Self-Inverse Interleavers for Turbo Codes
Amin Sakzad, Mohammad-Reza Sadeghi, Daniel Panario, Senior, IEEE, and Nasim Eshghi

Abstract—In this work we introduce and study a set of new interleavers based on permutation polynomials and functions with known inverses over a finite field \( \mathbb{F}_q \) for using in turbo code structures. We use Monomial, Dickson, Möbius and Rédei functions in order to get new interleavers. In addition we employ Skolem sequences in order to find new interleavers with known cycle structure. As a byproduct we give an exact formula for the inverse of every Rédei function. The cycle structure of Rédei functions are also investigated. Finally, self-inverse versions of permutation functions are used to construct interleavers. These interleavers are their own de-interleavers and are useful for turbo coding and turbo decoding. Experiments carried out for self-inverse interleavers constructed using these kind of permutation polynomials and functions show excellent agreement with our theoretical results.

Index Terms—Interleavers, turbo codes, permutation functions over finite fields.

I. INTRODUCTION

INTERLEAVERS play an important role in designing good turbo codes. In fact, interleavers have influence on various aspects of the performance of turbo codes. A huge number of studies have been done in this realm. There exist three types of well-known interleavers: random, pseudo-random and deterministic interleavers [27]. Each of them has their own drawbacks and benefits.

Since we need the de-interleaver for decoding, random interleavers need a remarkable amount of memory for their implementation and require good data structures for saving the permutation. This makes them less attractive. Pseudorandom interleavers are constructed [27] based on three major design criteria: \( s \)-random constraint, code Distance Spectrum Optimization (DSO) criterion [8], and Iterative Decoding Suitability (IDS) criterion [12]. These criteria result in different algorithms for constructing interleavers. For more information about pseudorandom interleavers, we refer the reader to [27] and its references. The first issue with pseudorandom interleavers is the need for memory in the interleavers with higher sizes. On the other hand, these kind of interleavers are known to produce turbo codes with good minimum distances.

The advantage of deterministic interleavers is that they have a simple structure that is easy to implement. Only some defining parameters of interleavers like the coefficients of polynomials or even rational functions are stored for encoding and decoding of deterministic interleavers. It means that, we do not need any huge extra memory in such interleavers for storing all the pairs of permuted indices. Let us assume that we want to use a permutation function \( f(x) = \frac{g(x)}{h(x)} \) such that \( \deg(g) = n_g \) and \( \deg(h) = n_h \). If \( f^{-1}(x) = \frac{g_1(x)}{h_1(x)} \) where \( \deg(g_1) = n_{g_1} \) and \( \deg(h_1) = n_{h_1} \), and \( n = \max \{n_g, n_h, n_{g_1}, n_{h_1} \} \), then we need to store at most \( 4n \) coefficients for deterministic interleavers. We note that \( n_g = n_{g_1} \) and \( n_h = n_{h_1} \) for self-inverse interleavers and the number of coefficients also reduce to \( 2n \). Furthermore, this number is \( 2q \) (for very large \( q \)) for random and pseudorandom interleavers. However deterministic interleavers do not have good performances when compared to \( s \)-random interleavers for large amounts of \( s \) [23–25].

The best performance for a deterministic interleaver in high dimensions (over 1000) is derived from a Takeshita-Costello interleaver [25]. They used a special construction of “self-inverse” interleaver of size 16384 that is only 0.7 dB away from capacity at a bit-error rate (BER) of \( 10^{-5} \). In lower dimension (below 1000), the quadratic permutation polynomial interleavers [21] are the best. The quadratic permutation polynomial interleavers have a good error performance, even exceeding the \( s \)-random interleavers in these low dimensions. Recent efforts [23, 24] in the field of deterministic interleavers have focused on permutation polynomials. The permutation polynomials used in these deterministic interleavers are all over the integer ring \( \mathbb{Z} \). So, it seems natural to search in the class of self-inverse permutation functions for finding good interleavers. Recent works concerning self-inverse interleavers like [5, 21, 23] motivate us to turn our attention to these types of permutation functions and interleavers. Using results from finite fields, we can construct permutation polynomials and permutation functions of \( \mathbb{F}_q \) that decompose in cycles of length 1 or 2 only. In other words, they can induce self-inverse permutation polynomials and functions. The self-inverse interleavers are of interest because the encoding and decoding processes are executed by a deterministic interleaver. Hence we can reduce needed space to implement our interleaver. Therefore, in these interleavers, the same structure and technology used for encoding can used for decoding.

There exist an extensive literature on permutation polynomials and permutation functions over a finite field \( \mathbb{F}_q \) [14, 16]. However, only very special cases of permutation functions are known. Some examples are permutation monomials, Dickson permutation polynomials, nonlinear transformation (Möbius) and Rédei permutation functions. All permutation polynomials over a fixed finite field \( \mathbb{F}_q \) under the compositional func-
tion operation form a noncommutative group. It has been shown \[16\] that this group is isomorphic to \(S_q\), the group of all bijective functions from \(\{0, 1, \ldots, q - 1\}\) onto itself. So we can determine the cycle structure of every permutation function just like we can do it for permutations in \(S_q\).

Permutation monomials \(x^n\) with a cycle of length \(j\) is treated in \[1]. Permutation monomials \(x^n\) with all cycles of the same length are characterized in \[20\]. The cycle structure of Dickson permutation polynomials \(D_n(x, a)\) where \(a \in \{0, \pm 1\}\) have been studied in \[15\]. Furthermore, the cycle structure of Möbius transformation is fully described in \[4\]. In this work we state an exact formula for the inverse of every Rédei function. Moreover, the cycle structure of Rédei functions is studied. More precisely, we characterize Rédei functions with a cycle of length \(j\) and then extend this to all cycles of the same length \(j\). An exact formula for counting the number of cycles of length \(j\) is also provided.

Skolem sequences and Langford sequences have been introduced and studied extensively \[2, 11\]. These sequences have applications in constructing cyclic Steiner triple systems and construction of code resistant to random interference \[4, 9\]. In this work we continue to find applications for these nice structured sequences. More specially, we use these sequences to produce self-inverse and non-self-inverse interleaver.

In this paper we also simulate and investigate the performance analysis of our new interleavers by constructing turbo codes with two identical systematic recursive convolutional codes. We run our program for various frame lengths from 256 to 1031 and we get better results in comparison with the other well-known self-inverse and non-self-inverse interleavers in \[23, 25\].

This paper is organized as follows. For making the paper self-contained, background on interleavers and permutation functions is given in Section \(\text{II}\). The general structure of our deterministic interleavers is explained and investigated in Section \(\text{III}\). The Monomial, Dickson, Möbius, Rédei and Skolem interleavers are studied in Section \(\text{IV}\). The self-inverse version of Rédei functions are introduced and used to construct self-inverse interleavers in this section. Several results about Rédei functions like their cycle structure and the number of cycles of certain length are also given. Simulation results are provided to show the performance analysis of the new theoretical deterministic self-inverse interleavers based on permutation functions and Skolem sequences in Section \(\text{V}\). Conclusions and further works are commented in Section \(\text{VI}\).

\section*{II. BASIC DEFINITIONS AND BACKGROUND}

\subsection*{A. Interleavers}

A classical structure of an encoder for turbo codes consists of an input sequence, two equal encoders and an interleaver, denoted by \(\Pi\). Let us consider a sequence of length \(N\) of information bits denoted by \(x = (x_0, \ldots, x_{N-1})\) fed into a turbo code. Since encoding is systematic, the information sequence \(x\) is the first output sequence. The first encoder generates its parity sequence. The interleaver reorders (or permutes) the \(N\) bits in the information block so that the second encoder receives a permuted sequence of the same size denoted by \(\bar{x} = (x_{\Pi(0)}, \ldots, x_{\Pi(N-1)})\) for feeding into the second encoder. So, the interleaver \(\Pi\) may be interpreted as a function which permutes the indices of components of \(x\). In other words, let \(I = \{0, 1, \ldots, N - 1\}\) be all indices of \(x\), then the interleaver \(\Pi\) can be pictured as a one-to-one and onto function of \(I\). The inverse function \(\Pi^{-1}\) is also necessary for decoding process when we implement a de-interleaver. An interleaver \(\Pi\) is called self-inverse if \(\Pi = \Pi^{-1}\).

\subsection*{B. Permutation functions}

Let \(p\) be a prime number, \(q = p^m\) and \(\mathbb{F}_q\) be the finite field of order \(q\). A permutation function over \(\mathbb{F}_q\) is a bijective function which maps the elements of \(\mathbb{F}_q\) onto itself. Clearly permutation functions have a functional inverse with respect to composition. Thus, for a permutation function \(P \in \mathbb{F}_q[x]\), there exists a unique \(P^{-1} \in \mathbb{F}_q[x]\) of degree less than \(q\) such that \(P(P^{-1}(x)) = P^{-1}(P(x)) = x\) (mod \(x^q - x\)) for all \(x \in \mathbb{F}_q\). A permutation function \(P\) is called self-inverse if \(P = P^{-1}\).

Let \(f\) be a primitive polynomial of degree \(m\) over \(\mathbb{F}_q[x]\). Assume that \(f(\alpha) = 0\). Since \(f\) divides \(x^{p^m-1} - 1\), we can represent \(\mathbb{F}_q\) as \(\mathbb{F}_q = \{0, \alpha^1, \ldots, \alpha^{p^m-2}, \alpha^{p^m-1}\}\), \(1\) where \(\alpha^{p^m-1} = 1\). The above representation (power representation) of \(\mathbb{F}_q\) is appropriate for actions like multiplication and raising to a power. Furthermore, for every \(\alpha^i, 0 \leq i \leq q - 1\), there exists a polynomial (in \(\alpha\)) representation with degree less than \(m\) which is adequate for addition and subtraction. Next, we review four well-known permutation functions on the finite field \(\mathbb{F}_q\). They are useful for constructing new deterministic interleavers.

- Monomials \[16\]: \(M(x) = x^n\) for some \(n \in \mathbb{N}\) is a permutation polynomial over \(\mathbb{F}_q\) if and only if \((n, q-1) = 1\). The inverse of \(M(x)\) is obviously the monomial \(M^{-1}(x) = x^m\) where \(nm \equiv 1 \pmod{q-1}\).
- Dickson polynomials of the 1st kind \[16\]:
\[
D_n(x, a) = \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{n}{n-p} \binom{n-p}{p} (-a)^p x^{n-2p} \tag{2}
\]
is a permutation polynomial over \(\mathbb{F}_q\) if and only if \((n, q^2 - 1) = 1\). Thus, for \(a \in \{0, \pm 1\}\), the inverse of \(D_n(x, a)\) is \(D_m(x, a)\) where \(nm \equiv 1 \pmod{q^2 - 1}\).
- Möbius transformation: The function
\[
T(x) = \begin{cases} \frac{ax+b}{cx+d} & x \neq -\frac{d}{c} \\ \frac{a}{c} & x = -\frac{d}{c} \end{cases} \tag{3}
\]
where \(a, b, c, d \in \mathbb{F}_q, c \neq 0\) and \(ad - bc \neq 0\) is a permutation function. It’s inverse is simply
\[
T^{-1}(x) = \begin{cases} \frac{dx-b}{-cx+a} & x \neq \frac{a}{c} \\ \frac{a}{c} & x = \frac{a}{c} \end{cases} \tag{4}
\]
Rédei functions \[18\]: Let \( \text{char}(\mathbb{F}_q) \neq 2 \) and \( a \in \mathbb{F}_q^* \) be a non-square element, then we define

\[
G_n(x, a) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} (a)^i x^{n-2i},
\]

(5)

\[
H_n(x, a) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i+1} (a)^i x^{n-2i-1}.
\]

(6)

The Rédei function \( R_n = \frac{G_n}{H_n} \) with degree \( n \) is a rational function over \( \mathbb{F}_q \). The Rédei function \( R_n \) is a permutation function if and only if \( (n, q + 1) = 1 \).

C. Skolem sequences

Let \( D \) be a set of integers. A Skolem-type sequence is a sequence with alphabet \( D \) where each element \( i \in D \) appears exactly twice in the sequence at positions \( a_i \) and \( a_i + i = b_i \). So, \( |b_i - a_i| = i \) for every \( i \in D \). These sequences might have empty positions, which we fill them with zeros. For more information about Skolem sequences, we refer the reader to \[11\]. Here we use Skolem sequences to construct self-inverse interleavers of a specific size. A partition of the set \([n]\) into \( n \) ordered pairs

\[
\{(a_i, b_i) : b_i - a_i = i, \ 1 \leq i \leq n\},
\]

implies a Skolem sequence of order \( n \). It is obvious that in order to generate Skolem sequence corresponding to this partition we must put integer \( i \in [n] \) in positions \( a_i \) and \( b_i \) of a sequence \( S = (s_1, \ldots, s_{2n}) \). A \( k \)-extended Skolem sequence of order \( n \) is a Skolem sequence of order \( n \) which contains exactly one hole in position \( k \). If \( k \) is in the penultimate position, the sequence is called a hooked sequence. A \((j, n)\)-generalized Skolem sequence of multiplicity \( j \) is a sequence \( S = (s_1, \ldots, s_t) \) of integers from \([n]\) such that for every \( i \in [n] \) there are exactly \( j \) positions in the sequence \( S \), let us say \( r_1, r_2 = r_1 + i, \ldots, r_j = r_1 + (j - 1)i \), such that \( s_{r_1} = s_{r_2} = \cdots = s_{r_j} = i \). It is easy to see that \( t = jn \) in this case. In the following we mention theorems from \[11\] which state necessary and sufficient conditions for the existence of the above defined Skolem sequences.

**Theorem 1.** A Skolem sequence of order \( n \) exists if and only if \( n \equiv 0, 1 \pmod{4} \). A hooked Skolem sequence of order \( n \) exists if and only if \( n \equiv 2, 3 \pmod{4} \). A \( k \)-extended Skolem sequence of order \( n \) exists if and only if \( n \equiv 0, 1 \pmod{4} \), when \( k \) is odd and \( n \equiv 2, 3 \pmod{4} \) when \( k \) is even.

**Theorem 2.** Let \( j = p^ct \), where \( p \) is the smallest prime factor of \( j \), and \( c, t \) are positive integers. Then a \((j, n)\)-generalized Skolem sequence exists if and only if \( n \equiv 0, 1, \ldots, p - 1 \pmod{p^{c+1}} \).

There is an efficient heuristic algorithm for obtaining a Skolem sequence of large order based on hill-climbing \[10\]. The sufficiency of the above existence theorems is usually proved by directly constructing the required sequence. New sequences can also be found by concatenating two or more existing sequences.

**Example 3.** The sequence \((4, 1, 1, 3, 4, 2, 3, 2)\) is a Skolem sequence of order 4. The sequence \((2, 5, 2, 6, 1, 1, 5, 3, 4, 6, 3, 0, 4)\) is a hooked Skolem sequence of order 6.

Several direct constructions of hooked extended Skolem sequences are provided in \[17\] in terms of unions or sums of two Skolem sequences. Pivoting and doubling are another techniques for constructing Skolem sequences \[11\]. When \( n > 5 \), we can use the explicit constructions below to find our ordered pairs for Skolem and hooked Skolem sequences. Every pair \((i, j)\) denotes the two different positions \( i, j \) in the Skolem sequences that contain \( k = |j - i| \), i.e. \( s_i = s_j = k \).

Based on Theorem 1 a Skolem sequence of order \( n \) exists if and only if \( n \equiv 0, 1 \pmod{4} \). Now, if \( n = 4u \), then

\[
\begin{align*}
(4u + r - 1, & \ 8s - r + 1) \quad 1 \leq r \leq 2u, \\
(r, 4u - r - 1) & \quad 1 \leq r \leq u - 2, \\
(u + r + 1, & \ 3u - r) \quad 1 \leq r \leq u - 2, \\
(u - 1, & \ 3u, (u, u + 1), \\
&(2u, 4u - 1), (2u + 1, 6u),)
\end{align*}
\]

are our appropriate ordered pairs. Let \( n = 4u + 1 \), then

\[
\begin{align*}
(4u + r + 1, & \ 8u - r + 3) \quad 1 \leq r \leq 2u, \\
(r, 4u - r + 1) & \quad 1 \leq r \leq u, \\
(u + r + 2, & \ 3u - r + 1) \quad 1 \leq r \leq u - 2, \\
(u + 1, & \ u + 2), (2u + 1, 6u + 2), \\
&(2u + 2, 4u + 1).
\end{align*}
\]

(8)

For \( n = 4u + 2 \) and \( n = 4u - 1 \) a hooked Skolem sequence of order \( n \) exist as follow respectively,

\[
\begin{align*}
(r, & \ 4u - r + 2) \quad 1 \leq r \leq 2u, \\
(4u + 2 + 3, & \ 8u - r + 4) \quad 1 \leq r \leq u - 1, \\
(5u + r + 2, & \ 7u - r + 3) \quad 1 \leq r \leq u - 1, \\
(2u + 1, & \ 6u + 2), (4u + 2, 6u + 3), \\
&(4u + 3, 8u + 5), (7u + 3, 7u + 4),
\end{align*}
\]

and

\[
\begin{align*}
(4u + r, & \ 8u - r - 2), \quad 1 \leq r \leq 2u - 2 \\
(r, & \ 4u - r - 1), \quad 1 \leq r \leq u - 2 \\
(u + r + 1, & \ 3u - r), \quad 1 \leq r \leq u - 2 \\
(2u + 1, & \ 6u - 1), (4u, 8u - 1), \\
(u - 1, & \ 3u, (u, u + 1), (2u, 4u - 1)).
\end{align*}
\]

(9)

By means of various types of Skolem sequences we introduce self-inverse Skolem interleavers of a specific length. It means that our interleavers have only cycles of length 2 or 1. In addition, we can generalize this to provide interleavers with cycles of length \( j \) or 1 by using generalized Skolem sequences. This is the first usage of Skolem sequences in constructing interleavers.

III. GENERAL STRUCTURE OF DETERMINISTIC INTERLEAVERS

If \( P \) is a permutation function over \( \mathbb{F}_q = \{0, \alpha^1, \ldots, \alpha^{q-2}, \alpha^{q-1}\} \), then for every \( 1 \leq i \leq q - 1 \) there exists a \( j \) such that \( P(\alpha^i) = \alpha^j, 1 \leq j \leq q - 1 \). Thus \( P^{-1}(\alpha^j) = \alpha^i \). In this manner, we can take this permutation
function \( P \) as a function which rearranges the powers of \( \alpha \). Therefore, every permutation function over the finite field \( \mathbb{F}_q \) can induce an interleaver as follows.

**Definition 4.** Let \( P \) be a permutation function over \( \mathbb{F}_q \). An interleaver \( \Pi_P : \mathbb{Z}_q \to \mathbb{Z}_q \) is defined by

\[
\Pi_P(i) = \ln(P(\alpha^i)) \tag{11}
\]

where \( \ln(.) \) denotes the discrete logarithm to the base \( \alpha \) over \( \mathbb{F}_q^* \) and \( \ln(0) = 0 \).

We can also define the interleaver \( \Pi_{P^{-1}} \) by means of \( P^{-1} \). Based on the above description we have the following propositions.

**Proposition 5.** There is a one-to-one correspondence between the set of all permutation functions over a fixed finite field \( \mathbb{F}_q \) and the set of all interleavers of size \( q \).

**Proof:** Let \( \mathcal{P} \) be the set of all permutation functions over \( \mathbb{F}_q \) and \( \mathcal{I} \) be the set of all interleavers of size \( q \). We present a one-to-one and onto function \( \Psi \) between \( \mathcal{P} \) and \( \mathcal{I} \). If \( P \in \mathcal{P} \), then we define \( \Psi(P) = \Pi_P \in \mathcal{I} \). In addition, let \( \Pi \in \mathcal{I} \) and \( \alpha \in \mathbb{F}_q \) be a primitive element, then for every \( i \in \mathbb{Z}_q^* \) there exists a unique \( j_i \in \mathbb{Z}_q^* \) for which \( \Pi(i) = j_i \). Now using an appropriate interpolation formula, we can construct a permutation function \( P_{\Pi}(x) \) which maps each \( \alpha^i \) to \( \alpha^{j_i} \) for \( i \in \mathbb{Z}_q^* \). It is clear that \( \Psi(P_{\Pi}) = \Pi \) and this completes the proof.

**Proposition 6.** Let \( P \) be a permutation function over \( \mathbb{F}_q \). Then, we have \( (\Pi_P)^{-1} = \Pi_{P^{-1}} \).

**Proof:** First suppose that for some \( 1 \leq i, j \leq q - 1 \), we have \( \Pi_P^{-1}(j) = i \). So we have \( \Pi_P(i) = j \). We can now write \( P_i = \alpha^j \) and \( P^{-1}_i = \alpha^i \). Therefore,

\[
\Pi_{P^{-1}}(j) = \ln(P^{-1}(\alpha^j)) = \ln(\alpha^i) = i.
\]

On the other hand if \( \Pi_{P^{-1}}(j) = i \), then we have \( P^{-1}(\alpha^j) = \alpha^i \). The last equality means that \( P(\alpha^j) = \alpha^i \) and \( \Pi_P(i) = j \). Hence \( \Pi_{P^{-1}}(j) = i \).

**Corollary 1.** Let \( P \) be a self-inverse permutation function over \( \mathbb{F}_q \). Then, we have \( \Pi_P = (\Pi_P)^{-1} \).

**Proof:** Since \( P = P^{-1} \) and \( (\Pi_P)^{-1} = \Pi_{P^{-1}} \), we get \( \Pi_P = (\Pi_P)^{-1} \).

We are going to construct interleavers based on self-inverse permutation functions over \( \mathbb{F}_q \). We need information on permutations that decompose on cycles of length 1 and 2 only.

**IV. DETERMINISTIC INTERLEAVERS OVER \( \mathbb{F}_q \)**

Given the above discussions, we now introduce interleavers based on permutation functions over finite fields. The only thing that we need is a set of permutation functions with known inverses. The classes of permutation functions with explicit inverse formula are cited in Section

**A. Monomial interleavers**

Let \( M(x) = x^n \) over \( \mathbb{F}_q \) and \((n, q - 1) = 1 \). Then

\[
\Pi_M(i) = \ln(M(\alpha^i)) = \ln((\alpha^i)^n) = ni \mod q - 1,
\]

for \( i \in \mathbb{Z}_q \). So, \( \Pi_M(x) = nx \mod q - 1 \) for \( x \in \mathbb{Z}_q \). Since \((n, q - 1) = 1 \), \( \Pi_M \) is a linear permutation polynomial.

**Definition 7.** Let \( M(x) = x^n \) be a permutation monomial over \( \mathbb{F}_q \) where \((n, q - 1) = 1 \). The \( \Pi_M \) as defined in (17) is called a monomial interleaver.

As we have seen from the above definition, a monomial interleaver is a linear permutation polynomial over the integer ring \( \mathbb{Z}_q \).

**Example 8.** Assume that \( n = 11 \) and \( q = 13 \). Since \((11, 12) = 1 \), the monomial \( M(x) = x^{11} \) is a permutation polynomial over \( \mathbb{F}_{13} \). Furthermore, \( 11.11 \equiv 1 \mod 12 \) and this means that \( M^{-1} = M \) is a permutation polynomial over \( \mathbb{F}_{13} \) and \( \Pi_M^{-1} = (\Pi_M)^{-1} \). Therefore, \( \Pi_M^{-1} = \Pi_M \) can act as the de-interleaver too. Since 2 is a primitive element of \( \mathbb{F}_{13} \), we get

\[
M(2^1) = 2^{11}, \quad M(2^2) = 2^{10}, \quad M(2^3) = 2^9,
M(2^4) = 2^8, \quad M(2^5) = 2^7, \quad M(2^6) = 2^6,
M(2^7) = 2^5, \quad M(2^8) = 2^4, \quad M(2^9) = 2^3,
M(2^{10}) = 2^2, \quad M(2^{11}) = 2^1, \quad M(2^{12}) = 2^0.
\]

we can interpret this interleaver and its de-interleaver as the following

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
0 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 12
\end{pmatrix}.
\]

Also we observe that the only three fixed points are 0, 2^6 = -1 (mod 13) and 2^{12} = 1 (mod 13).

We believe that having the same deterministic interleaver and de-interleaver results in a reduction of complexity in encoding and decoding process of turbo codes with an iterative decoding algorithm. Our approach based on Corollary 1 implies in the use of self-inverse permutation functions which have cycles of the same length \( j = 1 \) or 2. Such permutation monomials are obtained using the following Theorem from [20] for \( j = 2 \).

**Theorem 9.** Let \( q - 1 = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r} \). The permutation monomial \( M(x) = x^n \) of \( \mathbb{F}_q \) has only cycles of the same length \( j \) or 1 (fixed points) if and only if one of the following conditions holds for each \( 1 \leq l \leq r \):

- \( n \equiv 1 \mod p_l^{k_l} \)
- \( j = \text{ord}_{p_l^{k_l}}(n) \) and \( j|p_l - 1 \),
- \( j = \text{ord}_{p_l^{k_l}}(n) \), \( k_l \geq 2 \) and \( j = p_l \).

Recall that \( j = \text{ord}_{p_l^{k_l}}(n) \), if \( j \) is the smallest integer with the property \( n^j \equiv 1 \mod s \) We also use \( j = \text{ord}_{p_l^{k_l}}(n) \) as the least integer with \( n^j \equiv 1 \mod s \). The following corollary of the above theorem is useful for us since we concentrate on the case \( j = 2 \).

**Corollary 2.** Let \( q - 1 = p_0^{k_0} p_1^{k_1} \ldots p_r^{k_r} \) where \( p_0 = 2 \). The permutation polynomial of \( \mathbb{F}_q \) given by \( M(x) = x^n \)
decomposes in cycles of the same length \( j \) and \( \{0, 1, -1\} \) are the only fixed elements if and only if
- for \( k_0 \geq 2 \): \( j = 2 \) and \( n = q - 2 \) or \( n = \frac{q - 3}{2} \).
- for \( k_0 = 2 \): \( j = 2 \) and \( n = q - 2 \).

We note that if we use \( n = q - 2 \), we get some sort of symmetry in our permutation as in Example 8. Let \( n = q - 2 \), then we have \( \Pi_M(x) \) equals to

\[
\ln(M(\alpha^x)) = \ln(\alpha^{x(q-2)}) = x(q-2) \pmod{q-1}.
\]

It is easy to see that for every \( i \in \mathbb{Z}_q \) we have \( \Pi_M(i) = q-1-i \) and \( \Pi_M(q - 1 - i) = i \) because \( \Pi_M(i) = i(q-2) = i(q - 1 - 1) = i(q-1) - i = -i = q - 1 - i \pmod{q-1} \) and since the permutation is self-inverse, \( \Pi_M(q - 1 - i) = i \).

However, this is not the case for \( n = \frac{q-3}{2} \) in Corollary 2. In this situation \( \Pi_M(x) = x(\frac{q-3}{2}) \pmod{q-1} \). We do not see that symmetry in this case and it seems that these self-inverse monomial interleavers perform better than when we use \( n = q - 2 \).

B. Dickson interleavers

Let \((n, q^2 - 1) = 1\). It is known that \( D_n(x, a) \) for \( a \in \{0, \pm 1\} \) is a permutation polynomial over \( \mathbb{F}_q \) and has the compositional inverse \( D_n(x, a) \) where \( nm \equiv 1 \pmod{q^2-1} \) [16]. Now, we can define a set of deterministic interleavers

\[
\Pi_D^{(n,a)} : \mathbb{Z}_q \rightarrow \mathbb{Z}_q
\]

by \( \Pi_D^{(n,a)}(i) = \ln(D_n(a^i, a)) \).

**Definition 10.** Let \( D_n(x, a) \) be a first kind Dickson permutation polynomial over \( \mathbb{F}_q \) where \( a \in \{0, \pm 1\} \) and \((n, q^2 - 1) = 1\). Then, \( \Pi_D^{(n,a)} \) defined as in (11) is called a Dickson interleaver.

The proof of the following proposition is straightforward.

**Proposition 11.** Suppose that \( nm \equiv 1 \pmod{q^2-1} \). Then \( \Pi_D^{(n,a)} \) is the de-interleaver of \( \Pi_D^{(n,a)} \).

It is shown that \( \Pi_D^{(n,a)} \) has also a deterministic de-interleaver, which can reduce the complexity of iterative decoding of turbo codes dramatically.

**Example 12.** Let \( n = 19, q = 11 \) and \( a = 1 \). Then we get

\[
D_{19}(x, 1) = x^9 + 3x^7 + 9x^5 + 5x^3 + 5x \pmod{11}.
\]

Since \( (19, (11)^2 - 1) = (19, 120) = 1 \) and \( a = 1 \), \( D_{19}(x, 1) \) is a permutation polynomial over \( \mathbb{F}_{11} \) with compositional inverse \( D_m(x, 1) \) where \( 19m \equiv 1 \pmod{120} \). So \( m = 19 \), and it means that \( D_{19}(x, 1) \) is a self-inverse Dickson permutation polynomial over \( \mathbb{F}_{11} \). A Dickson interleaver \( \Pi_D^{(19,1)} : \mathbb{Z}_{11} \rightarrow \mathbb{Z}_{11} \) can be defined by \( \Pi_D^{(19,1)}(i) = \ln(D_{19}(2^i, 1)) \) where \( 2 \in \mathbb{F}_{11} \) is a primitive element. Thus, we have the following:

\[
\begin{align*}
D_{19}(0, 1) &= 0, & D_{19}(2, 1) &= 2, & D_{19}(2^2, 1) &= 3, \\
D_{19}(2^3, 1) &= 2^4, & D_{19}(2^4, 1) &= 2^9, & D_{19}(2^5, 1) &= 2^5, \\
D_{19}(2^6, 1) &= 2^6, & D_{19}(2^7, 1) &= 2^7, & D_{19}(2^8, 1) &= 2^8, \\
D_{19}(2^9, 1) &= 2^4, & D_{19}(2^{10}, 1) &= 2^{10}.
\end{align*}
\]

The above inequalities mean that \( \Pi_D^{(19,1)} \) permutes the elements of \( \mathbb{Z}_{11} \) as follow

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 & 4 & 10
\end{pmatrix}.
\]

The following two theorems are from [19].

**Theorem 13.** Let \( q-1 = p_{r_1}^{k_1} \cdots p_{r_r}^{k_r} \) and \( q+1 = p_{r_{r+1}}^{k_{r+1}} \cdots p_{r_s}^{k_s} \) be the prime factorization of \( q-1 \) and \( q+1 \) respectively. Suppose that \((n, q^2 - 1) = 1\). The Dickson permutation polynomial \( D_n(x, 1) \) over \( \mathbb{F}_q \) is the identity on \( \mathbb{F}_q \) or all the non-trivial cycles have length two if and only if one of the following holds for all \( 1 \leq l \leq r \), and one of the following conditions holds for all \( r+1 \leq l \leq s \):

1) Either
   - \( a \equiv 1 \pmod{p_{r_l}^{k_l}} \) and \( p_{r_l}^{k_l} = 2 \), or
   - \( b = \text{ord}_{p_l}^{\mathbb{F}_q}(n) \), and \( 4|/(p_l - 1) \).

2) Either
   - \( a \equiv 1 \pmod{p_{r_l}^{k_l}} \), or
   - \( b = \text{ord}_{p_l}^{\mathbb{F}_q}(n) \), \( p_l = 2 \), \( k_l \geq 2 \), and \( n \equiv -1 \pmod{p_{r_l}^{k_l}} \).

**Theorem 14.** Let \( q-1 = p_{r_1}^{k_1} \cdots p_{r_r}^{k_r} \) and \( q+1 = p_{r_{r+1}}^{k_{r+1}} \cdots p_{r_s}^{k_s} \) be the prime factorization of \( q-1 \) and \( q+1 \) respectively. Suppose that \((n, q^2 - 1) = 1\). The Dickson permutation polynomial \( D_n(x, -1) \) over \( \mathbb{F}_q \) is the identity on \( \mathbb{F}_q \) or all the non-trivial cycles have length two if and only if one of the following conditions holds for all \( 1 \leq l \leq r \), and one of the following conditions holds for all \( r+1 \leq l \leq s \):

1) Either
   - \( a \equiv 1 \pmod{p_{r_l}^{k_l}} \) and \( p_{r_l}^{k_l} = 2, 4 \), or
   - \( b = \text{ord}_{p_l}^{\mathbb{F}_q}(n) \), and \( 4|/(p_l - 1) \).

2) Either
   - \( a \equiv 1 \pmod{p_{r_l}^{k_l}} \), or
   - \( b = \text{ord}_{p_l}^{\mathbb{F}_q}(n) \), \( p_l \equiv 2 \), \( k_l \geq 2 \), and \( p_l = 2 \).

**Theorem 15.** Let \( D_n(x, 1) \) \((D_n(x, -1)) \) be a Dickson permutation polynomial over \( \mathbb{F}_q \) which meets the assumptions of Theorem 13 (Theorem 14) respectively. Then \( \Pi_D^{(n,1)} \) \((\Pi_D^{(n,-1)} \) respectively) is a self-inverse interleaver.

**Proof:** If \( D_n(x, 1) \) meets the assumptions of Theorem 13 then all the cycles of \( D_n(x, 1) \) have the same length 1 or 2. This means that \( D_n(x, 1) \) is a self-inverse permutation polynomial, that is, \( D_n(x, 1) = D_m(x, 1) \) where \( nm \equiv 1 \pmod{q^2-1} \). Using Theorem 6 completes the proof. The proof for \( D_n(x, -1) \) is identical.  

C. Möbius interleavers

**Definition 16.** Let \( T \) be a Möbius transformation over \( \mathbb{F}_q \); see [3]. Then, \( \Pi_T \) defined as in (11) is called a Möbius interleaver.

The inverse function of \( T \) is given in [4]. It is easy to see that \( T = T^{-1} \) when we have \( a = d, -b = b \) and \( c = -c \). Let
us assume that \( q = 2^n \), then we get \(-b = b \) and \( c = -c \) since we are in characteristic 2. So
\[
T(x) = T^{-1}(x) = \begin{cases} \frac{ax + b}{x^2 + c} & x \neq \frac{b}{c}, \\ \frac{b}{c} & x = \frac{b}{c}. \end{cases}
\]
(12)
where \( a^2 - bc \neq 0 \) and \( c \neq 0 \) is a permutation function over \( \mathbb{F}_{2^n} \).

**Example 17.** Let \( n = 3 \), \( a = \alpha^3 = d \), \( b = \alpha^2 \) and \( c = \alpha \). Then we get
\[
T(x) = \begin{cases} \frac{\alpha^3x + \alpha^2}{\alpha^2x + \alpha^3} & x \neq \alpha^2, \\ \frac{\alpha^2}{\alpha^3} & x = \alpha^2. \end{cases}
\]
It is clear that \( T \) is a permutation function over \( \mathbb{F}_{2^3} \) with compositional inverse \( T \). A Möbius interleaver \( \Pi_T : \mathbb{Z}_8 \rightarrow \mathbb{Z}_8 \) can be defined by \( \Pi_T(i) = \ln(T(\alpha^i)) \). Thus, we get
\[
T(0) = \frac{\alpha^2}{\alpha^2} = \alpha^{-1} = \alpha^6, \quad T(\alpha) = \frac{\alpha^3}{\alpha} = \alpha^{5} = \alpha^{-4} = \alpha^3, \\
T(\alpha^2) = \frac{\alpha^6}{\alpha^2} = \alpha^2, \quad T(\alpha^3) = \frac{\alpha^5}{\alpha} = \alpha^{4} = \alpha^{5}, \\
T(\alpha^4) = \frac{\alpha^6}{\alpha^2} = \alpha^4, \quad T(\alpha^5) = \frac{\alpha^7}{\alpha^3} = 1 = \alpha^7, \\
T(\alpha^6) = \frac{\alpha^1}{\alpha} = 0, \quad T(\alpha^7) = \frac{\alpha^0}{\alpha^3} = \alpha^5.
\]
The above equalities induce the following Möbius interleaver
\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
6 & 3 & 2 & 1 & 4 & 7 & 0 & 5
\end{pmatrix}.
\]
We note that in the above example 0 is not a fixed point. As a matter of fact, since we assume that all permutation polynomials have the property \( P(0) = 0 \), we expect to have \( \Pi(0) = 0 \). But this is not the case for interleavers derived from permutations like \( T \). The next theorem cited from [4] fully describes the cycle structure of \( T \) in terms of the eigenvalues of a matrix. Let \( T \) be as in (3). The coefficient matrix associated to the \( T \) is
\[
A_T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]
Therefore, the characteristic polynomial of \( A_T \) is a quadratic polynomial \( t \).

**Theorem 18.** Let \( T \) be the permutation defined by (3), and let \( t \) be the characteristic polynomial of the matrix \( A_T \) associated with \( T \). Let \( \alpha_1, \alpha_2 \in \mathbb{F}_q \) be the roots of \( t \).

1) Suppose \( t(x) \) is irreducible. If \( k = \text{ord} \left( \frac{\alpha_1}{\alpha_2} \right) = \frac{q + 1}{s} \), \( 1 \leq s < \frac{q + 1}{2} \), then \( T \) has \( s - 1 \) cycles of length \( k \) and one cycle of length \( k - 1 \). In particular \( T \) is a full cycle if \( s = 1 \).

2) Suppose \( t(x) \) is reducible and \( \alpha_1, \alpha_2 \in \mathbb{F}_q \) are roots of \( t(x) \) and \( \alpha_1 \neq \alpha_2 \). If \( k = \text{ord} \left( \frac{\alpha_1}{\alpha_2} \right) = \frac{q - 1}{s} \), \( s \geq 1 \), then \( T \) has \( s - 1 \) cycles of length \( k \), one cycle of length \( k - 1 \) and two cycles of length 1.

3) Suppose \( t(x) = (x - \alpha_1)^2 \), \( \alpha_1 \in \mathbb{F}_q \) where \( q = p^m \). Then \( T \) has \( p^{m-1} - 1 \) cycles of length \( p \), one cycle of length \( p - 1 \) and one cycle of length 1.

It is obvious that we are interested again on permutations with cycles of length 1 and 2 only. To this end, we use the cases of the above theorem:

1) We should have \( 2 = k = \text{ord} \left( \frac{\alpha_1}{\alpha_2} \right) \). But \( k = 2 \) iff \( (\alpha_1)^2 = (\alpha_2)^2 \) and \( \alpha_1 \neq \alpha_2 \) iff \( \alpha_1 = -\alpha_2 \) iff \( a + d = tr(A_T) = \alpha_1 + \alpha_2 = 0 \). Hence, \( t \) is irreducible and \( tr(A_T) = 0 \) iff \( k = 2 \) and we have \( \frac{q + 1}{2} - 1 \) cycles of length 2 and one cycle of length 1.

2) In a similar situation with case 1) we have that \( t \) is reducible and \( tr(A_T) = 0 \) iff \( k = 2 \) and we have \( s^2 - 1 \) cycles of length 2 and three cycles of length 1.

3) In the third case we have only cycles of length 1 and 2 when \( p = 2 \). As we mentioned in (12) in this case also we have \( tr(A_T) = a + d = 0 \). So \( a = d \) and it means that \( \alpha_1 = a \). Thus, \( a = d \) iff \( T \) has \( 2^{m-1} - 1 \) cycles of length 2 and two cycles of length one where \( q = 2^m \).

D. Rédei interleavers

**Definition 19.** Let \( R_n \) be a Rédei permutation function over \( \mathbb{F}_q \). The interleaver \( \Pi^n_R \) defined in (17) is called a Rédei interleaver.

Next we derive an inverse for every Rédei function \( R_n \). It is easily verified [3] that for all \( x, y \in \mathbb{F}_q \)
\[
R_n(R_m)(x, a) = R_{nm}(x, a),
\]
(14)
\[
R_n(x, a) = x \leftrightarrow n \equiv 1 \pmod{q + 1},
\]
(15)
\[
R_n \left( \frac{xy + a}{x + y}, a \right) = \frac{R_n(x, a)R_n(y, a) + a}{R_n(x, a) + R_n(y, a)}. \]
(16)
The next theorem is proved in [3].

**Theorem 20.** Let \( r \) be a rational function with coefficients in \( \mathbb{F}_q \) that satisfies
\[
r \left( \frac{xy + a}{x + y}, a \right) = \frac{r(x)y + a}{r(x) + r(y)}
\]
(17)
where \( a \) is a fixed element of \( \mathbb{F}_q \) and \( x \) and \( y \) are two unknowns. Then, if \( a \neq 0 \) and \( \text{char}(\mathbb{F}_q) \neq 2 \), \( r \) coincides with a Rédei’s function for some \( m \) (not necessarily relatively prime to \( q + 1 \)).

Now, by using the above characterization theorem, we find a general formula for the inverse of every Rédei function.

**Theorem 21.** Let \( R_n \) for some \( n \) be a Rédei function over \( \mathbb{F}_q \) where \( a \in \mathbb{F}_q \) is a non-square and \( (n, q + 1) = 1 \). Then \( R_n^{-1} = R_m \) for \( m \) satisfying \( nm \equiv 1 \pmod{q + 1} \).

**Proof:** First of all, it is clear that \( R_n \) has a compositional inverse \( R_n^{-1} \). Using (16) we have
\[
R_n \left( \frac{xy + a}{x + y}, a \right) = \frac{R_n(x, a)R_n(y, a) + a}{R_n(x, a) + R_n(y, a)}.
\]
Taking \( R_n^{-1} \) from both sides, we get
\[
xy + a = R_n^{-1} \left( \frac{R_n(x, a)R_n(y, a) + a}{R_n(x, a) + R_n(y, a)} \right).
\]
Let \( s = R_n(x, a) \) and \( t = R_n(y, a) \). Then applying (17) to \( R_n^{-1} \), we get
\[
R_n^{-1}(R_n(x, a))R_n^{-1}(R_n(y, a)) + a = \frac{R_n(x, a)R_n(y, a) + a}{R_n(x, a) + R_n(y, a)},
\]

implying that

\[
R_n^{-1}(s)R_n^{-1}(t) + a = R_n^{-1}\left(\frac{st + a}{s + t}\right)
\]

for all \(s, t \in \mathbb{F}_q\). Since \(R_n^{-1}\) satisfies all the conditions of Theorem 20, \(R_n^{-1}\) coincides with a Rédei function for some \(m\). So, \(R_n^{-1} = R_m\). Now, we are using (14) and (15) to get

\[
id_{g_4} = R_n(R_n^{-1}) = R_n(R_n) = R_{nm} \\
\Leftrightarrow \quad nm \equiv 1 \pmod{q + 1}.
\]

We note that \(R_n^{-1}\) is a rational function because every function from \(\mathbb{F}_q\) to itself can be interpreted as a polynomial with degree less than \(q\).

**Example 22.** Let \(q = 7\), \(n = 5\) and \(a = 3 \in \mathbb{Z}_7^*\) be a non-square. Since \(\text{char}(\mathbb{F}_7) \neq 2\) and \((5, 7 + 1) = 1\), we get

\[
G_5(x, 3) = x^5 + 30x^3 + 45x = x^5 + 2x^3 + 3x,
\]

\[
H_5(x, 3) = 5x^4 + 30x^2 + 9 = 5x^4 + 2x^2 + 2
\]

modulo 7. Therefore,

\[
R_5(x, 3) = \frac{G_5(x, 3)}{H_5(x, 3)} = \frac{x^5 + 2x^3 + 3x}{5x^4 + 2x^2 + 2}
\]

Thus, we have

\[
R_5(0, 3) = R_5(0, 3) = \frac{0}{2} = 0,
\]

\[
R_5(3^1, 3) = R_5(3, 3) = 3^6 + 2\times 27 + 9 = 2 = 5 = 3^6,
\]

\[
R_5(3^2, 3) = R_5(2, 3) = \frac{5 \times 16 + 8 + 4}{3} = \frac{4}{3} = 2^2 = 3^2,
\]

\[
R_5(3^3, 3) = R_5(6, 3) = -\frac{6}{2} = -3 = 3^4,
\]

\[
R_5(3^4, 3) = R_5(4, 3) = \frac{(3 - a)^5 + 2\times(27 - 9)}{5 \times 16 + 8 + 4 + 2} = \frac{5}{3} = 3^3,
\]

\[
R_5(3^5, 3) = R_5(5, 3) = \frac{(3^5 - 16 - 6)^2}{3} = \frac{-12}{6} = -2 = 3^5,
\]

\[
R_5(3^6, 3) = R_5(1, 3) = 6^2 = 3^4.
\]

So \(\Pi_5^5\) is

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & 6 & 2 & 4 & 3 & 5 & 1
\end{pmatrix}.
\]

We have \(R_5^{-1} = R_m\) where \(5m \equiv 1 \pmod{8}\). We have \(m = 5\) and this means that \(R_5^{-1} = R_5\) over \(\mathbb{F}_7\). Therefore, \(R_5\) is a self-inverse permutation function and \(\Pi_5^5\) is a self-inverse interleaver of \(\mathbb{Z}_7^*\).

Let \(\text{char}(\mathbb{F}_q) \neq 2\) and \(a \in \mathbb{F}_q^*\) be a non-square element. The numerator \(G_n(x, a)\) and the denominator \(H_n(x, a)\) are polynomials in \(\mathbb{F}_q[x]\) satisfying the equation

\[
(x + \sqrt{a})^n = G_n(x, a) + H_n(x, a)\sqrt{a}.
\]

Also, it is easy to see that

\[
(x - \sqrt{a})^n = G_n(x, a) - H_n(x, a)\sqrt{a}.
\]

For our interleaver construction we need the cycle structure of Rédei permutation function. We give that in the following theorem.
Theorem 23. Let $j$ be a positive integer. The Rédei function $R_n(x, a)$ of $\mathbb{F}_q^n$ with $(n, q + 1) = 1$ has a cycle of length $j$ if and only if $q + 1$ has a divisor $s$ such that $j = \text{ord}_s(n)$.

Proof: Let $R_n^{(j)}(x, a)$ denote the $j$-th iterate of $R_n(x, a)$ under the composition operation. We get

$$R_n^{(j)}(x, a) = x \iff R_n^{(1)}(x, a) = x \iff G_n^{(1)}(x, a) = x$$

where the first equivalence is derived from (14). Furthermore, we get the following identities for $x \in \mathbb{F}_q$:

$$(x + \sqrt{a})^n = G_n^{(1)}(x, a) \implies xH_n^{(1)}(x, a) = x$$

where $x$ is a primitive element of $\mathbb{F}_q^2$. Hence $y^{n' - 1} = H_n^{(1)}(x, a) \in \mathbb{F}_q$ and $y \in \mathbb{F}_q$. Let $y^{n' - 1} \in \mathbb{F}_q$ and this means that $y^{(n' - 1)(q - 1)} = 1$. So, $R_n(x)$ has a cycle of length $j$ if and only if $y^{(n' - 1)(q - 1)} = 1$ if and only if $q^2 - 1$, which is the size of the multiplicative group $\mathbb{F}_q^\times$, has a divisor $t$ such that $t|(n' - 1)(q - 1)$ if and only if $q + 1$ has a divisor $s$ where $n' \equiv 1 \pmod{s}$ and $j$ is the smallest integer with this property. □

Theorem 24. The number $N_j$ of cycles of length $j$ of the Rédei function $R_n$ over $\mathbb{F}_q$ with $(n, q + 1) = 1$ satisfies

$$jN_j + \sum_{i<j} iN_i + 1 = (n^j - 1, q + 1).$$

Proof: Based on the proof of Theorem 23, we are looking for $y \in \mathbb{F}_q^2$ where $y^{(n' - 1)(q - 1)} = 1$ and $y^{(n' - 1)} \in \mathbb{F}_q$. Let $\rho$ be a primitive element of $\mathbb{F}_q^2$. Let us assume that $s_0$ is a common divisor of $q + 1$ and $n' - 1$. Every $c$ with the property $(c, q + 1) = \frac{q + 1}{s_0}$ can raise to a cycle of length $j$

$$\left( y, R_n(y), R_n^{(2)}(y), \ldots, R_n^{(j-1)}(y) \right) = \left( \rho^c, R_n^{(c)}(\rho^c), R_n^{(2)}(\rho^c), \ldots, R_n^{(j-1)}(\rho^c) \right).$$

On the other hand $(c, q + 1) = \frac{q + 1}{s_0}$ implies that there exists $t_0 \in \mathbb{N}$ such that $c = \frac{q + 1}{s_0}$. Hence, we get

$$(\rho^c)^{n' - 1}(q - 1) = (\rho)^{c(n' - 1)(q - 1)} = (\rho)^{\\frac{n' - 1}{s_0}(q^2 - 1)t_0} = (1)^{\\frac{n' - 1}{s_0}t_0} = 1,$$

and

$$(\rho^c)^{n' - 1} = (\rho)^{c(n' - 1)} = (\rho)^{\\frac{n' - 1}{s_0}(q + 1)t_0} = (\rho_t + 1)^{\\frac{n' - 1}{s_0}t_0} \in \mathbb{F}_q,$$

where the last expression is true because the powers of $q + 1$ of $\rho$ forms $\mathbb{F}_q$ in $\mathbb{F}_q^2$. So, we are interested in the number of $c = \frac{q + 1}{s_0}$ such that $(c, q + 1) = \frac{q + 1}{s_0}$. That is the number of $(t_0, s_0) = 1$ and $t_0 \leq s_0$ which equals $\phi(s_0)$ where $\phi(.)$ denotes Euler’s function. Therefore, summing over all $\phi(s_0)$ with $s_0|(q + 1, n' - 1)$ gives the number of elements that contribute to the cycles of length $j$ and all its divisors $i$. We have

$$1 + \sum_{i<j} iN_i = \sum_{s_0|(q + 1, n' - 1)} \phi(s_0) = (q + 1, n' - 1).$$

The last equality is derived from the fact that for every $n$ we have $\sum_{d|n} \phi(d) = n$. We note that 1 in the left hand
side accounts for the element 0 since \( R_n(0) = 0 \) gives an extra fixed point and the above counting enumerates non-zero elements \( y \in \mathbb{F}_q^2 \).

**Corollary 3.** All cycles of the Rédei function \( R_n \) of \( \mathbb{F}_q \) with \((n, q + 1) = 1\) have cycles of length \( j \) or 1 if and only if for every divisor \( s \) of \( q + 1 \) we have \( n \equiv 1 \pmod{s} \) or \( j = \text{ord}_s(n) \).

One can insert \( j = 2 \) and produce a self-inverse Rédei function. Thus, \( R_n(x) \) is a self-inverse function if and only if

\[
\text{for all } s|q + 1, \quad \begin{cases} n \equiv 1 \pmod{s} \text{ or } n^2 \equiv 1 \pmod{s}. \end{cases}
\]

(19)

It is easy to see that (19) is equivalent to \( n^2 \equiv 1 \pmod{q + 1} \) and this simply shows that \( m = n \) for \( R_m(x) = R_n^{-1}(x) \).

Now we are able to find all self-inverse Rédei functions by using the above corollary. Also, one can apply self-inverse Rédei functions to produce self-inverse Rédei interleavers.

**Theorem 25.** Let \( q + 1 = p_0^{k_0}p_1^{k_1} \cdots p_r^{k_r} \). The permutation of \( \mathbb{F}_q \) given by the Rédei function \( R_n \) has cycles of the same length \( j \) or fixed points if and only if one of the following conditions holds for each \( 1 \leq i \leq r \):

- \( n \equiv 1 \pmod{p_i^{k_i}} \),
- \( j = \text{ord}_{p_i^{k_i}}(n) \) and \( j|p_i - 1 \),
- \( j = \text{ord}_{p_i^{k_i}}(n) \), \( k_i \geq 2 \) and \( j = p_i \).

**Proof:** The proof is analogous to the proof of Theorem 2 in [20]. Hence we omit it here.

**E. Skolem interleavers**

In this section we are going to construct some other types of self-inverse interleavers. In this case our underlying structure for construction are various types of Skolem sequences including \( k \)-extended, hooked and \((j, n)\)-generalized. Clearly, we shall utilize \((j, n)\)-generalized Skolem sequences to produce interleavers with cycles of length 1 and \( j \) only. First of all let us make a slight modification to every type of Skolem sequences to turn them into a consistent form for using in our construction method. Let us assume that \( S = (s_1, \ldots, s_t) \) be a Skolem sequence (not a generalized one but maybe \( k \)-extended or even hooked) over a set of integers \( D \). For every \( i \in [n] \) if \( \ell, 2 \leq \ell \leq t \), is the largest index such that \( s_{\ell} = i \) then convert \( s_{\ell} \) to \(-s_{\ell} \) i.e. put \(-i \) instead of \( i \) in the \( \ell \)th position. In the case that \( S \) is a \((j, n)\)-generalized Skolem sequence just insert \(-j \) instead of \( i \) in the \( \ell \)th position. These changed Skolem sequences are called modified Skolem sequences (modified generalized Skolem sequences, respectively).

Now, our strategy is to reorder any set of integers of length \( t \), let us say \( I = \{1, \ldots, t\} \), based on a modified (generalized) Skolem sequence of length \( t \), \( S = (s_1, \ldots, s_t) \).

Assume that \( S^m \) be a modified (generalized) Skolem sequence of order \( n \) with alphabet \([n] \). If \( i \in [n] \) repeats on positions \( u \) and \( v \) where \( u < v \), then \( s^m_u = i \) and \( s^m_v = -i \) and \( v - u = i \). Now, define the interleaver \( \Pi_S \) by sending \( u \) to \( v \). More precisely

\[
\Pi_S(u) = u + s^m_u = u + i = v.
\]

(20)

We observe that holes or zeros of our sequence produce fixed points. In other words, if \( s^m_h = 0 \) for some \( 1 \leq h \leq m \), then \( \Pi_S(h) = h + s^m_h = h + 0 = h \). If we imagine the indices of
our modified Skolem sequence as time and the amounts of our Skolem sequence as location (domain), then this interleaver may be interpreted as a combination of time and domain.

**Example 26.** The sequence

\[ S = (2, 5, 2, 6, 1, 1, 5, 3, 4, 6, 3, 0, 4) \]

is a hooked Skolem sequence. First we have to carry it to a modified hooked Skolem sequence. If we denote the modified version of \( S \) by \( S^m \), we get

\[ S^m = (2, 5, -2, 6, 1, -1, -5, 3, 4, -6, -3, 0, -4). \]

Thus, based on (20), we have

\[
\begin{align*}
\Pi_S(1) &= 1 + s_1^m = 3, \\
\Pi_S(2) &= 2 + s_2^m = 7, \\
\Pi_S(3) &= 3 + s_3^m = 1, \\
\Pi_S(4) &= 4 + s_4^m = 10, \\
\Pi_S(5) &= 5 + s_5^m = 6, \\
\Pi_S(6) &= 6 + s_6^m = 5, \\
\Pi_S(7) &= 7 + s_7^m = 2, \\
\Pi_S(8) &= 8 + s_8^m = 11, \\
\Pi_S(9) &= 9 + s_9^m = 13, \\
\Pi_S(10) &= 10 + s_{10}^m = 4, \\
\Pi_S(11) &= 11 + s_{11}^m = 8, \\
\Pi_S(12) &= 12 + s_{12}^m = 12, \\
\Pi_S(13) &= 13 + s_{13}^m = 9.
\end{align*}
\]

The above equalities induce the following Skolem interleaver

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
3 & 7 & 1 & 10 & 6 & 5 & 2 & 11 & 13 & 4 & 8 & 12 & 9
\end{pmatrix}
\]

**Theorem 27.** Let \( \Pi_S \) be an interleaver constructed using a modified Skolem sequence. Then \( \Pi_S \) is a self-inverse interleaver. Furthermore, if we use a modified \((j, n)\)-generalized Skolem sequence, then \( \Pi_S \) has only cycles of length \( j \) or 1.

**Proof:** Let \( S = (s_1, \ldots, s_m) \) and its modified version, \( S^m \), have been used to produce \( \Pi_S \). For every \( i \in D \) there exist indices \( u \) and \( v \) such that \( u < v \), \( s_u^m = i \), \( s_v^m = -i \) and \( v - u = i \). Based on the definition of \( \Pi_S \) we get

\[ \Pi_S(u) = u + s_u^m = u + i = v, \quad \Pi_S(v) = v + s_v^m = v - i = u, \]

and also if \( s_h^m = 0 \) for some \( h \) we have

\[ \Pi_S(h) = h + s_h^m = h + 0 = h. \]

Hence, \( \Pi_S \) is a self inverse interleaver. On the other hand let us suppose that \( S \) is a \((j, n)\)-generalized Skolem sequence and \( S^m \) be its modified version. For every \( i \in [n] \) there are indices \( r_1, r_2 = r_1 + i, \ldots, r_j = r_1 + (j - 1)i \) such that

\[ s_{r_1}^m = s_{r_2}^m = \cdots = s_{r_{j-1}}^m = i \]

and

\[ s_{r_j}^m = -(j - 1)i. \]

Since \( r_{w+1} = r_w + i \) and based on (20) we get

\[ \Pi_S(r_w) = r_w + s_{r_w}^m = r_w + i = r_{w+1} \]

for \( 1 \leq w \leq j - 1 \), and for \( w = j \) we have

\[ \Pi_S(r_j) = r_j + s_{r_j}^m = r_j - (j - 1)i = r_1. \]

It means that \((r_1, r_2, \ldots, r_j)\) builds a cycle of length \( j \) for \( \Pi_S \). More explicitly, \( \Pi_S(r_w) = r_{w+1} \) for \( 1 \leq w \leq j - 1 \) and \( \Pi_S(r_j) = r_1 \). What we did for holes (zeros) of a modified Skolem sequence, we can likewise do for generalized Skolem sequences.

\[ \square \]

**V. EXPERIMENTAL RESULTS AND DISCUSSION**

We simulate our theoretical results about self-inverse interleavers in practical situations. Given the size \( q \) we search for good self-inverse interleavers based on self-inverse permutation functions. We consider only turbo codes with component codes generated by \([1 5/7]\). We investigate the interleavers of size 256, 1019, 1021, 1024, and 1031. In addition, experimental results for size 1087 can be found in [22]. We choose 256, and 1024 since they are usually considered dimensions. Since every element in a finite field of characteristic 2 is a square we
cannot define Rédei functions for sizes 256 and 1024. Hence, we decided to select finite fields of sizes close to 1024, i.e. prime numbers like 1019, 1021, and 1031. So, we do not have simulation results for Rédei interleavers of sizes 256 and 1024. Instead we examine the performances of Rédei interleavers of sizes 1019, 1021, and 1031. It has to be noted that, if we replace [15/7] by other constituent codes such that [121/37] or even [125/37], we expect much better performances in the same frame sizes. It happens only because in these cases we use a convolutional code with larger memory.

In Fig. 2, 3, and 5 we demonstrate the performances of self-inverse Monomial, Dickson, Möbius, Skolem and the ones that have been reported in [23, 25]. We can say that, in low SNR region all the interleavers of size 256 have the same quality and for high SNRs, the Dickson and Möbius self-inverse interleavers outperform the best introduced self-inverse interleavers of the same frame size in this work. In addition, Möbius self-inverse interleaver has the best performance between other known interleavers in SNRs larger than 1.75 (dB). Also it is worth to note that in our simulations we found the bit error rate of $7.8 \times 10^{-6}$ for SNR equals 2.5 (dB). This shows an excellent improvement in high SNRs regions in comparison with other self-inverse and non-self-inverse interleavers cited in [23, 25].

In Fig. 2, 3 and 5 we demonstrate the performances of our self-inverse interleavers in different sizes. Obviously, the self-inverse Möbius and Rédei self-inverse interleavers which have been constructed based on permutation functions have the best performances among all other new introduced self-inverse interleavers.

Fig. 4 shows various self-inverse interleavers of size 1024 in comparison with two important interleavers reported from [23, 25]. It can be seen that the self-inverse Möbius interleaver of size 1024 outperform previous known self-inverse interleaver of the same size in [25] in close to capacity region and has the same curve as this interleaver for SNRs bigger than 1.5.

Overall, as we expected Skolem sequences and Monomial permutation polynomials are not appropriate for interleaving purposes in turbo codes. Because by our definition, Monomial interleavers turns back to be linear interleavers and these interleavers have a lower bound on their performance [25]. Furthermore, according to our construction method for Skolem sequences that has been described in [7], [8], [9] and [17] we have problematic “patterns” in terms of the parameters in [26]. These patterns produce turbo codes with poor minimum distance. However they may be useful for making interleavers with known cycle structure. We have the same possible option and characteristics for Dickson and Rédei permutations as well. More specifically, suppose that we are given a cycle structure $(i_1, \ldots, i_n)$ for a permutation $\sigma$ where $i_j$ denotes the number of cycles of length $j$ in $\sigma$. Assume that we want to construct an interleaver following the structure of the permutation $\sigma$. It is clear that $\sum_{j=1}^{n} ji_j = n$. Now we can construct such a permutation deterministically and in cases using our new interleavers based on permutation polynomials and functions and also Skolem sequences. In other words we need $i_j$ cycles of length $j$. So for example we can use a $(j, ji_j)$-generalized Skolem sequence or even employ Monomial, Dickson or Rédei permutations to produce these cycles. The cycle structure of Monomial, Dickson and Rédei permutations can be determined using results of Theorems [2, 13] and Corollary [5].

In Fig. 6 performance analysis of Rédei interleavers of sizes 1021 and 1031 has been reported. The Rédei function that we
have used to construct this interleaver has cycles of length $j$ or 1 only. We see that for $j = 3$ and length 1031 we obtain a coding gain of 1 (dB) in comparison to the self-inverse interleaver at bit error rate $10^{-5}$. In addition, the Rédei interleaver with cycles of length $j = 3$ and 1 only perform so much better for frame length $q = 1021$ rather than self-inverse Rédei interleavers of the same size. We increase $j = 3$ to $j = 9$ and we get quite the same performance as when $j = 3$ except in low SNR region where we outperform previously introduced Rédei interleavers. Hence, it seems possible that we can get better performances for turbo codes constructed except in low SNR region where we outperform previously needed to study this trade-off between the performance of turbo codes and interleavers with some given cycle structure.

VI. CONCLUSION AND FURTHER RESEARCH

We investigate some deterministic interleavers based on permutation functions over finite fields. Two well-known permutation functions are explained. Rédei functions are treated in detail. We derive an exact formula for the inverse of every Rédei function. The cycle structure of these functions are given. The exact number of cycles of a certain length $j$ is provided. Specifically, we focus on self-inverse permutation functions which can produce self-inverse interleavers and reduce the memory consumption considerably [21].

One can employ well-known permutation polynomials like monomials and Dickson polynomials to construct self-inverse permutation polynomials. The cycle structure of monomial and Dickson polynomials have been determined [1, 19]. We have also experimented with these self-inverse monomial and Dickson interleavers.

As we mentioned $s$-random interleavers are very interesting because of their strong randomness. One can use perhaps the algorithm to construct these interleavers along an interpolation formula to produce new permutation functions over finite fields. This can be viewed as an application of coding theory in finite field. However, obtaining a close formula for $s$-random based permutation functions in terms of coefficients and other defining parameters of a function over finite fields seems a hard problem but also a promising research topic.

APPENDIX A

PROOF OF THEOREM 25

This appendix contains the proof of Theorem 25 and it is given here for the referees only.

Lemma 1. If $n \equiv b \pmod{p^l}$, then $n^p \equiv b^p \pmod{p^{l+1}}$ for all $l \geq 1$.

Lemma 2. Let $j = \text{ord}_{p^l}(n)$. Then $j = \text{ord}_{p^{l+1}}(n)$ or $jp = \text{ord}_{p^{l+1}}(n)$.

Proposition 28. We have that $j = \text{ord}_{p^k}(n)$ and $jp - 1$ if and only if $j = \text{ord}_{p^l}(n)$ for all $1 \leq l \leq k$.

Lemma 3. Let $p = \text{ord}_{p^k}(n)$ for some $k \geq 1$. Then either $2 = p = \text{ord}_{p^l}(n)$ for $2 \leq l \leq k$ or $n \equiv 1 \pmod{p^l}$ for $1 \leq l \leq k$.

Lemma 4. Let $j = \text{ord}_s(n)$, $j = \text{ord}_l(n)$ and $(s, l) = 1$. Then $j = \text{ord}_{sl}(n)$.

Lemma 5. Let $j = \text{ord}_s(n)$, $n \equiv 1 \pmod{l}$ and $(s, l) = 1$. Then $j = \text{ord}_{sl}(n)$.

Proof of Theorem 25 (\iff) If $n \equiv 1 \pmod{p^h}$ for all $0 \leq l \leq r$, then $R_{n}(x)$ is the identity permutation. Suppose that $1 < j = \text{ord}_{p^h}(n)$ for some of the $i$s and $n \equiv 1 \pmod{p_i}$ for the others. Proposition 28 and Lemma 5 guarantee that $j = \text{ord}_{p_i}(n)$ or $n \equiv 1 \pmod{p_i^k}$ for all $0 \leq l \leq r$ and $1 \leq k \leq k_i$. Now, if $t \mid (q + 1)$, then by Lemmas 4 and 5 we have that, $j = \text{ord}_t(n)$ or $n \equiv 1 \pmod{t}$. Hence, by Corollary 3 all the cycles have length $j$ or 1.

(\implies) Suppose that all the cycles have the same length $j$. Then, by Corollary 3 $j = \text{ord}_t(n)$ or $n \equiv 1 \pmod{t}$ for all $t$ that divides $q + 1$. This holds in particular for $t = p^k_i$; $0 \leq l \leq r$. We only have to prove that, if $j = \text{ord}_{p^i}(n)$ then $j \mid (p^i - 1)$ or $j = p^i$; $k_i \geq 2$. Suppose that $1 \neq j = \text{ord}_{p_i^k}(n)$. If $k_i = 1$ then $j \mid (p_i - 1)$ and we are done. If $k_i \geq 2$ and $j \mid (p_i - 1)$, then Proposition 28 implies that $j \neq \text{ord}_{p_i^k}(n)$ for some $k < k_i$. Let $s$ be the largest one such that $j \neq \text{ord}_{p_i^k}(n)$. Then $n \equiv 1 \pmod{p_i^k}$ because otherwise, by Corollary 3 there would be a cycle of length different from $j$. By Lemma 1 $i^{p_i^k} \equiv 1 \pmod{p_i^{k+1}}$. But $j = \text{ord}_{p_i^{k+1}}(n)$ implies that $j \mid p_i$ and hence $j = p_i$.

REFERENCES


Fig. 6. Comparison graph for Rédei interleavers of lengths 1021 and 1031 for various $j \neq 2$. resistant to random interference”, Bell Syst. Tech. J. pp. 973-994, 1960.


