Polynomial fuzzy models for nonlinear control: a Taylor-series approach

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Abstract—Classical Takagi-Sugeno fuzzy models are formed by convex combinations of linear consequent local models. Such fuzzy models can be obtained from nonlinear first-principle equations by the well-known sector-nonlinearity modeling technique. This paper extends the sector-nonlinearity approach to the polynomial case. In this way, generalized polynomial fuzzy models are obtained. The new class of models is polynomial both in the membership functions and in the consequent models. Importantly, Takagi-Sugeno models become a particular case of the proposed technique. Recent possibilities for stability analysis and controller synthesis are also discussed. A set of examples shows that polynomial modeling is able to reduce conservativeness with respect to standard Takagi-Sugeno approaches as the degrees of the involved polynomials increase.

Index Terms—fuzzy modeling, fuzzy control, relaxed stability conditions, polynomial fuzzy systems, sum of squares

I. INTRODUCTION

Fuzzy control started as a heuristic methodology in the 1970’s, coding control rules by hand, trying to embed heuristics and reasoning into control loops. However, most of the heuristic results either (a) have no fundamental differences with standard PID regulators (the fuzzy-PD, fuzzy-PI and alike) or (b) they fuzzify operation rules for a complex plant, being one-of-a-kind tailored developments which have little interest for a broad audience. Due to these reasons, the emphasis on heuristics and reasoning in fuzzy control (popular in applications) has almost disappeared in research, in favor of rigorous mathematical tools, in order to guarantee control specifications expressed in terms of stability, performance, robustness to modeling errors, etc. For an overview of the current situation and perspectives of fuzzy control, the reader is referred to [1], [2].

For a quite general class of nonlinear systems, a systematic modeling methodology is available to exactly transform them into the so-called Takagi-Sugeno (TS) form [3]. The above methodology is known as sector-nonlinearity fuzzy modeling [4]. Hence, by using such models, current fuzzy control results allow solving nonlinear control problems with efficient semidefinite-programming tools (LMI [5], [6], sum-of-squares [7]). There are, however, some conservativeness issues in the overall procedure: such “performance gap” between fuzzy controllers and ideal non-linear ones is discussed in [8], [9].

Recently, sum-of-squares (SOS) ideas and software have appeared in literature [7], [10] to control polynomial systems. Such systems have the form \( \dot{x} = p(x) + q(x)u \) for some polynomial matrices \( p(x), q(x) \). They are, quite naturally, amenable to adaptation to the fuzzy case, as reported in some recent literature to be later cited. Indeed, the basics of the SOS formalism allows extending many linear control results to polynomial systems. As a natural extension, the ordinary Takagi-Sugeno models (convex sums of linear systems) may be extended to polynomial fuzzy models (convex sums of polynomial systems). Some modeling approaches to get such polynomial fuzzy models are those in [11] (where fuzzy polynomial-in-membership models are proposed) and [12] (where fuzzy polynomial-in-consequents ones appear) and its combination is hinted in [8], but almost no insight on how to obtain them from first-principle models is provided in the cited references.

This paper presents an extension of the sector-nonlinearity modeling technique which builds a family of progressively more precise\(^1\) polynomial fuzzy models, based on the Taylor series. The classical TS representation is a particular case of the proposed general approach. Some stability and stabilization problems can be successfully solved for the resulting polynomial fuzzy systems [8], [12], [13], [14], [15], by extending the seminal methodologies in [7], [10] to the fuzzy case. For completeness, in order to ease the understanding of the examples, the referred methodologies are briefly outlined in this paper.

Importantly, the technique being presented allows for asymptotically exact results for smooth nonlinear systems: if there is a smooth Lyapunov function for it, there will exist a polynomial Lyapunov function and a polynomial fuzzy model with a finite degree which will allow to prove stability of the original system (some extra assumptions apply). Asymptotic exactness applies only in compact regions of interest where the Taylor series approximation of the nonlinearities, as well as those of a valid Lyapunov function and its derivatives, converge uniformly. Control synthesis, however, requires an affine-in-control structure, as well as some additional artificial variables which introduce some conservativeness. Note, however, that there are many positive polynomials which are not SOS [16]: even if (asymptotically) there is no conservativeness in many polynomial fuzzy modeling cases, conservativeness remains in the SOS approach to polynomial fuzzy system analysis.

The structure of the paper is as follows: next section establishes notation and reviews classical sector-nonlinearity

\(^{1}\)The sector-nonlinearity technique is exact (no approximation involved) for any polynomial degree, including the classical TS models. Precise above must be understood as consequent models fitting more closely the nonlinearity being modeled, see example section.
Takagi-sugeno models. Section III introduces the polynomial fuzzy models and the Taylor-series based extension of the modeling methodologies. Section IV outlines recent developments in stability analysis and control design for polynomial fuzzy systems, to be used in the examples. Key remarks about locality, membership shape and asymptotical exactness are discussed in Section V. Illustrative examples appear in section VI. Some remarks and a brief discussion are provided in Section VII. A conclusion section closes the paper.

II. PRELIMINARIES: SECTOR-NONLINEARITY METHODOLOGY FOR TAKAGI-SUGENO MODELS

The sector-nonlinearity methodology [4] is a well-established technique able to obtain fuzzy models from a quite general class of nonlinear systems, as discussed below.

Definition 1 (TS representable system) Consider an \( n \)-th order dynamic system with \( p \) manipulated inputs so that:

1) The dynamics of the system can be expressed as:

\[
\dot{x} = f(x,u,z)
\]

where \( x(t) \in \mathbb{R}^n \) is the system state, \( u(t) \in \mathbb{R}^p \) are input variables, and \( z(t) \in \mathbb{R}^q \) is a vector of functions of time (interpretable as exogenous non-manipulated inputs or actual time-variance).

2) \( z \) takes values in a known compact set \( \Omega_z \).

3) \( x = 0, u = 0 \) is an equilibrium point for any value of \( z \), i.e., \( f(0,0,z) = 0 \) for all \( z \in \Omega_z \). Note that fixed equilibrium points may not exist in a general time-varying system, so this issue restricts the class of systems amenable to TS modeling (there are some options to relax the assumption, see later footnote 2).

4) \( f(x,u,z), u(t), z(t) \) fulfill the required Lipschitz conditions for existence and uniqueness of the solution of (1).

5) algebraic transformations allowing appearing (1), for some \( \xi_{ij} \), in the form:

\[
\dot{x}_i = \sum_{j=1}^{n} \xi_{ij}(x,z)x_j + \sum_{j=1}^{p} \xi_{ij}(x,z)u_j \quad i = 1, \ldots, n
\]

Lemma 1 ([4]) Consider a system fulfilling the conditions in the above definition, and a region of interest \( \Omega = \Omega_x \times \Omega_z \) (a priori defined). If \( \xi_{ij} \) are bounded in \( \Omega \), there exist functions \( \mu_i(x,z), i = 1, \ldots, r \) so that the system can be exactly represented (locally in \( \Omega \)) as:

\[
\dot{x} = \sum_{i=1}^{r} \mu_i(x,z)(A_ix + B_iu) \quad \forall(x,z) \in \Omega
\]

with the functions \( \mu_i \) belonging to the standard simplex:

\[
\Delta^{r-1} := \{ (\mu_1, \ldots, \mu_r) \in \mathbb{R}^r \mid 0 \leq \mu_i \leq 1, \sum_i \mu_i = 1 \}
\]

Such form is denoted as Takagi-Sugeno (TS) form [3] in fuzzy literature; functions \( \mu_i \) are denoted as (antecedent) membership functions and \( r \) is the number of “rules” of the TS model\(^2\).

Proof: Considering expression (2), each \( \xi_{ij} \) may be expressed as:

\[
\xi_{ij} = F_i(x,z)\xi_{ij,1} + \mu_i(x,z)\xi_{ij,2}
\]

\[
\mu_i(x,z) := \frac{\xi_{ij,1}(x,z) - \xi_{ij,2}}{\xi_{ij,1} - \xi_{ij,2}} \quad \mu_i^2 = 1 - \mu_i^1
\]

With any \( \xi_{ij,1}, \xi_{ij,2} \) such that

\[
\xi_{ij,1} \geq \sup_{(x,z) \in \Omega} \xi_{ij}, \quad \xi_{ij,2} \leq \inf_{(x,z) \in \Omega} \xi_{ij}
\]

so that the obtained \( \mu_i^1, \mu_i^2 \) lie in the closed interval \([0,1]\).

The bounds in (8) exist by assumption. For instance, if \( \Omega \) is compact and \( \xi_{ij} \) are continuous, such supremum and infimum are actually attained at minimum and maximum points.

It’s straightforward to realize that, after some manipulations, the system may be expressed as (3), with the number of “rules” being\(^3\) \( r = 2^n + p \). For details, see [4], [18].

Example 1 Consider a system \( \dot{x} = \sin(x) \) and \( \Omega = [-3,3] \).

Define \( \xi(x) = \sin(x)/x \) for \( x \neq 0 \), and \( \xi(0) = 1 \), so \( \dot{x} = \xi(x) \).

As \( \xi_1 = \min_{x \in \Omega} \xi(x) = \sin(3)/3 \) and \( \xi_2 = \max_{x \in \Omega} \xi(x) = 1 \), we may write \( \xi(x) \) as the interpolation between \( \xi_1 \) and \( \xi_2 \), i.e., denoting \( \mu_2 = 1 - \mu_1 \), we may write

\[
\dot{x} = \mu_1(x)\sin(x) + \mu_2(x)\sin(3)/3 \quad \forall x \in \Omega
\]

Indeed, from (7):

\[\mu_1(x) = \frac{\sin(x)/x - \sin(3)/3}{1 - \sin(3)/3}\]

The plot of \( \mu_1(x) \) appears in Figure 2 on page 8 (plus other more general polynomial models discussed later).

The methodologies stemming from Lemma 1 are nowadays classical and well-known, usually denoted as “sector-nonlinearity modeling. The reader is referred to [4] for further examples and details. It can be shown that the resulting models have a tensor-product structure; such structure can be used to relax the conservativeness of some fuzzy control design techniques [18].

\(^2\)If assumption 3 in Definition 1 does not hold, i.e., there are exogenous variables \( z \) which change the equilibrium point, they should be considered in the same way as \( u \), by forming a TS model in the form:

\[
\dot{x} = \sum_{i=1}^{r} \mu_i(x,z)(A_ix + B_iu + M_iz) \quad \forall(x,z) \in \Omega
\]

amenable to disturbance-rejection control solutions (details omitted for brevity). In fact, exogenous \( z \) and \( z \) have a very similar role regarding modeling; however, intentionally, \( \mu_i \) has not been made dependent on manipulated variables \( u \) in order to avoid algebraic loops in controller implementation.

\(^3\)Such a number of rules would be intractable if \( n + p \) were a large number. As the decomposition (3) is usually not unique, a decomposition should be found so that the resulting model is tractable. The number of rules will actually be much lower if:

- some \( \xi_{ij} \) are actually constant
- some nonlinearities appear several times in the system’s equations, allowing for, say, \( \xi_{23} \equiv \xi_{14} \)
- there is the option to treat some nonlinearities as uncertainty to reduce the number of rules [17]
Similar developments can be made with nonlinear output equations, \( y = h(x, u) \), if any. For convenience, shorthand \( \mu_i \) denoting \( \mu_i(x, z) \) will be frequently used in the sequel.

The objective of the paper is generalizing the above ideas to polynomial forms\(^4\) generalising (6).

**Discrete-time systems:** The sector-linearity modeling procedure also applies to discrete-time systems \( x_{k+1} = f(x_k, u_k, z_k) \) with obvious modifications, to obtain discrete TS models \( x_{k+1} = \sum_{i=1}^{I} \mu_i(x_k, z_k)(A_i x_k + B_i u_k) \). In the same way, most of the polynomial extensions in the rest of the paper can be adapted to the discrete case. Details are omitted for brevity.

**Linguistic interpretability:** The above procedure is a formal way of bounding nonlinear functions with linear ones. As (7) is an affine transformation of \( \xi_{ij} \), the membership functions inherit most of the properties of the nonlinearity they are modeling; in this way, non-smooth functions, those with multiple maxima and minima, etc. might give as a result non-convex membership functions which cannot be easily interpreted linguistically as “fuzzy numbers” [26]. This is a possible limitation of the TS approach regarding readability of the resulting models; definitely, it is not a limitation regarding accuracy or stability analysis and controller design. Nevertheless, if the function to be modeled is smooth and the region of interest is not too large, the obtained membership functions keep the linguistic interpretability (see Figure 2 and region of interest is not too large, the obtained membership models.\(^\text{5}\) i.e., a fuzzy model (in variables \( x, u \), \( z \) and size of \( \mu \)) assume perfect knowledge of the original nonlinear model. \(^\text{6}\)

Possible limitation of the TS approach regarding readability issues in TS fuzzy control [27] are out of the intended scope of this paper.

**Modeling error and uncertainty:** The above methodology assumes perfect knowledge of the original nonlinear model. A simple variation allows transforming “uncertain nonlinear models” to “uncertain TS models”.

Consider, for instance, \( \dot{x}_i = f(x, z_1, z_2) \) where \( z_1 \) denotes known variables and \( z_2 \) are “uncertain” ones.

The above expression can be fuzzified to TS considering \( z_2 \) to be known (if the assumptions in the previous Lemma hold). Then, the result is a fuzzy model in the form, for instance:

\[
\dot{x} = \sum_{i=1}^{I} \mu_i(z_1) \sum_{j=1}^{N} \mu_j(z_2) A_{ij} x
\]

i.e., a fuzzy model (in variables \( z_1 \)) with polytopic uncertainty\(^5\) arising from the unknown \( z_2 \). Note that if the uncertain variable \( z_2 \) changed the equilibrium point, it would appear as in (5).

The above idea on uncertainty modeling naturally extends to the polynomial cases in next sections, but details are left to the reader for brevity.

\(^4\)There exist other identification-based possibilities for fuzzy modeling (such as clustering [19], [20], approximate neuro-fuzzy systems in adaptive control [21], [22], or tensor-product approximations [23]). In some of them polynomial fuzzy systems might be obtained by modifying some of the regressors; intentionally, however, they are not being considered in the scope of this paper. Other conceptions of dynamic fuzzy systems, such as stochastic [24] or possibilistic [25], which might have polynomial extensions are also not considered in this work.

\(^5\)Other classes of uncertain TS models in literature (see [28], [29], [24]) replace \( A_i \) by \( A_i + \Delta A_i \) and \( B \) by \( B + \Delta B \) in (3). Some knowledge on the structure and size of \( \Delta A_i \) and \( \Delta B \) is usually assumed (such as \( \Delta A_i = H \delta E_i \), with \( H, E_i \) known and \( |\delta| < 1 \)). Actually, modeling issues in these literature references are usually overlooked.

III. POLYNOMIAL FUZZY MODELS

This section will extend the TS models (3) and (5) to polynomial consequents, as defined below.

**Definition 2 (Polynomial fuzzy system)** A polynomial fuzzy system is a system whose dynamics can be expressed as:

\[
\dot{x}_i = p_i(x, u, z, \mu) \quad i = 1, \ldots, n
\]

with \( p_i(x, u, z, \mu) \) being a polynomial in the variables \( x, u, z, \mu \) so that \( p_i(0, 0, 0, 0) = 0 \) for all \( \mu \in \Omega^{\mu} \) (i.e., steady-state equilibrium does not depend on the membership functions).

The meaning of \( x, u, z \) and \( \mu \) is the same as in the previous section.

An analogue definition would arise for discrete-time systems or those with output equations.

As memberships add one, without loss of generality, the polynomials \( p_i \) may be assumed to be homogeneous in \( \mu \), i.e., composed only of monomials whose degree in the variables \( \mu \) is the same. Indeed, any monomial in \( p_i \) can be multiplied by \( (\sum \mu_i)^q \), for any \( q \), in order to incorporate as many powers of \( \mu \) as required to make \( p_i \) homogeneous [30].

**A. Extension of the sector-linearity modeling**

Let us now consider some situations in which the sector-linearity methodology in Section II can be extended to a polynomial case.

**Lemma 2 Assume that:**

1) the equations of a nonlinear system (1) may be expressed, via some algebraic transformations, in the form:

\[
\dot{x}_i = p_i(x, u, z, \xi_1(x, z), \ldots, \xi_t(x, z)) \quad i = 1, \ldots, n
\]

where:

a) \( p_1, \ldots, p_n \) are polynomials in the variables \( x, u, z, \xi_1, \ldots, \xi_t \), with real coefficients

b) \( \xi_1, \ldots, \xi_t \) are some known nonlinear functions \( \mathbb{R}^{(n+q)^t} \mapsto \mathbb{R} \), which depend on the state \( x \) and exogenous variables \( z \);

2) \( x = 0, u = 0, z = 0 \) is an equilibrium point irrespective of \( \xi_t \), i.e., replacing \( \xi_t(x, z), i = 1, \ldots, t \) above by generic variables \( \varepsilon_t \), the following equilibrium condition holds for any \( \varepsilon_t^j \):

\[
p_i(0, 0, 0, \varepsilon_1^j, \ldots, \varepsilon_t^j) = 0 \quad i = 1, \ldots, n
\]

3) \( \xi_t(x, z), i = 1, \ldots, t, \) are bounded in a region of interest \( \Omega = \Omega_x \times \Omega_z \).

Then, if the above assumptions hold, the nonlinear system can be exactly expressed as a polynomial fuzzy system (locally in \( \Omega \)).

**Proof:** First, express each \( \xi_t(x, z) \) as an interpolation between any lower (\( \xi_t^1 \)) and upper (\( \xi_t^2 \)) bound in \( \Omega \) (which exist by assumption), i.e.,

\[
\xi_t^j = \mu_1^j \xi_t^1 + \mu_2^j \xi_t^2.
\]

Then, replacing the expressions for \( \xi_t \) in (11), a polynomial fuzzy system is obtained.
Note that the derivations of a standard fuzzy system in Section II are a particular case: (2) clearly fulfills the more general conditions (11)-(12).

**Example 2** Consider a nonlinear system

\[ \dot{x} = f(x, z) = -x^5 \sin^2(x) - \sin^3(z) \]

The function \( f(x, z) \) may be expressed with \( \xi_1(x, z) = \sin(x) \), \( \xi_2(x, z) = \sin(z)/z \) as \( \dot{z} = -x^5 \xi_1^2 - \xi_2^3 \). Modeled in the zone \(|x| < 1, |z| < 1\) we may write:

\[
\begin{align*}
\xi_1 &= \sin(1)\mu_{11}(x) + \sin(-1)\mu_{12}(x) \\
\xi_2 &= \mu_{21}(z) + \mu_{22}(z)\sin(1)
\end{align*}
\]

and obtain a polynomial fuzzy expression for \( f(x, z) \) so

\[ \dot{x} = -x^5(0.8415(\mu_{11}(x) - \mu_{12}(x)))^2 + (\mu_{21}(z) + \mu_{22}(z)0.8415)^3z \]

**Note:** if an affine structure in input \( z \) were desired (for instance, affine-in-control models are needed in order to apply results in Section IV-B), \( \xi_2(x, z) = \sin^3(z)/z \) would have been needed, instead of (14).

In this way, the first proposed extension to TS modeling is using the sector-nonlinearity technique only in non-polynomial nonlinearities, \( \xi \), to bound them by two constants.

The above is the basic underlying idea in the modeling approach outlined in [15].

However, there exists the possibility to bound a nonlinearity within two polynomials (instead of two constants): the result would still be a polynomial fuzzy system representation, but more general than the one above. This is the objective of the next subsection, where \( f(x) \) can correspond to any of the non-polynomial nonlinearities \( \xi \) in Lemma 2, so the above lemma keeps true, but in a more interesting and general way (its generality is later explored in Lemma 4 in Section V-C).

**B. Taylor-series based fuzzification**

**Lemma 3** Consider a sufficiently smooth function of one real variable, \( f(x) \), so that its Taylor expansion of degree \( n \) exists [31], i.e., there exists an intermediate point \( \psi(x) \in [0, x] \), so that:

\[
f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(0)}{i!} x^i + \frac{f^{(n)}(\psi(x))}{n!} x^n \tag{15}
\]

where \( f^{(i)}(x) \) denotes the \( i \)-th derivative of \( f \) and \( f^{(0)}(x) \) is defined, plainly, as \( f(x) \). Assume also that \( f^{(n)}(x) \) is continuous in a compact region of interest \( \Omega \). Then, an equivalent fuzzy representation exists in the form:

\[
f(x) = \mu_1(x) \cdot p_1(x) + \mu_2(x) \cdot p_2(x) \quad \forall x \in \Omega \tag{16}
\]

where \( \mu_1(x) + \mu_2(x) = 1 \) and \( p_1(x), p_2(x) \) are polynomials of degree \( n \). Furthermore, \( p_1 - p_2 \) is a monomial of degree \( n \).

**Proof:** Denote the Taylor approximations as:

\[
f_n(x) := \sum_{i=0}^{n-1} \frac{f^{(i)}(0)}{i!} x^i \tag{17}
\]

For later developments, denoting \( f_0(x) = 0 \), denote

\[
T_n(x) := \frac{f(x) - f_n(x)}{x^n} = \frac{f^{(n)}(\psi(x))}{n!} \tag{18}
\]

so the Taylor remainder is \( f(x) - f_n(x) = T_n(x)x^n \).

In the region of interest \( \Omega \), \( T_n(x) \) is bounded because \( f^{(n)}(x) \) is continuous in \( \Omega \), as assumed in the lemma.

Denoting:

\[
\psi_1 := \sup_{x \in \Omega} T_n(x), \quad \psi_2 := \inf_{x \in \Omega} T_n(x),
\]

we may write:

\[
T_n(x) = (\mu(x) \ast \psi_1 + (1 - \mu(x)) \ast \psi_2) \tag{19}
\]

with:

\[
\mu(x) := \frac{T_n(x) - \psi_2}{\psi_1 - \psi_2} \tag{20}
\]

Hence, as \( f(x) = f_n(x) + T_n(x)x^n \), it can be expressed as:

\[
f(x) = f_n(x) + (\mu(x) \ast \psi_1 + (1 - \mu(x)) \ast \psi_2)x^n \tag{21}
\]

so the polynomial consequent \( p_1 \) in (16) is given by \( p_1(x) = f_n(x) + \psi_1x^n \), and \( p_2(x) = f_n(x) + \psi_2x^n \).

**Corollary 1** If \( f(0) = 0 \), setting \( n = 1 \) we obtain the usual sector-nonlinearity methodology which bounds a function between two degree-1 polynomials, \( a_1x \leq f(x) \leq a_2x \), and generates the fuzzy model as the interpolation between them.

**Proof:** The corollary is evident from \( f_1(x) = f(0) = 0 \) and \( T_1 = f(x)/x \).

Note that, as in TS modeling, the representation (16) is exact, i.e., there is equality (no approximation involved) and there is no uncertainty in the membership functions, defined in (20).

Conceptually, the resulting membership functions can be thought of as entities capturing “the nonlinearity which cannot be described by a polynomial of a prescribed degree”. The idea generalizes the interpretation of classical Takagi-Sugeno memberships (they captured “all” the nonlinearity between some linear sector boundings).

As a conclusion, using the Taylor-based modeling for non-polynomial nonlinearities (say, trigonometric, exponential, etc.), any smooth nonlinear system can be exactly expressed as a fuzzy polynomial one in a compact domain \( \Omega \). Examples will be given in Section VI.

**Remark:** The Taylor-series approach above can be applied to any function which can be written as an expression tree with functions of one variable, addition and multiplication. For brevity, the idea is later illustrated in example 6 in Section VI.

**Remark:** Note that, as each nonlinearity results in a two-rule polynomial fuzzy description, the number of rules will still be a power of 2, keeping a final tensor-product structure [18]. Details are analogous to those in classical sector-nonlinearity models, omitted for brevity.
IV. STABILITY ANALYSIS AND CONTROLLER DESIGN

Techniques for Fuzzy Polynomial Systems

The results in section III-B allow obtaining polynomial fuzzy models of arbitrary degree of any smooth nonlinearity in the first-principle equations of a physical system (say, exponential, trigonometric, etc. functions). As mentioned in the introduction, some results have appeared recently dealing with stability and control design for fuzzy polynomial systems. For completeness of this paper, and in order to unify notation and clarify the methodology used in the example section, those results (some by the authors) will be reviewed and briefly outlined in this section.

The notation $\Sigma_a$ will be used to denote the set of SOS polynomials in the variables $a$, i.e., those expressed as the sum of squares of simpler polynomials. The key idea in the SOS approach is that an even-degree polynomial $p(a)$ is SOS if and only if there exist a vector of monomials $m(a)$ and a constant positive-definite matrix $H$ such that $p(a) = m(a)^T H m(a)$; in this way, SOS problems can be solved via semidefinite-programming tools, searching for such an $H$. The reader is referred to [7] for details. Evidently, all SOS polynomials are positive, but the converse is not true [16].

A. Stability Analysis

Consider a polynomial fuzzy system expressed as:

$$\dot{x}_i = p_i(x, \mu)$$  (22)

For operational purposes [8], the polynomials should be made homogeneous in $\mu$ and the membership functions, positive, will be described by the change of variable $\mu_i = \sigma^2_i$, resulting in a model $\dot{x} = \tilde{p}(x, \sigma)$.

Stability will be proved if a polynomial Lyapunov function $V(x)$ is found verifying:

$$V(x) - \varepsilon \in \Sigma_x$$

$$R(\sigma, x) := -\frac{dV}{dx} \tilde{p}(x, \sigma) \in \Sigma_x, \sigma$$  (23)

where $\varepsilon$ is a radially unbounded positive polynomial, usually $\Sigma_x x_j^2$ (but not necessarily so).

Indeed, setting $V(x)$ to be an arbitrary degree polynomial in the state variables ($\text{but not}$ in the memberships, in order to avoid the need of its derivatives), $\frac{dV}{dx}$ is also a vector of polynomials in the variables $x$ and $\sigma$. Hence $\frac{dV}{dx}$ denoted as $R(\sigma, x)$ in (24) is a polynomial. If $V$ is linear in some decision variables (the natural choice are the polynomial coefficients), so is $\frac{dV}{dx}$ and expressions (23) and (24) can be directly introduced into sum-of-squares programming packages [7] in order to get values of the decision variables fulfilling the above constraints. Examples of stability analysis of such systems appear in [8], [12], [15], and in the example section in this paper.

B. Fuzzy controller design

If the fuzzy polynomial system is affine in control, the procedures in [13] (which adapt [10] to the fuzzy case) may be readily applied. Indeed, consider an $n$-th order affine in control polynomial fuzzy system:

$$\dot{x} = \sum_{j=1}^r \mu_j(x)(A_j(x)\kappa(x) + B_j(x)u)$$  (25)

where $\kappa(x)$ is a vector composed of $t$ known polynomials on the state variables, say $\kappa = (x_1, x_2, x_1^2, x_1x_2, x_1^3, \ldots)^T$ and $A_j(x), B_j(x)$ are polynomial matrices with suitable dimensions. Consider now a fuzzy-polynomial control law [15]:

$$u := \sum_{j=1}^r \mu_j K_j(x)Q(x)\kappa(x)$$  (26)

where $K_j(x)$ and $Q(x)$ are to-be-computed polynomial matrices. Define a $t \times n$ Jacobian matrix $M(x)$ with elements:

$$M_{ij} := \frac{\partial K_i(x)}{\partial x_j}$$  (27)

so $\kappa = M(x)\dot{x}$, by chain rule. Define a candidate Lyapunov function:

$$V := \kappa^T P^{-1} (\dot{x}) \kappa$$

where $\dot{x}$ are the state variables whose time derivative does not explicitly depend on $u$, i.e., $\frac{dx}{dt} = 0$ (hence, the corresponding row of $B_j(x)$ is identically zero, for all $j$) and let $J$ denote the index set of those variables. Let $A_{ij}$ denote the $k$-th row of $A_j$.

Then, setting $P^{-1}(\dot{x}) = Q(x)$ in (26), we have:

$$\frac{dV}{dt} = \kappa^T dP^{-1} \dot{\kappa} + \kappa^T Q \dot{\kappa} + \dot{\kappa}^T Q^T \kappa =$$

$$= -\kappa^T Q \left( \sum_{k \in J} \frac{dP}{dx_k} \sum_{j=1}^r \mu_j A_{ij}^T \kappa \right) Q \kappa + 2 \kappa^T Q M \sum_{j=1}^r \mu_j (A_j \kappa + B_j u)$$  (28)

So, replacing the control action, and denoting $Q \kappa = v$ we get:

$$\frac{dV}{dt} = \sum_{j=1}^r \sum_{k \in J} \mu_j \frac{dP}{dx_k} \left( \sum_{j=1}^r \mu_j A_{ij}^T \kappa \right) - 2 \left( MA_j P + B_j K_i \right) v$$  (29)

The reader is referred to [13], [10], [15] for details. Now, the change of variable $\mu_i = \sigma^2_i$ is made and decision variables (coefficients of the polynomial elements in matrices $P$ and $K_i$) are sought to make the above expression a SOS polynomial in the variables $x, v, \sigma$. If the search is successful, a stabilizing controller has been found. Other state-feedback design criteria (such as $\mathcal{H}_\infty$, etc...) in [10] may also be adapted to the fuzzy polynomial case (details omitted for brevity). The paper [14] discusses guaranteed-cost control for polynomial fuzzy systems.

V. REDUCING THE CONSERVATIVENESS OF THE POLYNOMIAL STABILITY ANALYSIS

The above outlined techniques, appeared in recent literature, may be quite conservative. Two important considerations (locality and shape information) may be used to overcome part of the conservativeness. Also, when applying the Taylor-series approach in Section III, the conservativeness reduces as the polynomial degrees increase. Each of these issues will be analyzed below.
Note, however, that the results in this section apply to
stability analysis only: the change of variable $Qx = v$ destroys
the “shape” and “locality” structure in $v$ as it depends on
decision variables. Furthermore, the resulting expressions (29)
are double fuzzy summations, for which non-conservative
necessary and sufficient positivity conditions can be obtained
only asymptotically when a complexity parameter tends to
infinity (see Polya relaxations in [30], [7] and triangulation
approaches in [32]).

A. Local Stability

Sum of squares polynomials are globally non-negative
($p(x) \in \Sigma_x \Rightarrow p(x) \geq 0 \forall x$). However, as the proposed fuzzy-
polynomial modeling is, most of the times, local in a region
$\Omega$, further advantage can be taken from the knowledge of $\Omega$
to reduce conservativeness.

To that purpose, the so-called Positivstellensatz argumenta-
tion [7] extends the Lagrange multipliers and $S$-procedure
in LMIs to polynomial cases. Assume the region $\Omega$ is such
that a finite set $F = \{f_1(x), \ldots, f_2(x)\}$ of known polynomial
restrictions holds, i.e.,

$$\Omega \subset \{x \in \mathbb{R}^n \mid \forall f \in F \ f(x) \geq 0\} \quad (30)$$

Then, a sufficient condition for a polynomial $\pi(x)$ being
positive in $\Omega$ is that there exist multiplier SOS polynomials
$q_i(x), i = 1, \ldots, t$, with $q_i(x) \in \Sigma_x$, so that the sum-of-squares
condition below holds:

$$[\pi(x)]_F := \pi(x) - \sum_{i=1}^t q_i(x) \phi_i(x) \in \Sigma_x \quad (31)$$

where $\phi_i(x)$ are arbitrary polynomials composed by products
of those in $F$, say: $\phi_1 = f_1, \phi_2 = f_2, \ldots, \phi_t = f_1 f_2, \phi_t = f_1^2 f_2$.
Polynomials $\phi_i(x)$ must be $a priori$ defined, which is a source
of conservativeness.

Condition (31) is more relaxed than just requiring $\pi(x) \in \Sigma_x$
as, indeed, the latter results from $q_i(x) \equiv 0$ in (31). In this way,
conditions for a locally decreasing Lyapunov function may be
stated less conservatively.

Example 3 No Lyapunov function can be found for the system
$\dot{x} = -x + 10^{-8}x^3$ because trajectories diverge for $|x| > 10^4$.
However, let us consider the region of interest to be $\Omega :=
\{-1 \leq x \leq 1\}$. Then, $F = \{f_1(x) = 1-x, f_2(x) = 1+x\}$ fulfills
$F(x)$. Once $F$ is available, it is straightforward to find that
$V(x) = 0.5x^2$ is a Lyapunov function proving local stability.
Indeed, using $\phi_1(x) = f_1(x), \phi_2(x) = (1 - x^2), q_1(x) = 10^{-8}x^2$, the
sufficient condition below holds:

$$[\dot{V}(x)]_F = x(-10^{-8}x^2) - 10^{-8}x^2(1-x^2) = (1-10^{-8})x^2 \in \Sigma_x$$

A similar argumentation could be made if the cubic term were
fuzzy, such as $(10^{-8} \mu_1 + 10^{-10}(1 - \mu_1))x^3$ (details omitted for
brevity).

Remark: The above idea proves crucial in the presented
Taylor series context. Indeed, as higher-order polynomials can
never be bounded by linear terms, there may be quite a few
occasions where the Taylor expansions themselves do not
allow for proving global stability as in the above example.
However, higher-degree fuzzy models allow to significantly
improve results over plain Takagi-Sugeno models when locality
is correctly considered (see, for instance, example 4 in
Section VI).

B. Shape information

A second source of conservativeness in TS system analysis
is shape-independence (proving stability for any $\mu \in \Delta^{-1}$,
instead of only the particular one $\mu(x, z)$ from a system, which
are explicitly known).

Indeed, in TS and in polynomial fuzzy models the mem-
bership functions are themselves function of the states as (20)
clearly points out. However, the above SOS conditions (23)–
(24) do not take into account that fact, i.e., they are so-called
shape-independent conditions [11], [9].

As a result, the conditions may be very conservative: a
particular nonlinear system may be stable, but sufficient SOS
conditions (or LMI ones for classical TS systems) may fail to
prove that fact.

For instance, the system $\dot{x} = \mu_1(z) \cdot x + (1 - \mu_1(z)) \cdot (-x)$
cannot be proved stable for an arbitrary $\mu_1, 0 \leq \mu_1(z) \leq 1$
(it is unstable for $\mu_1(z) = 1$). However, it is stable for, say,
$\mu_1 = 0.2 + 0.2 \sin(x)$ as $x = (-1 + 2 \mu_1)x$ is, trivially, an ex-
ponentially stable first-order nonlinear system when $\mu_1 \leq 0.5$.
The shape-independence is responsible of some “strange”
issues in fuzzy control: even if exact, the TS representation is
not unique, and some choices may be better fated than other ones
in order to prove stability of a particular system (this applies to both linear TS models and polynomial ones).

In order to incorporate such “membership shape information”
only very recently some ideas have been put forward. In
ordinary TS models, a first approach to the problem appeared
with restrictions in the membership values, but unrelated to
the state [18], [33], [11]. Full membership-state constraints
appeared first in the conference papers [8], [34], [35].

In the context of polynomial fuzzy models, the results in
[35] can be used, if the information is in the form of a set of
inequalities:

$$\psi_k(\mu(x), x) \geq 0, k = 1, \ldots, q \quad (32)$$

known to hold for all $x$ in a region $\Omega$, being $\psi_k$ some poly-
nomials. Note that the above restrictions are a generalization
of (30).

For instance, consider the classical sector-nonlinearity mod-
eling of $\sin(x)$ in $[-1, 1]: \sin(x) = \mu_1 \cdot 0.8415x + \mu_2 x$, with
$\mu_2(x) = (\sin(x)/x - 0.8415)/(1 - 0.8415)$. If stability of a nonlinear system with $\sin(x)$ cannot be proved with such a
fuzzy model, there are two alternatives:

- re-model the sinusoid with a higher-degree polynomial
  (see Examples 2 and 4 in Section VI),
- considering, by inspection of the plot of $\mu_2(x)$ in $[-1, 1]$ that:

$$1 - 1.052x^2 \leq \mu_2 \leq 1 - x^2 \quad (33)$$

and adding such shape information to the SOS program
via a suitable set of Positivstellensatz multipliers and an
expression in the form (31).
In the context of this paper (higher-order polynomial modeling), shape information may be an alternative path to reducing conservativeness; indeed, shape-dependent conditions and higher-order Taylor series are conceptually close: both approaches make progressively preciser descriptions of a non-linearity, and they both reduce conservativeness at the expense of increasing computational complexity. As shape-dependent conditions for polynomial systems will not be used in this paper’s examples, the reader is referred to [35] for further details.

C. Asymptotic exactness

When some additional assumptions are fulfilled, conservativeness of polynomial fuzzy models and polynomial Lyapunov functions vanishes as the polynomial orders in both the Lyapunov function and the fuzzy models increase. The idea is formally stated below. Notation $\|x\|$ denotes the Euclidean norm.

**Lemma 4** Consider a dynamic nonlinear system $\dot{x} = f(x)$. Assume that there exists a Lyapunov function $V(x)$ (maybe non-polynomial) proving local asymptotic stability, in a compact region $\Omega$ and there exist $\gamma > 0$ and an integer $p \geq 1$ such that $V(x) = \frac{\partial V}{\partial x} f(x) < -\gamma \|x\|^{2p}$ for all $x \in \Omega$, $x \neq 0$. If:

1) $V(x)$ and $f(x)$ are infinitely differentiable in $\Omega$,

2) There exists a sequence of polynomial functions $V_n(x)$, which uniformly converges to $V(x)$ in $\Omega$ (such a sequence might be, for instance, the multivariate Taylor series expansion),

3) the sequence $\frac{\partial V_n}{\partial x}$ uniformly converges (so, it converges to $\frac{\partial V}{\partial x}$) in $\Omega$ and for any $\epsilon > 0$ there exists $n$ such that, defining $\alpha(x) := \frac{\partial V_n}{\partial x} \cdot \frac{\partial V}{\partial x}$, we have $\|\alpha(x)\| < \epsilon \|x\|^{2q}$, for some integer $q > p$.

4) Given a complexity parameter, to be denoted as $n$, there exists a sequence of exact polynomial fuzzy models $f(x) = \sum_{i=1}^{n} \mu_i \pi_i^{(n)}(x)$ in which each $\pi_i^{(n)}(x)$ uniformly converges to $f(x)$, when $n \to \infty$. Also, for any $\epsilon > 0$, there exists $n$ such that $\|\pi_i^{(n)}(x)\| := \|f(x) - \pi_i^{(n)}(x)\| < \epsilon \|x\|^{2q}$, $q \geq p$.

then, there exists a polynomial Lyapunov function $V(x)$ and a polynomial fuzzy model proving stability of $\dot{x} = f(x)$.

**Proof:** Note that, by assumption there exists an scalar $\gamma > 0$ so that $V < -\gamma \|x\|^{2p}$ for all $x \in \Omega$.

Consider a fixed model “rule” $i$. From the assumptions in conditions (3) and (4), we have:

$$\frac{\partial V_n}{\partial x} \pi_i^{(n)}(x) = \frac{\partial V}{\partial x} \alpha f(x) + \dot{\pi}_i^{(n)}(x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} \dot{\pi}_i^{(n)}(x) + \alpha f(x) + \alpha \dot{\pi}_i^{(n)}(x)$$

Hence, as $\Omega$ is compact and both $\frac{\partial V}{\partial x}$, $f(x)$ are continuous, they are bounded so $\|\frac{\partial V}{\partial x} \dot{\pi}_i^{(n)}(x) + \alpha f(x) + \alpha \dot{\pi}_i^{(n)}(x)\|$ (which grows as $O(\|x\|^{2q})$, $q \geq p$) can be made smaller than $\gamma \|x\|^{2p}$ by choosing a small enough $\epsilon$ in first place, so $\frac{\partial V}{\partial x} \pi_i^{(n)}(x) < 0$ holds.

Considering now all fuzzy model rules, choose the complexity parameter $n \geq \max_{i=1}^{m} n_0(i)$. Then, adding up for all $i$, we have: $\frac{\partial V}{\partial x} + \frac{\partial V}{\partial x} f = \frac{\partial V}{\partial x} (\sum \mu_i \pi_i^{(n)}) < 0$. Then, $\Pi = V_m$ is the sought Lyapunov function.

Condition 4 involves being able to set up tighter and tighter approximations in a compact region of interest (i.e., the membership functions interpolate between progressively closer bounds). Note that conditions 3 and 4 need a stricter version of uniform convergence: the sequences must be built in such a way so that the error is bounded by a polynomial of high-enough degree.

In order to ensure that condition 4 holds, if $x$ were a single variable, a Taylor series of $f$ with uniform convergence would be needed. In a multivariate case, if $f$ can be cast as an expression tree of functions of one variable, addition and product (see later Example 6), the complexity parameter may refer, for instance, to the degree of the Taylor approximations of each of the functions of one variable (details omitted for brevity).

**Remark:** Note that the asymptotical exactness lemma above applies only to stability analysis, and it does not take into account the number and degree of the Positivistellensatz multipliers which might be needed to prove local negativess of $\frac{\partial V}{\partial x} (\sum \sigma_i^2 \pi_i^{(n)})$ via SOS techniques. Furthermore, even if the polynomial approach to nonlinear control is non conservative (in the sense of the above lemma, i.e., there exists a polynomial Lyapunov function for smooth systems), the sum-of-squares approach to polynomial problems is intrinsically conservative, as not all positive multivariate polynomials are SOS [16].

VI. ADDITIONAL EXAMPLES

This section provides additional examples to illustrate the possibilities of the presented polynomial fuzzy modeling approach.

**Example 4** Consider a nonlinear system

$$\dot{x} = \begin{pmatrix} -\tanh (4x) & -(0.05 + 0.95 \sin(x^2)) x_1 x_3^3 \\ -(1 + \sin(x_2)^2) x_2 - x_1 x_2 \\ -x_3 + 3 x_1 x_3 + 3 (\sin(x_1) - x_3) \end{pmatrix}$$

(34)

to be modeled in the zone $\Omega = \{ -1 \leq x_1 \leq 1 \}$. Then, by computing maximum and minimum values in $\Omega$ we have

$$\sin(x_1) = 0.8415 \mu_{11} + (-0.8415) \mu_1 + \mu_1 + \mu_1 = 1$$

$$\sin(x_2)^2 = 0.708122 \mu_{11} - 1.41624 \mu_1 + 0.708122 \mu_{12}$$

and, also, as $\tanh x = ((\tanh x)/x_1) \cdot x_1^3$ we may model:

$$(\tanh x)/x_1 = 0.7616 \mu_{21} + 1 \cdot \mu_{22}$$

so $\tanh x = \mu_{21} \cdot 0.7616 x_1^3 + \mu_{22} x_1^3$. $\mu_{21} + \mu_{22} = 1$. In this way, a polynomial fuzzy model results by replacing the above expressions in the original equations (details omitted for brevity). Note that the modeling is not unique: for instance, $\tanh x_1$ may also be approximated by $(0.7616 \mu_{21} + 0 \cdot \mu_1) x_1$ (in fact, this would be the standard sector nonlinearity approach by bounding $0 \leq (\tanh x_1)/x_1 \leq 0.7617$ for $-1 \leq x_1 \leq 1$).

**Example 5** The objective of this example is considering the Taylor-series fuzzy modeling methodology for a sinusoid in the interval [-1,1].
Consider the Taylor series of the sinusoid around $x = 0$:

$$
sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
$$

then, we define, from the notation in Section III-B:

$$f_1 = 0, \quad T_1(x) = (\sin(x) - 0)/x$$

so by finding the maximum and minimum of $\sin(x)/x$ in $[-1,1]$, which are 1 and 0.8415, we may express:

$$\sin x = \mu_1(0 + 1 \cdot x) + \mu_2(0 + 0.8415x) \quad (35)$$

which is, in fact, coincident with the standard sector-linearity TS model in the previous example.

Using the cubic term in the Taylor series, the fuzzy model would be obtained by considering:

$$f_3 = 0 + x, \quad T_3(x) = (\sin(x) - x)/x^3$$

so $T_3(x)$ has maximum -0.1585 and minimum -0.1667 in the interval $[-1,1]$. Hence,

$$\sin x = \mu_1(0 + x - 0.1585x^3) + \mu_2(0 + x - 0.1667x^3) \quad (36)$$

If we proceed to 5th order:

$$f_5 = 0 + x - x^3/6, \quad T_5(x) = (\sin(x) - x + x^3/6)/x^5$$

In $[-1,1]$, $T_5(x)$ has maximum 0.0083336, minimum 0.0081376, so we get:

$$\sin x = x - x^3/6 + (\mu_1 \cdot 0.0083336x^5 + \mu_2 \cdot 0.0081376x^5) \quad (37)$$

where, for instance, $\mu_1$ would have the usual interpolation expression:

$$\mu_1(x) = (T_5(x) - 0.0081376)/(0.0083336 - 0.0081376) \quad (38)$$

Table 1 summarizes the different modeling choices. Note that the two bounding polynomials are closer and closer as their degree increases, as the maximum and the integral of their difference express.

In fact, both 3rd- and 5th-order bounds are very accurate (maximum error 0.0082 and 0.0002, respectively) in the chosen interval. In order to illustrate the differences between the various polynomial model choices, a similar procedure using as region of interest the wider interval $[-3.1,3.1]$ has also been carried out. The results are the upper and lower approximations depicted in Figure 1, for linear, 3rd and 5th order polynomials. The membership functions corresponding to the upper polynomial bound are plotted in Figure 2. At least in this example, the memberships keep a reasonable linguistic interpretability, allowing statements such as:

- If $x$ is low, $\sin(x) = p_1(x)$
- If $x$ is high, $\sin(x) = p_2(x)$

Example 6

Consider the function

$$f(x_1, x_2) = x_1 + x_2 \sin(x_2 e^{-x_1^2}) \quad (39)$$

to be modeled in the zone given by $|x_i| \leq 1$. Then, the argument to the sinusoid, $\xi = x_2 e^{-x_1^2}$, fulfills $\xi \in [-1,1]$. Hence, $\sin(\xi)$ may be modeled with, say, a third-order polynomial fuzzy model (36):

$$\sin \xi = \xi - (\mu_1 (0.1585 + \mu_2 0.1667) \xi^3) \quad (40)$$

On the other hand, $e^{-x_1^2}$ may be modeled, in the zone of interest, as $e^{-x_1^2} = \mu_21 + 0.3679 \mu_22$, or, taking 2nd order terms, as

$$e^{-x_1^2} = 1 - \mu_21 x_1^2 - \mu_22 0.632 x_1^4 \quad (41)$$

Replacing in (39) the sinusoid by (40), $\xi \equiv x_2 e^{-x_1^2}$ and later using (41), we get a polynomial fuzzy model in the form:

$$f(x_1, x_2) = x_1 + x_2^3 (1 - \mu_21 x_1^2 - \mu_22 0.632 x_1^4) - (\mu_1 0.1585 + \mu_21 0.1667)(x_2^3 (1 - \mu_21 x_1^2 - \mu_22 0.632 x_1^4))^3 \quad (42)$$

with $\mu_11 + \mu_12 = 1, \mu_21 + \mu_22 = 1$. The resulting model is a $2 \times 2$ tensor-product fuzzy system which can be flattened (conservatively) to a 4-rule fuzzy system.

A similar substitution procedure would allow the polynomial fuzzy modeling of any function which can be written as an expression tree involving functions of one variable, addition and multiplication.
Example 7 Consider the system:
\[
\begin{align*}
\dot{x}_1 &= -3x_1 + 0.5x_2 \\
\dot{x}_2 &= (-2 + 3 \sin(x_1))x_2
\end{align*}
\]  
with the objective of finding a decreasing Lyapunov function in the region \( \Omega = \{(x_1, x_2) \mid x_1 \in [-\pi, \pi], x_2 \in [-\pi, \pi]\} \) ensuring maximal decay rate. The region can be equivalently characterized as:
\[
\Omega = \{(x_1, x_2) \mid x_1^2 - \pi^2 \leq 0, x_2^2 - \pi^2 \leq 0\}
\]
Under usual TS modeling, as \( \sin(x_1) \) ranges between -1 and 1, we would obtain \( \dot{x} = \sum_{i=1}^{2} \mu_i A_i x \), with:
\[
A_1 = \begin{pmatrix} -3 & 0.5 \\ 0 & -5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -3 & 0.5 \\ 0 & 1 \end{pmatrix}
\]
and, as \( A_2 \) is not stable, the TS approach would fail in proving stability (unless, of course, shape-dependent conditions are used, as discussed in Section V-B).

However, using the Taylor expansion of order 1 of \( \sin(x_1) \), we can show that there exists a fuzzy model so that \( \sin(x_1) = \mu_1 (\sin(x_1)) + \mu_2 x_1 \). In this way, we obtain the fuzzy-polynomial model:
\[
\begin{align*}
\dot{x}_1 &= -3x_1 + 0.5x_2 \\
\dot{x}_2 &= -2x_2 + 3\mu_2 x_1 x_2
\end{align*}
\]
Replacing \( \mu_1 = \psi_1^2, \mu_2 = \psi_2^2 \), and noting that \( \psi_1^2 + \psi_2^2 = 1 \), we get:
\[
\begin{align*}
\dot{x}_1 &= (-3x_1 + 0.5x_2) \cdot (\psi_1^2 + \psi_2^2) \\
\dot{x}_2 &= -2x_2 (\psi_1^2 + \psi_2^2) + 3\psi_1^2 x_1 x_2
\end{align*}
\]
Looking for a quadratic Lyapunov function \( V(x) = p_{11}x_1^2 + p_{12}x_1 x_2 + p_{22}x_2^2 \), and a decay rate bound \( \gamma \), the condition
\[
Q(x) = -V - 2\gamma V > 0
\]
must be proved (see [5], [4]) in the region defined by:
\[
\begin{align*}
P_1(x, \mu) &= (\pi^2 - x_1^2) \cdot (\psi_1^2 + \psi_2^2) > 0 \\
P_2(x, \mu) &= (\pi^2 - x_2^2) \cdot (\psi_1^2 + \psi_2^2) > 0
\end{align*}
\]
Where \( P_1 \) and \( P_2 \) have been obtained from (45). Hence, a SOS program is set up in order to find decision variables proving
\[
\begin{align*}
V(x) - \varepsilon (x_1^2 + x_2^2) &\in \Sigma, \\
Q(x) - \eta_1 P_1 - \eta_2 P_2 &\in \Sigma, \\
\eta_1 &\in \Sigma_{\psi}, \quad \eta_2 \in \Sigma_{\psi}
\end{align*}
\]
where \( \eta_1 \) and \( \eta_2 \) are to-be-found polynomial Positivstellensatz multipliers of degree 2, and \( \varepsilon \) is set to 0.01.

As a result SOSTOOLS+SciDaMi are able to find a Lyapunov function proving a decay rate of \( \gamma = 0.272 \). A larger value of \( \gamma \) resulted into numerical problems or infeasibility.

If a third-order approximation of the sinusoid is used:
\[
x_1 - 0.16667x_1^3 \leq \sin(x_1) \leq x_1 - 0.1012x_1^3 \quad \forall x_1 \in [-\pi, \pi]
\]
we get a decay \( \gamma = 0.309 \) with a quadratic Lyapunov function.

Hence, this example has illustrated that polynomial fuzzy modeling may provide satisfactory results in situations where classical TS methods fail, and that increasing the precision (degree of Taylor expansion) of the polynomial fuzzy modeling may improve the achieved results.

Remark: stability and decay-rate performance have not been proved in the square zone where the SOS conditions hold, but in the largest invariant set \( V(x) \leq \nu_0 \), \( \nu_0 \) constant, contained in \( \Omega \). The obtained Lyapunov functions made the actual invariant region quite a small one in this particular example. This is, however, a well-known issue even in classical TS modeling approaches [8], [9].

Example 8 Consider the system, from [13]:
\[
\begin{align*}
\dot{x}_1 &= (-1 + x_1 + x_1^3 + x_1 x_2 - x_2^2) x_1 + x_1 u \\
\dot{x}_2 &= -\sin(x_1) - x_2
\end{align*}
\]
As discussed in the referred work, an 8-rule ordinary TS model with linear consequents can be obtained by applying the sector-nonlinearity approach. The vertex models are obtaining by computing the maximum and minimum of \( f_1(x) = -1 + x_1 + x_1^3 + x_1 x_2 - x_2^2, f_2(x) = x_1 \), and \( f_3(x) = \sin(x_1)/x_1 \) in a region of interest given by \( x_1 \in [-d_1, d_1], x_2 \in [-d_2, d_2] \). For details, see [13].

In the cited reference, a polynomial fuzzy model was obtained by applying the sector-nonlinearity to the only non-polynomial nonlinearity, i.e., to \( \sin(x_1) \), resulting in a 2-rule polynomial fuzzy system. In this case, the resulting model was in the form:
\[
\dot{x} = \sum_{i=1}^{2} A_i(x) x + B_i(x) u
\]
where
\[
A_1(x) = \begin{pmatrix} -1 + x_1 + x_1^3 + x_1 x_2 - x_2^2 & 1 \\
-1 & 1 \end{pmatrix},
A_2(x) = \begin{pmatrix} -1 + x_1 + x_1^3 + x_1 x_2 - x_2^2 & 1 \\
0.2172 & 1 \end{pmatrix}
\]
\[
B_1(x) = \begin{pmatrix} x_1 \\
0 \end{pmatrix},
B_2(x) = \begin{pmatrix} 1 \end{pmatrix}
\]
Then, a stabilizing regulator (26) was obtained. In fact, the resulting regulator was globally stabilizing, improving over the TS model which was only able to find a Lyapunov function for small enough \( d_1 \) and \( d_2 \). The reader is referred to [13] for details.
With minor modifications to the procedures in the cited work, a decay rate of 0.5135 can be proved (i.e., a quadratic Lyapunov function fulfilling $V \leq -2\gamma V$ where $\gamma$ is the decay rate).

The capabilities of the modeling procedure presented in this paper allow for proving better performance in control design. A 3rd-order representation of the sinusoid is:

$$\sin(x) = x + (\mu_1 * (-1/6) + (1 - \mu_1) * 0)x^3$$

(62)

because $(\sin(x) - x)/x^3$ ranges between $-1/6$ and zero for any $x$. Hence, a new 2-rule polynomial fuzzy system exactly representing the original nonlinear one is given by (58) with changed $A_i(x)$:

$$A_1(x) = \begin{pmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 \ 1 \end{pmatrix},$$

(63)

$$A_2(x) = \begin{pmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 \ -1 + x_2^2/6 \ 1 \end{pmatrix},$$

(64)

$$A_3(x) = \begin{pmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 \ -1 + x_2^2/6 \ 1 \end{pmatrix}.$$

(65)

This new model allows finding a fuzzy-polynomial controller which globally achieves a decay rate$^7$ of 0.999. The simulation of the resulting closed loop from initial conditions $(x_1 = 6.2, x_2 = 1)$ appears in Figure 3.

VII. DISCUSSION

This work has presented a Taylor-series based methodology for modeling any nonlinearity which can be expressed as an expression tree of functions of one variable, addition and multiplication. The examples have shown how results in stability and controller design improve as the Taylor series order increases.

There are, however, some pertinent remarks to the proposed modeling scheme.

First of all, recalling that there are significant conservativeness issues, discussed in Section V-C, even when there exists a polynomial fuzzy model and a (non-fuzzy) polynomial Lyapunov function.

Second, note that as the degree of the bounding polynomials representing a nonlinearity increases, the membership functions’ contribution to the overall model decreases, because the Taylor approximations get increasingly accurate$^8$, i.e., the bounding polynomials are closer and closer (see Table I and Figure 1). In this way, a “fuzzy” controller (whose control action depends on the computed $\mu_i$) will be less and less advantageous with respect to a plain “non-fuzzy” polynomial controller as the fuzzy information becomes less relevant. For instance, consider the bottom row of Table I: $p_1(x)$ and $p_2(x)$ are so close that, possibly, the modeling errors due to parametric and unstructured uncertainties will be much more relevant regarding how a controller behaves in practice; a non-fuzzy controller $u = K(x)Q(x)z$ would obtain very close results to those of (26).

This thought might be consider a drawback of the approach, but it can be reinterpreted: a converse idea to the above one is considering “fuzzy” models as complexity-reduction entities. For instance, consider a case where the actual nonlinearity of a system were, in fact, polynomial:

$$f(x) = 2x - 0.2x^3 + 0.03x^5 - 0.002x^8$$

If the system where $f$ appears is of high order, including 8th order polynomials in the model and Lyapunov functions might increase the computational cost and reach solver limits. However, in the zone $-2 \leq x \leq 2$ we have $1.424 \leq f(x)/x \leq 2$ so we may obtain an ordinary TS fuzzy model $\mu_1 \cdot 1.424 + \mu_2 \cdot 2x$. Using a 3rd-degree polynomial model, we get $2x + \mu \cdot (-0.016x^3) + (1 - \mu) \cdot (-0.2x^3)$, and so on. Sufficient performance for a particular application might be proved with, say, the TS or a 3rd-degree model with a reduced demand for computing resources.

The reasoning above may, of course, be extended to a smooth nonlinear function being expressed as an “infinite-degree polynomial”. That is what has actually been done in the examples: an infinite Taylor series has been bounded by two low-degree polynomials.

In summary, we may consider polynomial fuzzy modeling as a complexity reduction step: the TS sector-nonlinearity approach is the particular case providing the simplest models (polynomials of degree 1), but accuracy may be added at will by bounding nonlinearities with polynomials of higher degree (at the cost of increasing the associated computational cost).

Also, as commented in Section V-A, the introduction of higher-order terms usually needs some Positivstellensatz multipliers to actually improve over classical TS setups.

VIII. CONCLUSIONS

This paper has presented a general modeling methodology based on the Taylor series which extends the well-known Takagi-Sugeno sector-nonlinearity modeling to polynomial

7As the linearized model around the equilibrium point $(x = 0, u = 0)$ is $\dot{x}_1 = -x_1, \dot{x}_2 = -x_1 - x_2$, with decay rate equal to 1 (but not depending on $u$), no decay faster than 1 can be proved by any means for the example system in a neighborhood of the chosen equilibrium. Thus, the higher-order polynomial has proved the maximum possible decay-rate performance.

8assuming the modeling is carried out locally on a compact set where the Taylor series uniformly converges.
cases. The resulting models, denoted as polynomial fuzzy models, are polynomial in both the membership functions and the state variables. Stability analysis is directly amenable via sum-of-squares approaches, and so it is control design for affine-in-control polynomial systems.

By increasing the order of the polynomial approximation, the conservativeness of the results diminishes, at the expense of a higher computational cost: asymptotical exactness is achieved under additional assumptions regarding smoothness, compactness and uniform convergence (however, the gap between positive polynomials and SOS ones still remains). Improved results over TS modeling have been shown in the provided examples, both in stability analysis and control design. Note that locality and membership-shape information should be taken into account, in a general case, in order to obtain less conservative results.

REFERENCES