In this paper we study Arnold’s tongues in a $Z_2$-symmetric electronic circuit. They appear in a rich bifurcation scenario organized by a degenerate codimension-three Hopf–pitchfork bifurcation. On the one hand, we describe the transition open-to-closed of the resonance zones, finding two different types of Takens–Bogdanov bifurcations (quadratic and cubic homoclinic-type) of periodic orbits. The existence of cascades of the cubic Takens–Bogdanov bifurcations is also pointed out. On the other hand, we study the dynamics inside the tongues showing different Poincaré sections. Several bifurcation diagrams show the presence of cusps of periodic orbits and homoclinic bifurcations. We show the relation that exists between two codimension-two bifurcations of equilibria, Takens–Bogdanov and Hopf–pitchfork, via homoclinic connections, period-doubling and quasiperiodic motions.

1. Introduction

A periodic orbit, in a three-dimensional continuous parameterized autonomous system, exhibits a torus bifurcation (also called secondary Hopf bifurcation, or more properly, secondary Poincaré–Andronov–Hopf bifurcation) when its two Floquet multipliers cross the unit circle under generic conditions on the bifurcation parameter.

Alternatively, a fixed point in a planar parameterized map undergoes a Hopf bifurcation (also called Neimark–Sacker bifurcation) when the eigenvalues of the linearization at the fixed point cross the unit circle under generic perturbations.

In the above conditions, a curve of torus bifurcation occurs in a bidimensional parameter plane. On this curve, Floquet multipliers move along the unit circle. Each time they cross a root of unity, a pair of saddle-node bifurcation curves of periodic orbits emerge from such a point on the torus bifurcation curve. These curves limit locally the corresponding resonance region (Arnold’s tongue). For details, see, for instance, [Devaney, 1986; Guckenheimer & Holmes, 1986; Kuznetsov, 1998].

Resonance phenomena have been widely studied in maps, concretely in planar diffeomorphisms that correspond with the Poincaré map on a two-torus (see, e.g. [Arnold, 1983; Aronson et al., 1982]). Frouzakis et al. [1991] studied the bifurcation structure inside the resonance zone in an adaptively controlled system, modeled by a 3D map. Due to the two-dimensional character of the center manifold, the results obtained correspond to the dynamical behavior present in planar maps.
Similar phenomena have been found in nonautonomous two-dimensional systems. For instance, resonances in a periodically forced van der Pol circuit were considered in [Andronov et al., 1966] and in a predator–prey system by Kuznetsov et al. [1992]. Investigations in the field of electronic circuits modeled by piecewise-linear systems were carried out by Matsumoto et al. [1987].

On the other hand, in the autonomous case, Kirk [1993] studied the resonance zones for a three-dimensional system that corresponds to the normal form up to third-order of the codimension-two Hopf–saddle-node bifurcation (a zero and a pair of imaginary complex eigenvalues, also called Hopf–zero bifurcation). (See, e.g. [Kuznetsov, 1998] to find the references of the pioneer works of Gavrilov and Guckenheimer where the nondegenerate unfoldings of the Hopf–zero bifurcation are independently studied). Such a normal form is perturbed with nonsymmetric cubic terms, in order to break the axial symmetry (under rotations) and the invariance of the axis where the saddle-node bifurcation occurs. In this way a new phenomenon that cannot be presented in maps and forced systems, namely, the merging of tongues is found.

Nevertheless, some differences exist between resonance tongues in planar maps (or equivalently, in forced oscillators) and in three-dimensional autonomous systems.

On the one hand, in planar maps a resonance tongue $p:q$ can be defined as the locus in the parameter space where periodic points with rotation number $p:q$ exist. Therefore the value $p:q$ distinguishes one resonance tongue from the others. Although resonance tongues corresponding to different values of $p:q$ can overlap in parameter space, there is no path in parameter space where a periodic orbit corresponding to a resonance tongue becomes a periodic orbit corresponding to another different resonance tongue. In other words, a periodic orbit with four points cannot become a five-point orbit by moving the parameters (see, e.g. [Devaney, 1986; Kuznetsov, 1998]).

However, in the case of three-dimensional autonomous systems, resonance tongues are well defined only in a neighborhood of the torus bifurcation curve, that is, they are locally defined and is precisely in this zone where they behave as a diffeomorphism of the circle (see e.g. [Arrowsmith & Place, 1990] and [Devaney, 1986]). A resonance tongue is defined as the region of parameter space bounded by curves of saddle-node bifurcations of periodic orbits. But, as the periodic orbit can disappear in a homoclinic bifurcation inside the region between the saddle-node curves, such definition cannot be extended to the whole parameter space. Moreover, far from the torus bifurcation, the torus can disappear leaving free the periodic orbit. In this situation such a periodic orbit can modify its shape and then collapse in a saddle-node bifurcation with another periodic orbit coming from another resonance tongue. This phenomenon is called merging of resonance tongues (see [Kirk, 1993]).

The purpose of this paper is to study resonances in a realistic system (an electronic circuit modeled by a smooth three-dimensional autonomous system). Our numerical study will be guided by previous analytical results obtained by means of the powerful tools of the Bifurcation Theory. Specifically, the resonance tongues that appear in this work are organized by a curve of torus bifurcations, bounded by two codimension-two points that come from a nonlinear codimension-three degeneracy in a Hopf–pitchfork bifurcation (a zero and a pair of imaginary complex eigenvalues, in a system with symmetry to the change of sign, $\mathbb{Z}_2$-symmetry). The torus bifurcation curve that emerged from the Hopf–pitchfork point exhibits an angular degeneracy, that is, the argument of the Floquet multipliers fails to vary monotonically along the torus bifurcation curve. This angular degeneracy allows the presence of several closed Arnold’s tongues (see [Peckman et al., 1995]).

When we analyze the transition closed-to-open in the Arnold’s tongues we detect that the torus bifurcation curve splits since it collapses with a period-doubling curve. Note that the periodic orbit at such (at least) codimension-two point where the collapse occurs has a double $-1$ Floquet multiplier. Note that along this work we will find nondiagonalizable double $-1$ as well as double $+1$ Floquet multiplier points. In the second case, the periodic orbit exhibits a Takens–Bogdanov bifurcation (it occurs in nonsymmetric periodic orbits) whereas the double $-1$ case corresponds to a Takens–Khorozov–Bogdanov bifurcation (if the orbit has the symmetry to the change of sign). To distinguish them we will also say nonsymmetric (or quadratic) versus symmetric (or cubic) Takens–Bogdanov bifurcation, respectively. Moreover, in the cubic case two different situations might appear: homoclinic and heteroclinic types. See, for instance, [Chow et al., 1994; Guckenheimer & Holmes, 1986; Kuznetsov, 1998].
Therefore, after the collapse of the torus and the flip curves, two points of symmetric homoclinic-type Takens–Bogdanov bifurcations of the principal periodic orbit appear. Moreover, we find that to make possible such a transition it is necessary that the subharmonic periodic orbit that emerged in the resonance zone undergoes a torus bifurcation. This will lead to the appearance of new Takens–Bogdanov (nonsymmetric and symmetric of homoclinic type) bifurcations of subharmonic periodic orbits. This is the origin of an accumulation of symmetric homoclinic-type Takens–Bogdanov bifurcations of periodic orbits that will lead to a period-doubling cascade.

We also find, in this work, homoclinic connections related to the periodic orbits of the resonance. An important consequence of the opening of Arnold’s tongues is the apparition of homoclinic connections organizing the branches of the corresponding periodic orbits. Therefore, we put in evidence the relation between the torus bifurcation curve and the homoclinic orbits: the periodic orbit born in the resonant torus wiggles in a branch organized by a homoclinic connection.

With this work we provide (for the first time, as far as we know) a continuous model that exhibits four of the five different scenarios for torus bifurcation curves in a bifurcation set proposed by Peckman et al. [1995].

The work is organized as follows. In Sec. 2 we show the system under study, summarize the bifurcation behavior it exhibits and focus on the torus bifurcation curve detecting an angular degeneration. The analysis of Arnold’s tongues is described in Sec. 3. First we consider the dynamics inside them by means of several Poincaré sections. Secondly, we study their evolution moving a parameter. Later, we show the existence of Takens–Bogdanov bifurcations of periodic orbits and finally, we consider the presence of homoclinic connections organizing the bifurcation diagrams. Finally, conclusions appear in Sec. 4.

2. Description of the Model

The great diversity of dynamical behaviors shown by some electronic circuits means these devices are widely studied. One of the causes of systems having a variety of solutions, both periodic and aperiodic, is the presence of homoclinic connections of Shil’nikov type as well as the existence of torus bifurcations. Our work is developed in this context.

The electronic oscillator we study, is shown in Fig. 1. In the coupling, through the nonlinear conductance \( G_2 \), it consists of two circuits: an \( RCL \) parallel one (nonlinear conductance \( G_3 \), inductance \( L \) and capacity \( C \)) and an \( RC \) parallel one (nonlinear conductance \( G_1 \) and capacity \( C_0 \)). Different choices for the nonlinear conductances \( G_1, G_2 \) and \( G_3 \) give rise to diverse particular systems, that have been previously studied (see [Freire et al., 1993; Algaba et al., 1998b] and references therein).

The corresponding state equations of the circuit, taking the voltages on the capacitors and the current across the inductance as state variables, are

\[
\begin{align*}
C_0 \frac{dv_1}{d\tau} &= -i_1(v_1) + i_2(v_2 - v_1), \\
C \frac{dv_2}{d\tau} &= -i_L - i_2(v_2 - v_1) - i_3(v_2), \\
L \frac{di_L}{d\tau} &= v_2,
\end{align*}
\]

where \( i_j \) represents the current–voltage characteristics of conductance \( G_j \), \( j = 1, 2, 3 \), respectively, which we assume are odd functions:

\[
i_j(v) = \mu_j v + A_3^{[j]} v^3, \quad j = 1, 2, 3.
\]

Now, we rescale the time, variables and parameters of the system according to

\[
\omega = \frac{1}{\sqrt{LC}}, \quad t = \omega \tau, \quad x = v_1, \quad y = v_2, \quad z = \frac{i_L}{\omega C},
\]

\[
r = \frac{C_0}{C}, \quad \nu = \frac{\mu_1}{\omega C}, \quad \beta = \frac{\mu_2}{\omega C}, \quad \gamma = \frac{\mu_3}{\omega C},
\]

\[
A_3 = \frac{A_3^{[1]}}{\omega C}, \quad B_3 = \frac{A_3^{[2]}}{\omega C}, \quad C_3 = \frac{A_3^{[3]}}{\omega C}.
\]

The system (1) becomes

\[
\begin{align*}
r \ddot{x} &= - (\nu + \beta)x + \beta y - A_3 x^3 + B_3 (y - x)^3, \\
\dot{y} &= \beta x - (\beta + \gamma)y - z - B_3 (y - x)^3 - C_3 y^3, \\
\dot{z} &= y,
\end{align*}
\]

where dots mean derivative respect to time.

As the parameters \( \nu, \beta \) and \( \gamma \) correspond (adequately scaled) to the linear approximations to the \( i-v \) characteristics of the conductances \( G_1, G_2 \) and \( G_3 \) and their variation is easily handled in practice, they constitute a natural choice for bifurcation parameters.
We now summarize the results for local bifurcations that system (2) exhibits in the parameter space \((\nu, \beta, \gamma)\). The origin \((0, 0, 0)\) is always an equilibrium point due to the symmetry the system exhibits to the change of the sign in the state variables \((Z_2\text{-symmetry}): (x, y, z) \to (-x, -y, -z)\). A pitchfork bifurcation of equilibria occurs on the plane \(\nu + \beta = 0\), in such a way that two nontrivial equilibria also exist for \(\nu + \beta < 0\). The origin as well as the nontrivial equilibria undergo Hopf bifurcations (note that the Hopf bifurcation should be called more accurately Poincaré-Andronov-Hopf bifurcation). Their study, including possible degeneracies, is carried out in \([\text{Algaba et al.}, 1998b]\).

System (2) exhibits two different kinds of Takens-Bogdanov bifurcations (nondiagonalizable double-zero eigenvalue) of equilibria. Note that due to the symmetry the system with the equilibria at the origin will undergo a symmetric (or cubic) Takens-Bogdanov bifurcation, that should be called more accurately as Takens-Khorozov-Bogdanov bifurcation (see \([\text{Chow et al.}, 1994]\)). The first one, of homoclinic type, occurs on the straightline \(\nu = -\sqrt{\gamma}, \beta = \sqrt{\gamma}, \gamma \neq \sqrt{\gamma}\). The second one, of heteroclinic type, appears on the straightline \(\nu = \sqrt{\gamma}, \beta = -\sqrt{\gamma}, \gamma \neq \sqrt{\gamma}\). The study of this codimension-two bifurcation of equilibria in system (2) is also performed in \([\text{Algaba et al.}, 1998b]\). Note that the Takens-Bogdanov of equilibria that appears in the present paper corresponds to the homoclinic case.

As the aim of this work is the study of resonance zones, we are specially interested in the torus bifurcation since Arnold’s tongues emerge from the locus where that bifurcation occurs. As is well known, a torus bifurcation curve emerges from a Hopf-pitchfork bifurcation (the linearization matrix has eigenvalues \(\pm \omega_0 i, 0\) when it is of type VIa or VIIa (following the notation of [Guckenheimer & Holmes, 1986]). Then it will be very interesting to determine when the Hopf-pitchfork bifurcation, that in system (2) occurs on the segment \(\nu = \gamma, \beta = -\gamma\), \(|\gamma| < \sqrt{\gamma}\), can give rise to the presence of torus bifurcation curves. The study of the Hopf-pitchfork bifurcation in system (2) appears in \([\text{Algaba et al.}, 1999a, 1999b, 2000]\). It is shown that, for

\[
\gamma = \gamma_c = \pm \sqrt{\frac{qr^2}{1 + rq}}, \text{ where } q = \sqrt{\frac{B_3 + C_3}{A_3 + B_3}},
\]

a nonlinear degeneracy occurs (codimension-three) in the Hopf-pitchfork bifurcation (the coefficient of \(r^3\) of the component of the normal form with respect to \((\partial/\partial r)\) vanishes) in such a way that the case VIa appears when \(\gamma^2 > \gamma_c^2\).

For \(\nu \approx \gamma_c, \beta \approx -\gamma_c, \gamma \approx \gamma_c\), system (2) can be put in correspondence with the unfolding (see \([\text{Algaba et al.}, 1998a]\)):

\[
\begin{align*}
\dot{r} &= r(\varepsilon_1 + \varepsilon_3 r^2 + z^2 + cr^4), \\
\dot{z} &= z(\varepsilon_2 + dr^2 + az^2), \\
\dot{\theta} &= \omega_0 + \alpha_1 r^2 + \alpha_2 z^2.
\end{align*}
\] (3)

The bifurcation set of this unfolding, if we do not consider the azimuthal component, is studied in \([\text{Algaba et al.}, 1999a]\). In the work the consequences for the dynamics of the three-dimensional systems are also pointed out. Of particular interest to us is the appearance of a codimension-two bifurcation of periodic orbits that belongs to the unfolding (3). This bifurcation corresponds to a diagonalizable double +1 Floquet multiplier of the
periodic orbit emerged from the Hopf bifurcation of the nontrivial equilibria. In the following, we will refer to this Hopf–saddle-node bifurcation of periodic orbits as H. In fact, the torus bifurcation curve that emerged from the Hopf–pitchfork point ends at A.

We finish this summary of local bifurcations of equilibria exhibited by system (2) noting that a triple-zero degeneracy occurs at $(\pm \sqrt{r}, \pm \sqrt{r}, \pm \sqrt{r})$ when the curves of Takens–Bogdanov and Hopf–pitchfork collapse.

For all the paper we fix the parameter values $r = 0.6, A_3 = 0.3286, B_3 = 0.9336$ and $C_3 = 0$, in accordance with other previous works, for example, [Freire et al., 1993] and [Champneys & Rodríguez-Luis, 1999]. For these values, a nonlinear degeneracy in the Hopf–pitchfork bifurcation takes place at the point $(\gamma_c, -\gamma_c, \gamma_c)$ of the $(\nu, \beta, \gamma)$-parameter space, where $\gamma_c \approx \pm 0.451915$. For this critical value, the coefficients in (3) are $a \approx -13.993$, $c = +1$, $d = -1$, $\omega_0 = 0.6325$, $\alpha_1 = -0.2814$ and $\alpha_2 = -22.143$.

In this way, we choose $\gamma = -0.6$ to obtain a parameter plane where the toroidal attractors exist. In fact, for this value of $\gamma$ the Hopf–pitchfork is of type VIa.

Using numerical continuation methods (see [Doedel et al., 1995]) we can extend the bifurcation curves detected in the local study. A partial bifurcation set for $\gamma = -0.6$ is shown in Fig. 2. We find the presence of four codimension-two points, that correspond to the following bifurcations: Hopf–pitchfork of the origin, HZ, cubic homoclinic-type Takens–Bogdanov of the origin, TB, degenerate Hopf of the origin, H, and Hopf–saddle-node of periodic orbits, A (note that three of them, TB, HZ and H, are on the curve of Hopf bifurcation of the origin, H).

With respect to the codimension-one bifurcations present in that figure, the following comments are in order. The Hopf curve H emerges subcritical from TB and becomes supercritical at H. A pitchfork bifurcation PI occurs in a straightline passing through TB and H. Above this curve the origin is the only equilibrium, below it two nontrivial equilibria also exist. A curve of saddle-node bifurcations of periodic orbits, sn, and a curve of homoclinic connections of the origin, Hom, emerge from TB. For this value of $\gamma$, the homoclinic curve Hom has a turning point and does not intersect with H (such an intersection would correspond to a Hopf–Shil’nikov point, see [Hirschberg & Knobloch, 1993; Champneys & Rodríguez-Luis, 1999], that exists if we increase $\gamma$). Later it finishes in a T-point close to TB (we have not drawn it because it is almost undistinguishable with TB in the parameter window showed). Note that a T-point is a codimension-two heteroclinic connection where the one-dimensional manifolds of two saddle equilibria are connected and the loop is closed with the intersections of the two-dimensional manifolds (see [Glendinning & Sparrow, 1986; Bykov, 1993; Fernández-Sánchez et al., 1999; Champneys & Rodríguez-Luis, 1999]). The curve of Hopf of the nontrivial equilibria (supercritical), h, joins TB and HZ. A curve of period-doubling bifurcations, PD, is drawn. It is related with the presence of Shil’nikov homoclinic connections (the transition $\delta = 1$, see [Belyakov, 1984; Gaspard et al., 1984; Glendinning & Sparrow, 1984]), occurs on the curve Hom at a point very close to TB). The curve of pitchfork bifurcations of periodic orbits PPO2 joins HZ and A. Below A another curve of pitchfork PPO1 exist. Finally, the torus bifurcation curve HH emerges from HZ and ends at A. Between all these curves, we obviously fix our attention on the torus bifurcation curve HH.

We remark that the period-doubling and the torus bifurcations present in the above bifurcation set are exhibited by the nonsymmetric periodic orbit emerged in the Hopf bifurcation of the
nontrivial equilibria, $h$. Along this work we will show that the interaction between the curves $PD$ and $HH$ imply the presence of complex dynamical behavior, namely, Takens–Bogdanov bifurcations of periodic orbits.

3. Analysis of Arnold’s Tongues

The Floquet multipliers of the periodic orbit exhibiting the torus bifurcation evolve along the unit circle in the way shown in Fig. 3(a), where the argument (angle between the horizontal axis and the Floquet multiplier of the complex conjugate pair with positive imaginary part) is drawn versus parameter $\beta$. When the argument fails to vary monotonically we say that the torus bifurcation has an angular degeneracy (see [Peckman et al., 1995]), as occurs in our case where the curve has a parabola-like shape and reaches a maximum at an angular degeneration point $D$ (where the Floquet multipliers of the periodic orbit reverse their direction of movement on the unit circle). The appearance of this angular degeneracy is a consequence of the existence of the point $A$ (since the argument is zero at both endpoints of $HH$).

The value of the argument at this angular degeneracy point $D$ ($\approx 35.3$ degrees) indicates to us that resonances $1:p$ and $2:q$ will appear on the $HH$ curve only for $p \geq 11$ and $q \geq 21$. Moreover, Arnold’s tongues corresponding to the same resonance have two tips on the curve $HH$ for two different values of the parameters.

To see the evolution of the resonance zones we have to detect the periodic solution that has emerged on the torus. The analytical study of the Hopf–pitchfork bifurcation combined with the numerical computation of the Floquet multipliers along the torus bifurcation curve provide the resonance zone at first approximation. We proceed in the following way. First, we fix the parameters, close to the $HH$ curve, in the zone where the quasi-periodic motion exists. Here, we construct the Poincaré section of the torus, that defines a diffeomorphism of the circle onto itself. Then, we compute the powers of this diffeomorphism (taking into account of the Farey’s order of rational numbers) and this leads to the closest resonance present for the values of the parameters fixed. Finally we search for the corresponding periodic orbit with the help of the Poincaré section. The numerical continuation of this periodic orbit, performed with the AUTO code (see [Doedel et al., 1995]), provides the limits of the zone where it exists (saddle-node bifurcation curves).

Obtained in that way, several Arnold’s tongues corresponding to these $1:p$ resonances are shown in Fig. 3(b) in a neighborhood of the $HH$ curve. The shape of these tongues far from the $HH$ curve is presented in Fig. 3(c), where the asymptotic behavior of all of them can be observed (see [Broer et al., 1993; Kirk, 1993; Broer et al., 1998]).

In Fig. 3(d) we show, for $\gamma = -0.6$, the angular degeneracy point $D$ on the torus bifurcation curve $HH$ as well as the first resonance zones ($1:p$ and $2:q$) emerged close to it. Numerical computation suggests that, since the first three are closed zones $(2:21, 1:11, 2:23)$, the point $D$ is of banana type (see [Peckman et al., 1995]).

In this work we will focus on the first two lower resonances $1:p$, namely, $1:11$ and $1:12$ resonances. We will refer to both cases to illustrate that the behavior found is generic in the tongues of the weak resonances $1:p$ in this circuit. Moreover, for the first value of $\gamma$ fixed, they correspond respectively to a closed resonance zone $(1:11)$ and to the first open resonance zone $(1:12)$ in the parameter plane $\nu-\beta$ [see Fig. 3(e)].

We intend to study the evolution of the Arnold’s tongues corresponding to such resonances taking $\gamma$ as control parameter.

Note that the saddle-node curves that emerged from the two tips corresponding to the same value of the argument on the curve $HH$ joins giving rise to the closed resonance region $1:11$. However, there are three curves of saddle-node bounding the open region $1:12$ because only one of the curves of the upper tip joins with a curve emerged in the lower tip.

Note that inside the resonance zone $1:11$ the periodic orbits do not exhibit any bifurcation. Both are hyperbolic, one of saddle type and the other repulsive. Then an $o$-shaped isola bifurcation diagram appears in this situation.

3.1. Dynamics inside the tongues

In order to study the dynamics inside the tongues we have taken several Poincaré sections in two horizontal slices (in the $\nu-\beta$ plane) of the tongue $1:11$, specifically for $\beta = 0.529$ and $\beta = 0.524$ [see Fig. 4(a)]. This numerical work has been carried out with the program DSTOOL (see [Guckenheimer & Kim, 1992]). The first slice (A–C) is taken close to the tip of the resonance tongue and above the
Fig. 3. For $\gamma = -0.6$. (a) The arguments of the characteristic multipliers versus $\beta$ along the torus bifurcation curve, with an angular degeneration point $D$. Arnold’s tongues for the resonances $1:p$, with $p = 11, 12, 15, 20, 25$: (b) close to the HH curve; (c) in a wider region where their asymptotic behavior is observed. (d) Arnold’s tongues for the first resonances $1:p$ and $2:q$, namely $2:21, 1:11, 2:23, 1:12$ and $2:25$. (e) Detail of Arnold’s tongues for the resonances $1:11$ (closed resonance zone) and $1:12$ (first open resonance zone).
Fig. 4. For $\gamma = -0.6$. (a) Zoom of the upper tip in the tongue $1:11$. Poincaré sections with the plane $y = 0$, corresponding to the horizontal slice $A-C$, in the region: (b) $A$; (c) $B$; (d) $C$.

curve where the Floquet multipliers of the repulsive periodic orbit are equal. This curve has not been drawn in this figure because it is undistinguishable with the frontier of the tongue, due to the narrow range of the parameters.

In the first horizontal slice $\beta = 0.529$, regions $A$ and $C$ are outside the tongue. Concretely, the Poincaré section corresponding to the zone $A$ [see Fig. 4(b)] shows that the trajectories are repelled in counterclockwise from the invariant circle outside as well as inside it. In particular, the inner trajectories go towards an attractive periodic orbit inside the torus. When crossing the border curve of the resonance $1:11$, in the zone $B$ [see Fig. 4(c)], a period 11 saddle-node pair appears on the invariant circle. The repulsive invariant set is now formed by the stable manifolds of the saddle periodic orbits (labeled $S$) together with the repulsive periodic orbits (of node type, labeled $N$). Finally, increasing $\nu$ further, the saddle and node periodic orbits approach each other on the repulsive invariant set and then collide in another saddle-node bifurcation, when we cross the other side of the resonance tongue. In the zone $C$ the situation is similar to that in zone $A$ but in this case the orbit rotation is clockwise [see Fig. 4(d)]. These are
Fig. 5. For $\gamma = -0.6$. (a) Zoom of the tongue $1:11$ close to the section A–G. Poincaré sections (details of the lower part of the repulsor) with the plane $y = 0$, corresponding to the section A–G, in the region: (b) B; (c) C; (d) D; (e) E; (f) F.
the behaviors expected in the vicinity of the tip of weak resonances (see [Aronson et al., 1982]).

In the second horizontal slice, further down in the resonance tongue ($\beta = 0.524$), a richer variety of Poincaré sections is found (A–G). Along the one-parameter cut two outer zones to the tongue appear, A and G, with analogous behavior as in zones A and C of the first slice. In Fig. 5(a) we show a zoom of this second slice of Fig. 4(a). In this partial bifurcation set of the tongue 1 : 11 we see two very narrow regions C and E, where homoclinic tangencies occur, h1 and h2. These tangencies separate the different behaviors found in regions B, D and F.

In Figs. 5(b)–5(f) we show the Poincaré sections, with the plane $y = 0$, inside the tongue; specifically, we show a zoom of the lower part of the repulsor. Three of them are structurally stable (zones B, D and F) and the other two are structurally unstable since they are related to homoclinic tangles (zones C and E). In this sequence, we can observe how the periodic orbits come from the inner part (zone B) to the outer part (zone F) of the repulsive invariant set that, in fact, is losing differentiability when moving the parameter $\nu$ towards the interior of the tongue.

When crossing the curve of saddle-node bifurcations and entering zone B [see Fig. 5(b)], two
periodic orbits emerge, one saddle $S$ and the other repulsive of node type. This last one quickly becomes of focus type $F$, as the curve where the transition node–focus occurs is very close to the bifurcation curve. The appearance of the periodic orbits occurs inside the repulsor, and we can see how the unstable manifold of the saddle periodic orbit confines the focus periodic orbit in its interior. An analogous situation occurs in zone $F$ [see Fig. 5(f)], but in this case the periodic orbits are outside the invariant circle.

Increasing $\nu$, we enter in zone $C$ [see Fig. 5(c)] where a homoclinic interaction between the manifolds of the saddle periodic orbit occurs. In this way, the invariant circle disappears. Note that in this zoom we cannot see the folds of the unstable manifold of the saddle orbit when it approaches the adjacent saddle orbit.

The following zone, $D$ [see Fig. 5(d)], is the widest of the inner zones in the tongue. In this situation, the frontier of the basin of attraction of the stable periodic orbit, that was inside the repulsor, is formed now by the stable manifolds of the saddle orbit together with the periodic orbit of focus type. In this case the repulsive invariant set is also present but it has lost its smoothness. This behavior is analogous to the one obtained in region $B$ for the first slice, but now the repulsive periodic orbit is of focus type instead of a node.

Finally, in zone $E$ [see Fig. 5(e)] we find another homoclinic connection that, probably, originates the unstable invariant circle we observe in zones $F$ and $G$. Note that lines $h1$ and $h2$ of Fig. 5(a) are really regions in the parameter plane $\nu-\beta$. That is, after the homoclinic tangency $h1$, as the parameter $\nu$ is varied, manifolds intersect transversally generating a complex geometric structure known as a homoclinic tangle (see e.g. [Guckenheimer & Holmes, 1986]); finally the manifolds become tangent again on the opposite side. Such global bifurcations generically occur over an interval of parameter values. In our case, the details occur over an interval in parameter space that lies below our computational resolution, so these bifurcations are depicted as lines in the parameter plane. The dynamical behavior observed in this slice coincides with the one obtained for the resonance $1:5$ in Fig. 7(b) of [Frouzakis et al., 1991], in absence of a saddle-node bifurcation of invariant circles. Remark that we do not dispose of an analytical expression of the Poincaré sections (we detect homoclinic connections of periodic orbits) whereas [Frouzakis et al., 1991] have a diffeomorphism.

Note that, in our slice, the sequence of crossing the regions $B-C-D-E-F$ has a certain symmetry with respect to the central situation $D$. The difference between the dynamics in $C$ and $E$ is the change in the direction of rotation. If we compare the behavior found in $B$ and $F$ we observe not only the change in the direction of rotation but also the change in the relative position of periodic orbits and the repellor. Comparing with the schematic representation of the internal bifurcation structure expected in a typical resonance tongue in the Bogdanov map (see Fig. 20 in [Arrowsmith et al., 1993]), in our slice the sequence of crossing the regions $F-E-D-C-B$ corresponds to the situation $j'-i'-h-i-j$ in the quoted work.

### 3.2. Evolution of Arnold’s tongues

Now we move $\gamma$ to see the evolution of the Arnold’s tongue $1:11$. When $\gamma$ decreases we detect that for

![Fig. 7. Qualitative partial bifurcation set, in the $(\nu, \beta, \gamma)$ parameter space that explains why the torus bifurcation curve splits. The convention of symbols used is the following: Filled circles are points on the Hopf-pitchfork curve $HZ$, filled squares correspond to points on the Takens–Bogdanov curve $TBS$, filled triangles stand for points on the Hopf-saddle-node of periodic orbits curve $A$ and inverted filled triangles indicate points on the angular degeneracy curve $D$.](image)
$\gamma_c \approx -0.69205$ the torus bifurcation curve collapses at a point $TBS$ with a period-doubling curve (corresponding to the asymmetric periodic orbit emerged in a Hopf bifurcation of the nontrivial equilibria).

For $\gamma < \gamma_c$, the torus bifurcation curve appears broken in two parts, as is shown in Fig. 6(a) for $\gamma = -0.69217$. The first part of the torus bifurcation curve, $HH1$, joints the Hopf–pitchfork point $HZ$ with the point $TBS1$ (where the periodic orbit has a nondiagonalizable double $-1$ Floquet multiplier). On the other hand, the second part of the torus bifurcation curve, $HH2$, connects the points $TBS2$ and $A$. At these codimension-two points, the periodic orbit has a nondiagonalizable double $-1$ and a diagonalizable double $+1$ Floquet multiplier, respectively. In fact, $TBS1$ and $TBS2$ correspond to cubic homoclinic-type Takens–Bogdanov bifurcations of periodic orbits (see [Kuznetsov, 1998]).

This splitting of the torus bifurcation curve provokes the disappearance of the angular degeneration point $D$, as can be seen in Fig. 6(b) where we have represented how the Floquet multipliers of the periodic orbit exhibiting the torus bifurcation evolve along the unit circle (argument versus parameter $\beta$).

We show in Fig. 7 a qualitative partial bifurcation set, in the $(\nu, \beta, \gamma)$-parameter space, that explains why the torus loci changes because of the presence of Takens–Bogdanov bifurcations of periodic orbits, we will focus on Arnold’s tongues and their evolution. We will see that two different types of Takens–Bogdanov bifurcations of periodic orbits will be also present, including a cascade of one of them.

Once we have explained how the torus loci changes because of the presence of Takens–Bogdanov bifurcations of periodic orbits, we will see how the Arnold’s tongues corresponding to the $1:11$ resonance emerge from $HH1$ and $HH2$, respectively. Note that the right branch of folds that emerges from the $1:11$ resonance on $HH1$ crosses this curve when

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![Fig. 8. Arnold’s tongue for the 1:11 resonance for: (a) $\gamma = -0.60184$; (b) $\gamma = -0.6018$.](image-url)
it moves away from its starting point on HH1 [see Fig. 6(c)]. The same comment is valid for the right curve of saddle-node bifurcations that started at the 1:11 resonance on HH2 [see Fig. 6(d)]. This crossing implies that periodic orbits (of approximately eleven times the period of the principal periodic orbit) exist on both sides of the HH1 and HH2 curves. This fact is not a consequence of the torus curve break (since it also occurs, for example, for $\gamma = -0.65$ when there is only one torus bifurcation curve) but to the evolution of the curves HH and the frontier of the resonance zone 1:11 when varying $\gamma$ (that is, in the three-parameter space, the surfaces HH and SN$_{11}$ intersect independently of the collision of the surface PD with HH).

The continuation of these saddle-node curves of periodic orbits [see Fig. 6(a)] shows that both right branches are connected whereas the left branches are disconnected. Recall that for $\gamma = -0.6$ the 1:11 resonance zone was a closed region. Moreover, in this situation the repulsive periodic orbit undergoes a period-doubling bifurcation, contrary to the behavior found for $\gamma = -0.6$. See this flip curve PD$_{11}$ in Figs. 6(c) and 6(d). We have not drawn it in Fig. 6(a) as it would not be distinguishable with the saddle-node curves that limit the resonance zone 1:11.

Our task now will be to describe the transition process between the situations found for $\gamma = -0.69217$ and $\gamma = -0.6$, that is, to see how the
resonance zones evolve from open to closed. To do that, we will increase the parameter $\gamma$ from $\gamma = -0.69217$. In this way we see for $\gamma = -0.60184$ how the upper and lower left branches of the resonance zone approach [see Fig. 8(a)] and after the intersection of the curves two different resonance regions appear for $\gamma = -0.6018$ [see Fig. 8(b)]. Note that the saddle-node curves that bound the closed resonance zone emerge from the curve $\text{HH}$ but the curves of the open zone are not related to $\text{HH}$.

Consequently, for the initial value $\gamma = -0.6$, there exists not only the closed resonance zone $1:11$ drawn in Fig. 2 but also an open zone we show in Fig. 9(a). In fact, this open resonance zone exists even before the closed resonance zone (this one appears for the first time when $\gamma \approx -0.5957$).

### 3.3. Takens–Bogdanov bifurcations

Recall that inside the closed resonance zone $1:11$ the periodic orbits do not exhibit any bifurcation. Both are hyperbolic, one of saddle type and the other repulsive. However, the study of the periodic orbits inside the open resonance zone $1:11$ for $\gamma = -0.6$ shows that the stability of one of them changes with respect to the periodic orbits inside the closed resonance zone for the same value of the parameter $\gamma$. Now one periodic orbit is saddle and the other one is attractive, but this last one undergoes a period-doubling bifurcation when crossing the curve $\text{PD}_{11}$.

Obviously, the change in the stability of one of the periodic orbits in the resonance zones $1:11$ indicates that, as $\gamma$ varies, some additional bifurcation has to be present on the saddle-node curve that bounds the open resonance zone. This bifurcation is necessary to make possible the contact between the open and the closed resonance zones $1:11$. To understand this situation we investigate the partial bifurcation set inside this region for values of $\gamma$ close to the value where both regions join.

Moving again the control parameter $\gamma$ we see in Fig. 9(b) that a closed torus bifurcation curve $\text{HH}_{11}$ of the 11-period orbit appears in the parameter plane (this curve does not exist for $\gamma = -0.60128$). As $\gamma$ decreases this closed curve approaches the saddle-node curve that bounds the open region of the resonance $1:11$, in such a way that the contact between both curves originates two (quadratic) Takens–Bogdanov points of periodic orbits $\text{TB}_{11}$ and $\text{TB}_{21}$ [see Fig. 9(c)]. Later [see Fig. 9(d)], when the torus bifurcation curve $\text{HH}_{11}$ contacts transversally with the curve of period-doubling $\text{PD}_{11}$, it splits in two curves $\text{HH}_{11}$ and $\text{HH}_{21}$. Two new Takens–Bogdanov points (in this case cubic homoclinic-type) of periodic orbits, $\text{TBS}_{11}$ and $\text{TBS}_{21}$, appear. Note that both curves $\text{HH}_{11}$ and $\text{HH}_{21}$ connect two Takens–Bogdanov points that are placed respectively on the saddle-node curve $\text{SN}_{11}$ and on the period-doubling curve $\text{PD}_{11}$.

We have checked if the presence of closed torus bifurcation curves also occurs for other resonances. The answer is affirmative, but a difference may appear. In the above case (resonance $1:11$), the torus bifurcation curve first collapses with $\text{SN}_{11}$ and later with $\text{PD}_{11}$, whereas in other resonances the torus bifurcation curve first collapses with the period-doubling curve and later with the saddle-node curve. This is illustrated for the $1:12$ resonance in Fig. 10. The curve $\text{HH}_{12}$ starts and ends at two symmetric homoclinic-type Takens–Bogdanov points of periodic orbits, $\text{TBS}_{12}$ and $\text{TBS}_{22}$.

Peckman et al. [1995] proposed several possible scenarios, all involving secondary Hopf bifurcations and most involving Takens–Bogdanov points, where angular degeneracy points are present. In the paper, the authors wonder if there is some model that presents such a behavior in relation to the torus bifurcation curve and its resonance tongues [see their Figs. 5(d) and 5(e)]. The three-dimensional autonomous model we consider exhibits four of the five possible situations for the global continuation of a secondary Hopf bifurcation curve in a two-parameter family:

1. continuation in each direction terminates at a quadratic Takens–Bogdanov point (nondiag-
For $\gamma = -0.6$: (a) Arnold’s tongues for the resonance $1:12$. (b) Details inside the resonance zone $1:12$ of the upper region. (c) Projection on the $x-z$ plane of the 12T torus that exists for $\nu = -0.5929$, $\beta = 0.5221$. (d) Zoom of the Poincaré section with the plane $y = 0$ of the strange attractor existing for $\nu = -0.5929$, $\beta = 0.52207$ (we show one of the twelve intersections).

1. continuation in each direction terminates at a diagonalizable double $+1$ Floquet multiplier) [see Fig. 9(c)];
2. continuation in each direction terminates at a cubic homoclinic-type Takens–Bogdanov point (nondiagonalizable double $-1$ Floquet multiplier). We have detected this case for the resonance $1:12$ (see Fig. 10);
3. continuation in one direction terminates at a quadratic Takens–Bogdanov point, continuation in the other direction terminates at a cubic homoclinic Takens–Bogdanov point [see Fig. 9(d)];
4. continuation forms a closed curve [see Fig. 9(b)].

Once we have shown the evolution of the Arnold’s tongues from closed-to-open (moving the parameter $\gamma$) and we have described how the Takens–Bogdanov points of periodic orbits appear, we will describe the bifurcation set in a neighborhood of these Takens–Bogdanov points. To illustrate that this dynamical behavior occurs generically for all the weak resonances $1:p$ (not only for $1:11$) we will now consider the resonance $1:12$, for $\gamma = -0.6$. We commented before that it is the first resonance that appears in an open region, because the contact between the open and closed regions had previously occurred following the steps described for the $1:11$ resonance.

In Fig. 11(a) we show the principal torus bifurcation curve $HH$ and the Arnold’s tongues of the
Fig. 12. For $\beta = 0.51$ and $\gamma = -0.6$: (a) Isola bifurcation diagram in the 1:11 closed tongue; (b) Bifurcation diagram in the 1:12 tongue organized by two homoclinic connections; (c) Phase portrait of the homoclinic orbit to the nontrivial equilibria that organizes the right branch of periodic orbits in (b) ($\nu \approx -0.625911$). (d) Bifurcation diagram in the 1:12 tongue for $\nu = -0.61$, $\gamma = -0.6$.

The 1:12 resonance $\text{SN}_{12}$. We see how the torus bifurcation curves $\text{HH}_{12}$ and $\text{HH}_{24}$ connect the saddle-node curves $\text{SN}_{12}$ with the period-doubling curve $\text{PD}_{12}$. To see clearly what bifurcations are present we show in Fig. 11(b) a zoom of the region where the curve $\text{HH}_{12}$ exists. We observe a cascade of torus bifurcations $\text{HH}_{12}$, $\text{HH}_{24}$, ..., connecting $\text{SN}_{12}$ with $\text{PD}_{12}$, $\text{PD}_{12}$ with $\text{PD}_{24}$, ... Therefore, we have found a cascade of Takens–Bogdanov bifurcations of periodic orbits. The first point $\text{TB}_{12}$ is of quadratic type whereas the other Takens–Bogdanov points ($\text{TBS}_{12}$, $\text{TBS}_{24}$, ...) are of cubic homoclinic-type. Further work is needed to study the different attractors that coexist in this region of the parameter plane ($\nu, \beta$). As a first result, we show in Fig. 11(c) the projection onto the $x$-$z$ plane of the invariant attracting set that appears when crossing the curve $\text{HH}_{12}$. Its disappearance gives rise to a strange attractor we can see by means of a zoom of a zone of its Poincaré section with the plane $y = 0$ [see Fig. 11(d)].

3.4. Homoclinic connections

In this final part of the work we will show several bifurcation diagrams of the periodic orbits that exist inside the resonance zones 1:11 and 1:12 for $\gamma = -0.6$.

Now we will see how many kinds of bifurcation diagrams we find for the periodic orbits that
exist inside the resonance zones 1:11 and 1:12 for $\gamma = -0.6$. In particular we are interested in verifying if some global bifurcation (homoclinic connection, . . . ) organizes such diagrams.

First we consider the bifurcation diagram (Period versus $\nu$) for $\gamma = -0.6$ and $\beta = 0.51$. In Fig. 12(a) we show an $o$-shaped isola that we obtain inside the closed resonance zone 1:11. In fact, for any value of $\beta$ we fix for the bifurcation diagram inside the closed zone, an $o$-shaped isola will appear.

However, if we consider for the same values of $\gamma$ and $\beta$ the bifurcation diagram corresponding to the open resonance zone 1:12 [see Fig. 12(b)], we see that the branches of periodic orbits are organized by homoclinic connections of the nontrivial equilibria. The phase portrait of one of these homoclinic orbits is drawn in Fig. 12(c).

On the other hand, if we represent bifurcation diagrams of Period versus $\beta$ (vertical slices in the parameter plane $\nu$–$\beta$) in the open resonance zone and we fix, for instance, $\gamma = -0.6$ and $\nu = -0.61$, we obtain a more complicate isola that appears in Fig. 12(d). We observe several extra folds due to the presence of other saddle-node bifurcation curves. Concretely, four curves of saddle-node bifurcations of 12T orbits, for this value of the parameter, are all inside the resonance zone 1:12. A detailed study of the structure of these curves, where several cusps are present, is left for the future. For instance, it will be interesting to classify the cusps according to Broer et al. [1998] to see the connection between different resonances.

4. Conclusions

For the first value of the parameter $\gamma$ considered [see Fig. 2(a)], the torus curve joins two codimension-two points: HZ (Hopf–pitchfork of equilibria) and A (Hopf–saddle-node of periodic orbits). The organizing centre HZ exhibited a nonlinear degeneracy that implies the appearance of the new important point A (see [Algaba et al., 1999a]).

On the one hand, we have found the splitting of the torus bifurcation curve by means of a collision with the curve of period-doubling bifurcations of periodic orbits. In this manner, two points of cubic homoclinic-type Takens–Bogdanov bifurcations of periodic orbits appear [points TBS1 and TBS2 in Fig. 6(a)]. A qualitative sketch of the bifurcation set showing the curve of angular degeneracies $\mathcal{D}$, that connects the degenerate Hopf–pitchfork point DHZ with the point TBS at $\gamma = \gamma_c$, has been presented (see Fig. 7). Further work has to be done to complete the understanding of this situation.

On the other hand, the presence of an angular degeneracy along the torus bifurcation curve has also been pointed out. In the first moment the resonance zones are closed due to this degeneracy and, moreover, implies that the Arnold’s tongues corresponding to the same resonance are connected. In fact, some of them are doubly connected and then closed resonance zones appear. We have shown, moving a parameter, the evolution of the resonance zones, that is, how they evolve from closed-to-open. Homoclinic connections have been detected when the resonance zone is open. The existence of two different types of Takens–Bogdanov bifurcations of the resonance periodic orbits have been found. The first one (quadratic case) appears on a saddle-node curve whereas the second one (cubic homoclinic case) occurs in a cascade on the period-doubling curves.

Further work is also needed in this direction, checking, for instance, if the same behavior is present in Arnold’s tongues corresponding to strong resonances, namely 1:2, 1:3 and 1:4. We also intend to study more deeply the relation between open resonance zones with homoclinic orbits to equilibria, that appear to increase the parameter $\gamma$.

In this work we have shown how the dynamics corresponding to two codimension-two bifurcation points of equilibria (Takens–Bogdanov and Hopf–pitchfork) are globally related by means of the curves of saddle-node bifurcations of periodic orbits. These saddle-node curves appear from the torus bifurcation curve (starting at the Hopf–pitchfork point) and from a Shil’nikov-type homoclinic connection (such a curve of homoclinic orbits emerged from the Takens–Bogdanov point of equilibria). We have found that the existence of a cascade of cubic homoclinic-type Takens–Bogdanov bifurcations of periodic orbits is necessary to allow the connections between the saddle-node curves. Obviously, an interesting task to be developed in the future is the study of the triple-zero linear degeneracy of equilibria: at this codimension-three bifurcation the Takens–Bogdanov and Hopf–zero bifurcations of equilibria coalesce and other global behavior are also present. Some results are presented in [Freire et al., 2000a, 2000b] for systems without symmetry.

We conjecture that the complex dynamical behavior (existence of different kinds of strange
and the bifurcations (existence of successive torus and period-doubling bifurcations, Takens–Bogdanov of periodic orbits) found in this circuit is generic. It will be necessary to study other autonomous three-dimensional systems to confirm it.

Finally, we remark that some of the features (or other similar) present in our system have been found in a very recent model, the fattened Arnold family of annulus diffeomorphisms analyzed by Broer et al. [1998].

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