Observability and Controllability of Neuronal Network Motifs

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Observability and controllability are vitally important in networks, but almost all of the present theory was developed for linear networks without symmetries. To advance beyond the study of such generic networks, we quantify observability and controllability in small (3 node) nonlinear neuronal networks as a function of 1) the connection topology and symmetry, and 2) the measured and controlled nodes. We find that typical observability metrics for 3 neuron motifs range over several orders of magnitude, depending upon topology, and for motifs containing symmetry we find that the network observability and controllability decreases. Networks containing dynamical synchrony facilitated by sufficiently strong coupling also cause decreases in observability and controllability, and that under certain connection motifs, an optimal network coupling strength is found for maximum observability and controllability. Additionally, the number of connections into and out of a network node contributes to the observability (controllability) of the full system as measured (controlled) from that node. Our findings demonstrate that such nonlinear networks are partially observable (controllable), and suggest their potential efficacy in reconstructing network dynamics from limited measurement data. Brain networks are characterized by highly overrepresented small motifs, often with retained topological symmetries. Observability and controllability are essential to the design of predictive network observer models and controllers of brain and their use in experimental settings is a subject for future experimental work.

INTRODUCTION

An observer model of a natural system has many useful applications in science and engineering, including understanding and predicting weather or controlling dynamics from robotics to neuronal systems [1]. A fundamental question that arises when utilizing filters to estimate the future states of a system is how to choose a model and measurement function that faithfully captures the system dynamics and can predict future states [2, 3]. An observer is a model of a system or process that assimilates data from the natural system being modeled [4], and reconstructs unmeasured or inaccessible variables. In linear systems, the key concept to employ a “well designed” observer is observability, which quantifies whether there is sufficient information contained in the measurement to adequately reconstruct the full system dynamics [5, 6].

An important problem when studying networks, is how best to observe and control the entire network when only limited observation and control input nodes are available. In recent work [7, 8], Liu and colleagues found a novel method for computing the minimum set of control (observation) nodes for various types of generic linear networks with linear (or linearizable) nodal dynamics. In classic work, Lin [9] described the topologies of graph directed linear networks that were structurally controllable. Incorporating Lin’s framework, Liu et al (2011) described an efficient strategy to count the number of control points required for a complex network, which have an interesting dependence on time constant [10]. Structural observability is dual to structural controllability [11]. In [8], the requirements of structural observability incorporated explicit use of transitive components of directed graphs - fully connected subgraphs where paths lead from any node to any other node - to identify the minimal number of sites required to observe from a network.

All of these prior works depend critically on the dynamics being linear and generic, in the sense that network connections are essentially random. Joly [12] showed that transitive generic networks with nonlinear nodal dynamics are observable from any node. Nevertheless, symmetries are present in natural networks, as evident from their known structures as well as the presence of synchrony. Recently, Golubitsky et al [13] proved the rigid phase conjecture - that the presence of synchrony in networks implies the presence of symmetries and vice versa. In particular, synchrony is an intimate component of brain dynamics in normal and pathological brain dynamics [13]. Of particular interest when designing observers and controllers for networks containing topological symmetries, is the role the symmetries and network coupling strengths play when reconstructing or controlling net-
work dynamics.

The intuition here is straightforward: consider 3 linear systems with identical dynamics (diagonal terms of the system matrix $A$ in $\dot{x} = Ax$), if the coupling terms are identical (off-diagonal terms of $A$), it’s easy to show that the resulting observability of individual states becomes degenerate as the rows and columns of the system matrix are linearly dependent under elementary matrix operations. Basically, the system model under similar initial conditions and the same inputs gives the same outputs, so there’s no additional information to distinguish one system’s state from another leading to a system maneuver. Thus, the coupling in a network can have a profound impact on its observation and control.

In this article, we extend the exploration of observability and controllability to network motifs with explicit nonlinearities and symmetries. We further explore the effect of coupling strength within such networks, as well as spatial and temporal effects on observability and controllability.

**BACKGROUND**

From the theories of differential embeddings [16] and nonlinear reconstruction [17] [18] we can create a nonlinear measure of observability comprised of the measurement function and its higher Lie derivatives employing the differential embedding map [19]. The differential embedding map of an observer provides the information contained in a given measurement function and model, which can be quantified by an index [20] [21]. Computed from the Jacobian of the differential embedding map, the observability index is a matrix condition number which quantifies the perturbation sensitivity (closeness to singularity) of the mapping created by the measurement function used to observe the system. There is a dual theory for controllability, where the differential embedding map is constructed from the control input function and its higher Lie brackets with respect to the nonlinear model function [22] [23]. Singularities in the map cause information about the system to be lost and observability to decrease. Additionally, the presence of symmetries in the system’s differential equations makes observation difficult from variables around which the invariance of the symmetry is manifested [24] [25]. We extend this analysis to networks of ordinary differential equations and investigate the effects of symmetries on observability and controllability of such networks as a function of connection topology, measurement function, and connection strength.

**Linear Observability and Controllability**

In the early 1960s, Rudolph Kalman introduced the notions of state space decomposition, controllability and observability into the theory of linear systems [5]. From this seminal work comes the classic concept of linear observability for a linear time-invariant (LTI) dynamic system, which defines a ‘yes’ or ‘no’ answer whether a state can be reconstructed from a measurement using a rank condition check.

A dynamic model for a linear (time-invariant) system can be represented by

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

(1)

where $x \in \mathbb{R}^n$ represents the state variable, $u \in \mathbb{R}^m$ is the external input to the system and $y \in \mathbb{R}^p$ is the output (measurement) function of the state variable. The linear observability matrix is defined as [26]

$$O = \begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{n-1}
\end{bmatrix}$$

(2)

The finite limit of the matrix comes from the Cayley-Hamilton theorem [26]. In a similar fashion, the linear controllability matrix is defined [26] as

$$Q = [B, AB, A^2 B, \ldots, A^{n-1} B].$$

(3)

**Differential Embeddings and Nonlinear Observability**

From early work on the nonlinear extensions of observability in the 1970s, [22] [23] showed that the observability matrix for nonlinear systems could be expressed using the measurement function and its higher order Lie derivatives with respect to the nonlinear system equations. The core idea is to evaluate a mapping $\phi$ from the measurements to the states $\phi : \mathbb{R}^p \rightarrow \mathbb{R}^n$. In particular, Hermann and Krener [24] showed that the space of the measurement function is embedded in $\mathbb{R}^p$ when the mapping from measurement to states is everywhere differentiable and injective by the Whitney Embedding Theorem [16] [17]. In other words, an embedding is a map involving differential structure that does not collapse points or tangent directions [18], thus a map $\phi$ is an embedding when the determinant of the map Jacobian $\det(\frac{\partial \phi}{\partial x})_{x \in \mathbb{R}^n}$ is non-vanishing (everywhere differentiable) and one-to-one (injective). In a recent series of papers [19] [24] [27], Letellier et al. computed the nonlinear observability matrices for the well-known Lorenz and Rössler systems [28] [29] and...
demonstrated that the order of the singularities present in the observability matrix (and thus the amount of inter-
section between the singularities and the phase space
trajectories) was related to the decrease in observability.

For a nonlinear system, we replace \( A \mathbf{x} \) in (1) by a non-
linear vector field \( A_{NL}(\mathbf{x}) \), we assume that the scalar
measurement function is taken as \( y(t) = C \mathbf{x} \) and the
system equations comprise the nonlinear vector field
\( f(\mathbf{x}) = A_{NL}(\mathbf{x}) \) (note: if there is no external input, then
\( B \mathbf{u} = 0 \)). Differentiating \( y(t) \):

\[
\dot{y}(t) = \frac{d}{dt}y(x) = \frac{d}{dt}Cx = \frac{dC}{dx}A_{NL}(x) = L_f y(x)
\]

where \( L_f y(x) \) is the Lie derivative of \( y \) along the vec-
tor field \( f \). The differential embedding map \( \phi \) is defined
as the Lie derivatives \( L_f^0 y(x) \ldots L_f^{n-1} y(x) \), where the su-
perscripts represent the order of the Lie derivative from
0 to \( n - 1 \), where \( n \) is the order of the system matrix
\( A_{NL}(x) \). Taking the Jacobian of the map we arrive at the
observability matrix:

\[
O = \frac{\partial \phi}{\partial \mathbf{x}} = \begin{bmatrix}
\frac{\partial L_f^0 y(x)}{\partial \mathbf{x}} \\
\frac{\partial L_f^1 y(x)}{\partial \mathbf{x}} \\
\vdots \\
\frac{\partial L_f^{n-1} y(x)}{\partial \mathbf{x}}
\end{bmatrix}
\]

which reduces to (2) for linear system representations.

Observability/Controllability Index

In systems with real numbers, calculation of the
Kalman rank condition may not yield an accurate mea-
sure of the relative closeness to singularity (condition-
ing) of the observability matrix. It was demonstrated in
\cite{20} that the calculation of a matrix condition num-
ber \cite{15} would provide a more robust determination of the
ill-conditioning inherent in a given observability matrix,
since condition number is independent of scaling and is
a continuous function of system parameters (and states
in the generic nonlinear case). We will use the inverted
form of the observability index \( \delta(x) \) given in \cite{20} so that
\( 0 \leq \delta(x) \leq 1 \)

\[
\delta(x) = \frac{\sigma_{\min}[O^T O]}{\sigma_{\max}[O^T O]}
\]

where \( \sigma_{\min} \) and \( \sigma_{\max} \) are the minimum and maximum
singular values of \( O^T O \) respectively and \( \delta(x) = 1 \) indi-
cates full observability while \( \delta(x) = 0 \) indicates no ob-
servability \cite{30}. Similarly the controllability index is just
\cite{8} with the substitution of \( Q \) for \( O \).

Observability and Controllability of 3-Node Fitzhugh-Nagumo Network Motifs

Fitzhugh-Nagumo System Dynamics

The Fitzhugh-Nagumo (FN) equations \cite{31} \cite{32}, com-
prise a general representation of excitable neuronal mem-
brane. The model is a 2-dimensional analogue of the
well known Hodgkin-Huxley model \cite{33} of an axonal ex-
citable membrane with ionic currents and voltage gated
ion channels. The FN model can exhibit a variety of
dynamical modes which include active transients, limit
cycles, relaxation oscillations with multiple time scales,
and chaos \cite{31} \cite{34}. A nonlinear connection function will
be used to emulate properties of neuronal synapses.

The system dynamics at a node are given by the (local
2nd order) state space:

\[
\begin{align*}
\dot{v} &= c(v - v^3/3 - w + f_{NL}(v^*, d) + I) \\
\dot{w} &= v - bw + a
\end{align*}
\]

where \( v \) represents membrane voltage, \( w \) is recovery, \( d \) the
inter-nodal distance, \( v^* \) the voltage of neighbor nodes,
square wave input current \( I = 0.25 \sum_{n=-\infty}^{\infty} \{\omega t - n T\} + 1 \) \cite{20} 
\( (\omega = 2\pi/5 \) and \( T = 16/2 \), and the system pa-
rameters \( a = 0.7, b = 0.8, c = 10 \). As defined above in Eqns.
\cite{5} and \cite{7}, the observability and controllability matrices
are a function of the states which means a dependance
on the particular trajectory taken in phase space. In the

Lie Brackets and Nonlinear Controllability

The nonlinear controllability matrix is given in \cite{22} and
is constructed from the input function \( g = B \mathbf{u} \) in system
\cite{1} and its Lie brackets with respect to the nonlinear
vector field \( f(x) = A_{NL}(x) \). The Lie bracket is defined as,

\[
(ad_f^m g) = [f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g
\]

where \( (ad_f^m g) \) is the adjoint operator and the super-
scripts represent the order of the Lie bracket from 1 to \( n \),
where \( n \) is the order of the system matrix \( A_{NL}(x) \). From
this definition, we construct the nonlinear controllability
matrix in a similar fashion to the observability matrix,

\[
Q = [g, [f, g], [f, [f, g]], \ldots, [f, (ad_f^{n-1} g)]]
\]
following analysis, we are interested in directed information flow between nodes as a function of various topological connection motifs, connection strengths and input forcing functions (which provide different trajectories through phase space). Each motif is representative of a unique combination of directed connections between the 3 nodes with and without latent symmetries. The nonlinear connection function commonly used in neuronal modeling 35 takes the form of the sigmoidal activation function of neighboring activity (hyperbolic tangent) and an exponential decay with inter-nodal distance to convey the connection coupling strength. We utilize various coupling strengths to determine the effects on the observability (controllability) of the network. Our hyperbolic tangent function takes the form:

\[ f_{NL}(v^*, d) = \frac{k}{2} \left( \tanh\left( \frac{v^* - h}{2w} \right) + 1 \right) e^{-d v^*} \]  

(10)

The sigmoid parameters \( k = 1, h = 0, w = 1/4 \), are set such that \( \tanh \) outputs the interval \([0,1]\) for the input interval \([-2, 2]\), which is the range of the typical FN voltage variable. In this configuration inputs from neighboring nodes act in an excitatory-only manner, while the driving input current was a square wave applied to all three nodes to provide a limit cycle regime to the network; for the limit cycle regime generated in the original paper by FitzHugh 31, the driving current input was constant \( I = -0.45 \) (with the system parameters mentioned above) which we will also explore. Chaotic dynamics were generated with a slightly different square wave input 31 \( I = 0.1225 \sum_{n=-\infty}^{\infty} \tau(\omega t - nT) + 1 \) (with \( \omega = 2\pi/1.23 \) and \( T = 2.7891 \)) also applied to all three nodes.

Network Motifs and Simulated Data

As we are interested in the effect of connection topology on observability and controllability, we study the simplest nontrivial network: a 3-node network. Such small network motifs are highly overrepresented in both neuronal and other complex system networks 36 37. For each network motif shown in FIG. 1 we compute the observability (controllability) indices for various measurement nodes, connection strengths, and driving inputs (dynamic regimes). Measurements of \( v \) for each motif were from one of the nodes 1, 2, or 3. Simulated network data were used to compute the observability (controllability) index for two cases: 1) where the system parameters for all 3 nodes and connections were identical, and 2) where the nodes had a heterogeneous (10% variance), symmetry-breaking set of coupling parameters. To create simulated data, the full six-dimensional FN network equations were integrated from the same initial conditions with the same driving inputs for each node via a Runge-Kutta 4th order (RK4) method with \( \Delta t = 0.04 \) for 12000 timesteps (with the initial transient discarded) in MATLAB® for each test case: 1) limit cycle and 2) chaotic dynamical regimes with a) identical and b) heterogeneous coupling (nodal parameters remain identical throughout). Convergence of solutions was achieved when \( \Delta t \) was decreased to 0.004. Data were then imported into Mathematica® and inserted into symbolic observability and controllability matrices (computed for each node), which then were numerically computed to obtain the observability (controllability) indices for each coupling strength. The indices were then averaged over the integration paths starting from the random initial conditions. These calculations are summarized in FIGS. 1, 5 and 7 for observability and in FIGS. 2, 6 and 8 for controllability.

Construction of Differential Embedding Map and Lie Brackets

As an example case we begin constructing the observability matrix for motif 1 (shown in FIG. 1), the FN network equations form the nonlinear vector field \( f \):

\[
\begin{align*}
    f_1 &= c(v_1 - \frac{v_3}{3} + w_1 + f_{NL}(v_2, v_3, d_{12}, d_{13})) \\
    f_2 &= \frac{1}{c}(v_1 - a + b w_1) \\
    f_3 &= c(v_2 - \frac{v_3}{3} + w_2 + f_{NL}(v_1, v_3, d_{21}, d_{23})) \\
    f_4 &= \frac{1}{c}(v_2 - a + b w_2) \\
    f_5 &= c(v_3 - \frac{v_3}{3} + w_3 + f_{NL}(v_1, v_2, d_{31}, d_{32})) \\
    f_6 &= \frac{1}{c}(v_3 - a + b w_3)
\end{align*}
\]  

(11)

and the measurement function for node 1 in motif 1 is \( y = Cx = [1, 0, 0, 0, 0, 0] \) \( x = v_1 \). We construct the differential embedding map by taking the Lie derivatives from \( \phi_{f}^{y}(x) = v_1 \) to \( \phi_{f}^{y^{-1}}(y(x)) \) as:

\[
\begin{align*}
    \phi_{f}^{(1)} &= y \hspace{1cm} f_1 = c(v_1 - \frac{v_3}{3} + w_1 + f_{NL}(v_2, v_3, d_{12}, d_{13})) \\
    \frac{\partial f_2}{\partial v_1} &= f_1 + \frac{\partial f_2}{\partial v_2} f_2 + \frac{\partial f_2}{\partial v_3} f_3 + \ldots + \frac{\partial f_2}{\partial v_N} f_N \\
    \frac{\partial f_4}{\partial v_1} &= f_1 + \frac{\partial f_4}{\partial v_2} f_2 + \frac{\partial f_4}{\partial v_3} f_3 + \ldots + \frac{\partial f_4}{\partial v_N} f_N \\
    \frac{\partial f_5}{\partial v_1} &= f_1 + \frac{\partial f_5}{\partial v_2} f_2 + \frac{\partial f_5}{\partial v_3} f_3 + \ldots + \frac{\partial f_5}{\partial v_N} f_N \\
    \frac{\partial f_6}{\partial v_1} &= f_1 + \frac{\partial f_6}{\partial v_2} f_2 + \frac{\partial f_6}{\partial v_3} f_3 + \ldots + \frac{\partial f_6}{\partial v_N} f_N
\end{align*}
\]  

(12)

where \( \frac{\partial f_i}{\partial v_j} \) is the partial derivative of the \( i \)-th row of the embedding map \( \phi \), with respect to the \( j \)-th state variable. We obtain the observability matrix by taking the Jacobian of \( \phi \). In this FN network the observability matrix is dependent on the state variables and is thus a function of the location in phase space as the system evolves in time. Letellier et al. 27 used averages of the observability index over the state trajectories in phase space as a qualitative measure of observability. We adopt this convention when computing observability of various network motifs. The indices are computed for each time point in the trajectory, and then the average is taken over all of the trajectories.
Constructing the nonlinear controllability matrix for motif 1 begins with the input function $g = Bu = [1, 0, 0, 0, 0, 0]^T$ and its Lie bracket with respect to the nonlinear vector field $f$ in (11),

$$[f, g] = \frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial x} g = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} g_1 - \ldots - \frac{\partial f_1}{\partial x_6} g_6 \\ \frac{\partial f_2}{\partial x_1} g_1 - \ldots - \frac{\partial f_2}{\partial x_6} g_6 \\ \frac{\partial f_3}{\partial x_1} g_1 - \ldots - \frac{\partial f_3}{\partial x_6} g_6 \\ \frac{\partial f_4}{\partial x_1} g_1 - \ldots - \frac{\partial f_4}{\partial x_6} g_6 \\ \frac{\partial f_5}{\partial x_1} g_1 - \ldots - \frac{\partial f_5}{\partial x_6} g_6 \\ \frac{\partial f_6}{\partial x_1} g_1 - \ldots - \frac{\partial f_6}{\partial x_6} g_6 \end{bmatrix}$$

where $\frac{\partial g}{\partial x} = 0$ because $g$ is the same at each node. From the definitions in equations (6 and 7), we construct the nonlinear controllability matrix as the control input function and its higher Lie Brackets with respect to the nonlinear vector field system equations,

$$Q = [g, [f, g], (ad_f, g), (ad_f, g), (ad_f, g), (ad_f, g)]$$

RESULTS

The plots of observability and controllability display solid colored lines corresponding to identical motif connections while the dashed lines correspond to heterogeneous motif connections; each of the colored curves correspond to observation (control) from the node of the same color in the motif diagram for each plot.

Motifs with Symmetry (1, 3 and 7)

For motif 1 (in FIGS. 1, 3, 5, 6, 7 and 8), the data show that a system with full $S_3$ symmetry (due to the connection topology and identical nodal and coupling parameters) generates zero observability (controllability) over the entire range of coupling strengths (solid lines). This effect of symmetry is evidenced by the non-zero indices for the second calculation (dashed lines), where a symmetry breaking parameter variation is present. Of particular interest is the substantial loss of observability (controllability) as the coupling strengths increase to critical levels for systems containing latent structural symmetries in the presence of heterogeneity (motifs 1 and 3). That is, increasing the coupling strengths when recording (stimulating) from any node in motif 1 or node 2 in motif 3, degrades observability (controllability) as coupling strength increases. A study of the 3D phase plots of the FN voltage variable in motif 3 (as a function of coupling strength for chaotic dynamics) reveals a blowout bifurcation at lower values of coupling strengths (FIGS. 2 and 3), and in contrast leads to generalized synchrony and the subsequent loss of observability (controllability) at higher levels of coupling strength (motif 3 as observed (controlled) from node 2 in FIGS. 4 and 5). This is demonstrated in motif 3 (FIG. 2), where a bifurcation in the dynamics causes the wandering trajectories at weak coupling strengths to collapse onto the attractor at stronger coupling strengths. The cyclic symmetry of motif 7 doesn’t cause loss of observability (controllability) as the chirality of the valence 1 connectivity (single input, single output) of this motif precludes the possibility of input (output) ambiguation of the signals between the nodes.

Motifs without Symmetry

Local output symmetries occur in motifs 2, and 6 when controlling from the first and second node respectively (green and blue solid lines), which is remedied by the disambiguating effect of parameter variation (dashed lines). Additionally, as in the motifs with symmetry, the broken local symmetries lose controllability as coupling strength increases. In the cases where the indices are zero without symmetries (motifs 5, 6, and 8 in FIGS. 1, 3, 5, 6, 7 and 8), the motif must contain an isolated or unmeasurable node(s). From the viewpoint of observability this means that information from the isolated node(s) cannot reach the measured node (11, 12): for controllability, this means that the isolated node(s) is not reached by the controlled node (9) (this nodal isolation is exemplified in motif 8, where the network is only observable from node 1, and only controllable from node 3). Additionally, the plots in FIGS. 7 and 8 (and to some extent plots in FIGS. 5 and 6) show counter-intuitively that as coupling strength increases the observability (controllability) indices increase to an optimal value, and then begin to decrease as coupling strength increases past this critical coupling value. Networks with with constant input (FIGS. 7 and 8) traverse different paths through phase space than the same networks with pulsed input (FIGS. 5 and 6); evident in constant input network results possessing a more distinct peak in observability and controllability across all motifs than the pulsed input networks even though different inputs produce qualitatively similar dynamics (limit cycle behavior).

Observability and Controllability Index Distribution

Log-scaled histograms (FIG. 9) of the index distributions reveal that the local observability (controllability) along the trajectories in phase space are close to a log-normal distribution. After removing zeros from the data, these log-normal distribution fits were computed and verified with the $\chi^2$ test metric for all of the observability and controllability computation cases that contained an adequate number of data points to accurately compute the fit (over 90% of the data). The $\chi^2$ test for goodness of fit confirmed that the data come from a log-normal distribution with 95% confidence.
Fitzhugh-Nagumo Chaos Dynamics (Pulsed Input Forcing) – Observability

FIG. 1. Calculation of observability indices for each of the 8 FN network motifs for a chaotic dynamical regime, as measured from each node (green=1, blue=2, red=3). Solid lines represent the observability indices computed with identical coupling strengths while dashed lines are those computed with symmetry breaking heterogeneous couplings. The calculations show the effect of symmetries in the network topology and choice of network coupling strength on observability. Chaotic regime attained with input current square wave $I = 0.1225\sum_{n=-\infty}^{\infty} \lfloor (\omega t - nT) \rfloor + 1$ (with $\omega = 2\pi/1.23$ and $T = 2.7891$) applied to all three nodes in equation (9).

Motif 3 Phase Space - Chaos
K=0.004 K=0.21366 K=0.47574 K=0.73782 K=0.9999

Motif 3 Phase Space - Chaos

FIG. 2. FN nodal dynamics (green=1, blue=2, red=3) for the chaotic regime and the 3-dimensional phase space for $v$, showing trajectories in motif 3 as measured from node 2 for a range of connection strengths (weak to strong coupling from left to right respectively). In the third row, heterogeneous network coupling (second row of nodal dynamics) in black and identical network coupling in light blue (first row of nodal dynamics). Asterisks mark locations in phase space where observability is higher than the mean for the trajectory (the identical network coupling has zero observability and therefore displays no asterisks). The broken symmetry of the heterogeneous network has trajectories that visit locations in the phase space that vary widely in observability with a log-normal distribution.
FIG. 3. Calculation of controllability indices for each of the 8 FN network motifs for a chaotic dynamical regime, as measured from each node (green=1, blue=2, red=3). Solid lines represent the controllability indices computed with identical coupling strengths while dashed lines are those computed with symmetry breaking heterogeneous couplings. The calculations show the effect of symmetries in the network topology and choice of network coupling strength on controllability.

FIG. 4. FN nodal dynamics (green=1, blue=2, red=3) for the chaotic regime and the 3-dimensional phase space for $w$, showing trajectories in motif 3 as measured from node 2 for a range of connection strengths (weak to strong coupling from left to right respectively). In the third row, heterogeneous network coupling (second row of nodal dynamics) in black and identical network coupling in light blue (first row of nodal dynamics). Asterisks mark locations in phase space where controllability is higher than the mean for the trajectory (the identical network coupling has zero controllability and therefore displays no asterisks). The broken symmetry of the heterogeneous network has trajectories that visit locations in the phase space that vary widely in controllability with a log-normal distribution.
FIG. 5. Calculation of observability indices for each of the 8 FN network motifs for a typical limit cycle dynamical regime, as measured from each node (green=1, blue=2, red=3). Solid lines represent the observability indices computed with identical coupling strengths while dashed lines are those computed with symmetry breaking heterogeneous couplings. The calculations show the effect of symmetries in the network topology and choice of network coupling strength on observability.

FIG. 6. Calculation of controllability indices for each of the 8 FN network motifs for a typical limit cycle dynamical regime, as measured from each node (green=1, blue=2, red=3). Solid lines represent the controllability indices computed with identical coupling strengths while dashed lines are those computed with symmetry breaking heterogeneous couplings. The calculations show the effect of symmetries in the network topology and choice of network coupling strength on controllability.
FIG. 7. Calculation of observability indices for each of the 8 FN network motifs for the constant input limit cycle dynamical regime, as measured from each node (green=1, blue=2, red=3). Solid lines represent the observability indices computed with identical coupling strengths while dashed lines are those computed with symmetry breaking heterogeneous couplings. The calculations show the effect of symmetries in the network topology and choice of network coupling strength on observability.

FIG. 8. Calculation of controllability indices for each of the 8 FN network motifs for the constant input limit cycle dynamical regime, as measured from each node (green=1, blue=2, red=3). Solid lines represent the controllability indices computed with identical coupling strengths while dashed lines are those computed with symmetry breaking heterogeneous couplings. The calculations show the effect of symmetries in the network topology and choice of network coupling strength on controllability.
FIG. 9. The histogram of the log-scaled controllability indices for motif 1 with heterogeneous (HET) coupling and chaotic dynamics

**DISCUSSION**

Despite the growing importance of exploring observability and controllability in complex graph directed networks, there has been little exploration of nonlinear networks with explicit symmetries. We here report the first exploration of symmetries in nonlinear networks, and show that observability and controllability are a function of symmetry, the spatial location of nodes sampled or controlled, and the time evolution of the system.

In networks with structural symmetries, breaking symmetry through randomly altering the coupling strengths established substantial observability or controllability that was absent in the fully symmetric case. In cases of true structural symmetry, increasing the overall level of coupling strength could re-establish the underlying topological symmetry and destroy the observability (controllability). Such dynamic modulation of observability and controllability through coupling strength suggests that there are optimum values of coupling for such systems. Furthermore, they imply that when using an observer model for observation or control, that optimizing the coupling strength both within the observer model, and between the observer model and the underlying measured system, become important parameters to take into account in real systems.

Observation (control) in motifs 2, 3, 4, 5 and 6 suggests a relationship between the degree of connections into and out of a node and its effective observability (controllability). In general, the more direct connections into an observed node, the higher the observability from that node, and the duality suggests that the more direct number of outgoing connections from a controlled node leads to higher controllability than from other less connected nodes. The high degree ‘hub’ nodes were not the most effective driver nodes in complex networks using linear theory [7], and extending nonlinear results to more complex networks with symmetries is a challenge for future work.

When observing kinematics and dynamics of rigid body mechanics obeying Newton’s laws with $SE(3)$ group symmetry, such symmetries must be preserved in constructing an observer (controller) [39]. In the observation of graph directed networks containing transitive networks, one can observe from any point equivalently within such transitive components [8]. In the control of graph directed networks, the minimum number of control points are related to the maximal matching nodes [7]. In [40], contraction theory was used to determine symmetric synchronous subspaces - these spaces actually correspond to our regions without observability or controllability. In fact, the proof of observability is that initial conditions and trajectories do not contract [12]. Furthermore, it is clear that the groupoid input equivalence classes (such as our motifs 6 and 7, see [41] figure 21) are not equivalently observable or controllable - note that only 1 node can serve as an observer node in motif 6 regardless of coupling strength (our Figure 1). Indeed, whether virtual networks [40] with particular groupoid equivalent symmetries serve as detectors of observability and controllability remains unresolved at this time.

Our deep knowledge of symmetries and observers in classical mechanics [39] do not readily translate to graph directed networks. Further development of a theory of observability and controllability for nonlinear networks with symmetries is a vital open problem for future work.

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