tr>
<tr>
<td>volatility of stock</td>
<td>$(\sigma_{S_1}, \sigma_{S_2}, \sigma_{S_3})$</td>
<td>(0.1,0.08,0.1)</td>
</tr>
<tr>
<td>time horizon</td>
<td>$T$</td>
<td>20</td>
</tr>
<tr>
<td>initial money</td>
<td>$x_0$</td>
<td>1</td>
</tr>
<tr>
<td>initial value of inflation index</td>
<td>$I(0)$</td>
<td>1</td>
</tr>
</tbody>
</table>
4.1. Sensitivity analysis of the optimal investment strategies. Firstly, we study the impact of parameters on the optimal reinsurance and investment strategies. Figure 1 shows that we invest a lot in the zero-coupon bond, increasing greatly over time, i.e., the zero-coupon bond curve goes fast from 0.55 at time 0 to 0.95 at time 20, and however the TIPS has been relatively stable. But Figure 2 shows that the proportion of zero-coupon bond is in fact decreasing, while the proportion of stock is increasing slowly. The reinsurance proportion also increases slowly, which means that we are dividing less insurance risk as time goes by. An interesting thing is that the risk of inflation is not such important to us and we only need to short a few TIPS to hedge the risk of inflation. The proportions of bond, stock and TIPS changes a little in the figure.

Figure 3 shows the optimal mean allocation and reinsurance strategies when $\gamma = 4$. In this case, the money invested in TIPS is the largest, increasing fast from 0.41 to 0.89. Besides, the proportion of TIPS also increases largely. The mean allocations of bond stays the same after increasing for a while via comparing with the case when $\gamma = 2$. We also
have little demands of bond and stock. The difference between the above two cases lies in the risk aversion $\gamma$. The $\gamma$ increases makes the insurer more sensitive to the risk in the market. So the insurer will buy more TIPS to hedge the risk of inflation and divide more insurance risk to the reinsurer. Since TIPS is also correlated with the nominal interest rate, we can in fact hedge a part of interest risk and so the mean allocations of zero coupon bond decreases when $\gamma$ increases. The two figures also describe the effect of $\gamma$ on the optimal reinsurance. Higher $\gamma$ means higher aversion of the risk, so the insurer will expect to reduce more of its insurance risk and thus will purchase more reinsurance business.

4.2. **Sensitivity analysis of the optimal reinsurance strategy.** We are also interested in how the parameters effect the reinsurance strategy. Figure 5 shows the relationship between optimal reinsurance policy and the expectation of one claim $\mu_1$. The reinsurance strategy increases as $\mu_1$ increases, making the insurer has more risk of insurance. In fact, we can observe from the formula of wealth that with larger $\mu_1$, compared
to the risk we shall take from insurance, we can get more income from
the premium. Concerning these, we will take more insurance risk.

As the insurer control its insurance risk by reinsurance, the optimal
reinsurance strategy will also depend on the variance of claim \( \mu_2 \). \( \mu_2 \)
means the risk of insurance. The reinsurance strategy decreases as the
variance of claim \( \mu_2 \) increases, which is shown by Figure 6. In other
words, if the risk of insurance becomes larger, the insurer should give
more risk to the reinsurer to gain the optimal wealth. Besides, the pa-
rameter \( \theta \) is also an important factor that affect the reinsurance strategy.

\( \theta \) measures the cost to hedge the risk of insurance. With higher \( \theta \), the
insurer should cost more to hedge the risk of insurance, i.e., the insurer
will take more risk of insurance by himself. So the reinsurance strategy
is positively correlated with the safety loading of reinsurer \( \theta \), which is
shown by Figure 7. We can also see from the above figures that there
is a positive relationship between time \( t \) and the reinsurance strategy.

4.3. Sensitivity analysis of the optimal utility. This section presents
how the parameters effect the optimal utility. The optimal utility is
given by proposition(3.4). We can see from the expression of optimal invest and reinsurance strategies that the real interest rate $r_r$ is not correlated with these two. In the market with different real interest rate, we will adopt the same optimal strategies. But $r_r$ in fact effects the optimal utility. Figure 8 states that the optimal utility is positively correlated with $r_r$. When real interest rate becomes bigger, the real money we own is bigger and so we can get a bigger utility. Besides, if the initial nominal interest rate $r_0$ increases, we can make more money by investment. So we will also get bigger utility, as shown by Figure 9. The parameter $\lambda$ means the intensity of claims. If $\lambda$ increases, we may have more claims in a fixed time horizon. So we will lost more money and the utility will decrease.

5. Conclusion

In this paper, we consider the optimal reinsurance and investment problems under stochastic nominal interest rate and stochastic inflation index. The nominal interest rate is assumed to follow Ornstein-Uhlenbeck process and the inflation index is given by the Fisher equation. The surplus process of the insurer is given by the classical Lundberg model and approximated by a diffusion process. We can invest in the nominal risk bond, zero coupon bonds, TIPS and a stock to hedge the risk. Because the original problem is not self-financing, we introduce an auxiliary self-financing problem and solve it by stochastic dynamic programming. Finally, we get the optimal reinsurance and investment strategies under maximizing CRRA utility in Section 3. The optimal strategies consist of a strategy to gain the optimal utility, the optimal investment to hedge the risk of inflation and an investment in zero coupon bonds to offset the effect of outcome of the wealth. We also find that the real interest rate has no effect on the optimal reinsurance and investment strategies. Moreover, we give some sensitivity analysis to show the economic behavior of the optimal strategies and optimal utility.

6. Appendix

6.1. Proof of Proposition 3.4. The boundary condition for $V(t, y, r_n, I)$ is $V(T, y, r_n, I) = \frac{1}{1-\gamma} (\frac{y}{T})^{1-\gamma}$. We guess that the $V(t, y, r_n, I)$ has the
following form
\[ V(t, y, r_n, I) = \frac{1}{1 - \gamma} \left( \frac{y}{T} \right)^{1-\gamma} h(t, r_n), \text{ and } h(T, r_n) = 1. \] (6.1)

Substituting (6.1) into (3.9), we see the optimal strategies \( u^*(t) \) is
\[ u^*(t) = \frac{y}{\gamma} \Sigma^{-1} \sigma \Lambda + (1 - \frac{1}{\gamma}) y \Sigma^{-1} \sigma I + \frac{1}{\gamma} h_{rn} y \Sigma^{-1} \sigma r. \] (6.2)

Next we substitute the above formula into (3.4), and then we find that \( h(t, r_n) \) satisfies the following equation:
\[
\begin{align*}
&\frac{h_t}{h} + \frac{h_{rn}}{h} [ab - ar_n + \gamma - (1 + 1) \sigma^T r \sigma I] - \frac{1}{2\gamma} \frac{h}{h} r_{rn} r_n^T \sigma r - \frac{1}{\gamma} \Lambda^T r + \\
&\quad \frac{1}{\gamma} \frac{2}{\gamma} \sigma^T r \sigma r + \frac{1}{\gamma} h_{rn} h \sigma^T r \sigma r + (1 - 1)(-r + \sigma I, \lambda r + \sigma I, \lambda I) + \\
&\quad \frac{1}{2}(\gamma - 2)(\gamma - 2) \sigma^T I \sigma I - (\gamma - 1) \left[ \frac{1}{2\gamma} \Lambda^T \Lambda + \frac{1}{2\gamma} \frac{1}{\gamma} \Lambda^T I \sigma I + (1 - 1) \Lambda^T I \sigma I \right] = 0.
\end{align*}
\]

The solution for \( h(t, r_n) \) must be the following form:
\[ h(t, r_n) = \exp[q_1(t) r_n + q_2(t)], \] (6.3)

and the \( q_1(t) \) and \( q_2(t) \) satisfy the boundary conditions \( q_1(T) = q_2(T) = 0 \). Thus we derive the explicit forms of \( q_1(t) \) and \( q_2(t) \) are
\[
\begin{align*}
q_1(t) &= 0, \\
q_2(t) &= \int_t^T (\gamma - 1)(-r(s) + \sigma I, \lambda r + \sigma I, \lambda I - \frac{1}{2\gamma} \Lambda^T \Lambda - (1 - 1) \Lambda^T I \sigma I) ds.
\end{align*}
\]

Once we get the explicit form of \( h(t, r_n) \), the explicit form of \( u^*(t) \) and \( V(t, y, r_n, I) \) can be easily derived, so the proposition is proved.

6.2. Proof of Proposition 4.1. The proof is a little tedious and difficult. To calculate \( E[Y^*(t)] \), firstly we have the following lemma.

**Lemma 6.1.** \( r_n(t) \) satisfies the following equations:
\[
\begin{align*}
r_n(t) = & (r_0 - b) \exp(-at) + b - \sigma r_n \exp(-at) \int_0^t \exp(as) dW_{r_n}(s) , \quad (6.4) \\
\int_0^t r_n(s) ds = & (r_0 - b) \frac{1 - \exp(-at)}{a} + bt - \int_0^t \sigma B_1(t-s) dW_{r_n}(s). \quad (6.5)
\end{align*}
\]

Hence \( \int_0^t r_n(s) ds \) is a random variable with normal distribution, i.e.,
\[
\int_0^t r_n(s) ds \sim N[(r_0 - b) \frac{1 - \exp(-at)}{a} + bt, \left( \frac{2}{a^2} \frac{- \exp(-2at)}{a^2} \right)].
\]
We can easily find that the solution of the Ornstein-Uhlenbeck equation is lognormal distributed. So the expression is the form of the first formula. For the second formula, we have

\[
\int_0^t r_n(s)ds = \int_0^t [(r_0-b) \exp(-as) + b - \sigma_n \exp(-as) \int_0^s \exp(au) dW_{r_n}(u)] ds
\]

\[
= (r_0-b) \frac{1 - \exp(-at)}{a} + b - \sigma_n \int_0^t \exp(-as) \int_0^s \exp(au) dW_{r_n}(u) ds
\]

\[
= (r_0-b) \frac{1 - \exp(-at)}{a} + bt - \int_0^t \sigma_B(t-s) dW_{r_n}(s).
\]

(6.6)

And so the distribution of \(\int_0^t r_n(s) ds\) follows.

Next, we derive the mean of \(Y^*(t)\). In fact, \(Y^*(t)\) has the following expression:

\[Y^*(t) = Y_0 \exp \left\{ \frac{1}{\gamma} \Lambda^T \Lambda t + (1 - \frac{1}{\gamma}) \sigma_I^T \Lambda t - \frac{1}{2} \left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_I^T \right] \left[ \frac{1}{\gamma} \Lambda + (1 - \frac{1}{\gamma}) \sigma_I \right] t \right\} + \int_0^t r_n(s) ds + \left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_I^T \right] W(t).\]

So

\[E[Y^*(t)] = Y_0 \exp \left\{ \frac{1}{\gamma} \Lambda^T \Lambda t + (1 - \frac{1}{\gamma}) \sigma_I^T \Lambda t - \frac{1}{2} \left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_I^T \right] \left[ \frac{1}{\gamma} \Lambda + (1 - \frac{1}{\gamma}) \sigma_I \right] t \right\} \cdot E \left\{ \int_0^t r_n(s) ds + \left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_I^T \right] W(t) \right\}
\]

\[= Y_0 \exp \left\{ \left( \frac{1}{\gamma} - \frac{1}{2\gamma^2} \right) \Lambda^T \Lambda t + (1 - \frac{1}{\gamma})^2 \Lambda^T \Lambda t - \frac{1}{2} \left( 1 - \frac{1}{\gamma} \right)^2 \sigma_I^T \sigma_I t \right\} \cdot E \left\{ \int_0^t r_n(s) ds + \left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_I^T \right] W(t) \right\}.
\]

(6.7)

So, we only need to calculate \(E\left\{ \exp \left\{ \int_0^t r_n(s) ds + \left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_I^T \right] W(t) \right\} \right\}\).

Denote \(Q_t = \int_0^t r_n(s) ds + \left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_I^T \right] W(t)\). In fact, \(Q_t\) is normal distribution random variable and \(\exp \left\{ \int_0^t r_n(s) ds + \left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_I^T \right] W(t) \right\}\) is lognormal distributed. So

\[E \left\{ \exp \left\{ \int_0^t r_n(s) ds + \left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_I^T \right] W(t) \right\} \right\} = E \left[ \exp(Q_t) + \frac{1}{2} \text{Var}(Q_t) \right].
\]

(6.8)
where
\[
\mathbb{E}\{Q_t\} = \mathbb{E}\left[ \int_0^t r_n(s) \, ds \right] = (r_0 - b) \frac{1 - \exp(-at)}{a} + bt,
\]
\[
\text{Var}\{Q_t\} = \text{Var}\left[ \int_0^t r_n(s) \, ds + \left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_T^2 \right] W(t) \right]
\]
\[
= \text{Var}\left[ \int_0^t r_n(s) \, ds \right] + \text{Var}[\left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_T^2 \right] W(t)]
\]
\[+ 2 \text{Cov}\left[ \int_0^t r_n(s) \, ds, \left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_T^2 \right] W(t) \right].
\]
Because
\[
\text{Var}\left[ \left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_T^2 \right] W(t) \right] = \left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_T^2 \right] \left[ \frac{1}{\gamma} \Lambda + (1 - \frac{1}{\gamma}) \sigma_I \right] t,
\]
\[
\text{Var}\left[ \int_0^t r_n(s) \, ds \right] = \frac{\sigma_n^2}{a^2} \left[ t + \frac{2 \exp(-at)}{a} - \frac{\exp(-2at)}{2a} - \frac{3}{2a} \right],
\]
and
\[
\text{Cov}\left[ \int_0^t r_n(s) \, ds, \left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_T^2 \right] W(t) \right]
\]
\[
= \mathbb{E}\left[ \int_0^t r_n(s) \, ds \cdot \left[ \frac{1}{\gamma} \Lambda^T + (1 - \frac{1}{\gamma}) \sigma_T^2 \right] W(t) \right]
\]
\[
= \left[ \frac{1}{\gamma} \lambda_n + (1 - \frac{1}{\gamma}) \sigma_I \right] \mathbb{E}\left[ \int_0^t r_n(s) \, ds \cdot W_{r_n}(t) \right]
\]
\[
= -\left[ \frac{1}{\gamma} \lambda_n + (1 - \frac{1}{\gamma}) \sigma_I \right] \mathbb{E}\left[ \int_0^t \sigma_{B_1}(t-s) \, dW_{r_n}(s) \int_0^t \, dW_{r_n}(s) \right]
\]
\[
= -\left[ \frac{1}{\gamma} \lambda_n + (1 - \frac{1}{\gamma}) \sigma_I \right] \int_0^t \sigma_{B_1}(t-s) \, ds
\]
\[
= -\sigma_n \left[ \frac{1}{\gamma} \lambda_n + (1 - \frac{1}{\gamma}) \sigma_I \right] \left[ \frac{t}{a} - \frac{1}{a^2} + \frac{e^{-at}}{a^2} \right],
\]
we finally obtain
\[
\mathbb{E}\left[ Y^*(t) \right] = Y_0 \exp \left\{ (r_0 - b) \frac{1 - \exp(-at)}{a} + bt + \frac{1}{\gamma} \Lambda^T \Lambda t \right\}
\]
\[+ \left( 1 - \frac{1}{\gamma} \right) \sigma_T^2 \Lambda t + \frac{\sigma_n^2}{2a^2} \left[ t + \frac{2 \exp(-at)}{a} - \frac{\exp(-2at)}{2a} - \frac{3}{2a} \right]
\]
\[+ \sigma_n \left[ \frac{1}{\gamma} \lambda_n + (1 - \frac{1}{\gamma}) \sigma_I \right] \left[ \frac{t}{a} - \frac{1}{a^2} + \frac{e^{-at}}{a^2} \right].
\]

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References


