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Secure Weakly Connected Domination in the Join of Graphs

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Abstract

In this paper, we take a look at the secure weakly connected domination in the join of graphs. In particular, we obtain the bounds for the secure weakly connected domination number of the join and, give necessary and sufficient conditions for the join to have secure weakly connected domination number equal to 1, 2 and 3.

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1 Introduction

Let $G = (V(G), E(G))$ be a connected undirected graph. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $X \subseteq V(G)$, the *open neighborhood* of X is $N(X) = \bigcup_{v \in X} N(v)$ and the *closed neighborhood* of X is $N[X] = \bigcup_{v \in X} N[v]$.

The subgraph $\langle C \rangle$ of G *induced* by C is the graph having vertex-set C and whose edge set consists of those edges of G incident with two elements of C. A graph is called *connected* if every two vertices are joined by a path; otherwise, it is *disconnected*.

A set S is a *dominating set* of G if for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $uv \in E(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. A dominating set $C \subseteq V(G)$ is called a *weakly connected dominating set* of G if the subgraph $\langle C\rangle_w = (N_G[C], E_w)$ weakly induced by C is connected, where E_w is the set of all edges with at least one vertex in C. The *weakly connected domination number* of G, denoted by $\gamma_w(G)$, is the smallest cardinality of a weakly connected dominating set of G.

A set S is a *secure dominating set* of G if S is a dominating set of G and for every $u \in V(G)\backslash S$, there exists $v \in S$ such that $uv \in E(G)$ and $(S\backslash\{v\}) \cup \{u\}$ is a dominating set of G. The *secure domination number* of G, denoted by $\gamma_s(G)$, is the smallest cardinality of a secure dominating set of G. A set S is a *secure weakly connected dominating set* of G if S is a weakly connected dominating set of G and for every $u \in V(G)\backslash S$, there exists $v \in S$ such that $uv \in E(G)$ and $(S\backslash\{v\}) \cup \{u\}$ is a weakly connected dominating set of G. The *secure weakly connected domination number* of G, denoted by $\gamma_{sw}(G)$, is the smallest cadinality of a secure weakly connected dominating set of G.

The concept of weakly connected domination is discussed in [2] [3], and [4]. Another domination parameter is the secure domination which was discussed in [1] and [5]. A combination of these two concepts give rise to a new variant of domination called secure weakly connected domination. The *join* of two graphs G and H, denoted by $G+H$, is the graph with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

2 Results

Remark 2.1 *Let G be a graph of order n. Then $\gamma_{sw}(G) = 1$ if and only if $G = K_n$.*

If G and H are complete graphs, then $G + H$ is a complete graph. The next corollary follows from Remark 2.1.

Corollary 2.2 *Let G and H be a graphs. Then $\gamma_{sw}(G + H) = 1$ if and only if G and H are complete graphs.*

Lemma 2.3 *Let G and H be non-complete graphs. Then $2 \leq \gamma_{sw}(G+H) \leq 4$.*

Proof: Let $S = \{u, v, x, y\}$, where $u, v \in V(G)$ and $x, y \in V(H)$. Then S is a weakly connected dominating set of $G + H$. Let $w \in V(G + H)\backslash S$. Then

$w \in V(G)$ or $w \in V(H)$. Assume that $w \in V(G)$. Then $wx \in E(G+H)$ and $(S\backslash\{x\} \cup \{w\} = \{u,v,w,y\}$ is a weakly connected dominating set of $G+H$. Hence, S is a secure weakly connected dominating set of $G+H$. Thus, $\gamma_{sw}(G+H) \leq |S| = 4$. Since $|V(G+H)| > 2$, $2 \leq \gamma_{sw}(G+H)$. Therefore, $2 \leq \gamma_{sw}(G+H) \leq 4$. □

Theorem 2.4 *Let G and H be non-complete graphs. Then $\gamma_{sw}(G+H) = 2$ if and only if one of the following holds:*

(*i*) $\gamma_s(G) = 2$ *or* $\gamma_s(H) = 2$.

(*ii*) $\gamma(G) = 1$ *and* $\gamma_s(H) = 1$.

(*iii*) $\gamma_s(G) = 1$ *and there exists* $v \in V(H)$ *such that* $\langle V(H)\backslash N_H[v]\rangle$ *is complete.*

(*iv*) $\gamma_s(H) = 1$ *and there exists* $u \in V(G)$ *such that* $\langle V(G)\backslash N_G[u]\rangle$ *is complete.*

(*v*) *There exists* $u \in V(G)$ *and* $v \in V(H)$ *such that* $\langle V(G)\backslash N_G[u]\rangle$ *and* $\langle V(H)\backslash N_H[v]\rangle$ *are complete subgraphs of G and H, respectively.*

Proof: Suppose $\gamma_{sw}(G+H) = 2$. Let $S = \{u,v\}$ be a secure weakly connected dominating set of $G+H$. Consider the following cases:

Case 1. $S \subseteq V(G)$.

Clearly S is a secure dominating set of G. Thus, $\gamma_s(G) \leq |S| = 2$. Since G is non-complete, $\gamma_s(G) \neq 1$. Hence, $\gamma_s(G) = 2$.

Case 2. $S \subseteq V(H)$.

This is similar to Case 1. Thus, $\gamma_s(H) = 2$.

Case 3. $u \in V(G)$ and $v \in V(H)$. Consider the following subcases:

Subcase 3.1. u is a dominating vertex of G and v is a dominating vertex of H.

Then $\gamma(G) = 1$ and $\gamma(H) = 1$.

Subcase 3.2. u is a dominating vertex of G and v is not a dominating vertex of H.

Then $\gamma(G) = 1$ and $V(H)\backslash N_H[v] \neq \varnothing$. Suppose $\langle V(H)\backslash N_H[v]\rangle$ is not complete. Then there exists $x,y \in V(H)\backslash N_H[v]$ such that $xy \notin E(H)$. Since $ux \in E(G+H)$ and $vx \notin E(G+H$, $(S\backslash\{u\}) \cup \{x\} = \{v,x\}$ is not a dominating set of $G+H$. This contradicts the assumption. Hence, $\langle V(H)\backslash N_H[v]\rangle$ is complete.

Subcase 3.3. u is not a dominating vertex of G and v is a dominating vertex of H.

As in Subcase 3.2, $\gamma_s(H) = 1$ and there exists $u \in V(G)$ such that $\langle V(G)\backslash N_G[u]\rangle$ is complete.

Subcase 3.4 u is not a dominating vertex of G and v is not a dominating vertex of H.

From the proof of Subcase 3.2, it follows that $\langle V(G)\backslash N_H[u]\rangle$ and $\langle V(H)\backslash N_H[v]\rangle$ are complete subgraphs of G and H, respectively.

For the converse, suppose first that $\gamma_s(G) = 2$. Let $S = \{x, y\}$ be a secure dominating set of G. By definition of $G + H$, S is a secure dominating set of $G + H$. Suppose $xy \notin E(G)$. Pick $z \in V(H)$. Then $z \in N_{G+H}[S]$ and $[x, z, y]$ is a path in $\langle S\rangle_W$. Thus, S is weakly connected. Let $w \in V(G + H)\backslash S$. If $w \in V(G)$, then since S is a secure dominating set of G, either $xw \in E(G)$ or $yw \in E(G)$. Suppose $xw \in E(G)$. Then $(S\backslash\{x\}) \cup \{w\} = \{y, w\}$ is a weakly connected set of $G + H$. If $w \in V(H)$, then $xw, yw \in E(G + H)$. Hence, $(S\backslash\{x\}) \cup \{w\} = \{y, w\}$ is a weakly connected set of $G + H$. Thus, S is a secure weakly connected dominating set of $G + H$. Similarly, if $\gamma_s(H) = 2$, then S is a secure weakly connected dominating set of $G + H$.

Secondly, suppose that $\gamma(G) = 1$ and $\gamma(G) = 1$. Let $S = \{a, b\}$, where a and b are dominating vertices of G and H, respectively. Clearly, S is a secure weakly connected dominating set of $G + H$.

Thirdly, suppose $\gamma(G) = 1$ and there exists $v \in V(H)$ such that $\langle V(H)\backslash N_H[v]\rangle$ is complete. Let $S = \{u, v\}$. Then S is a weakly connected dominating set of $G + H$. Let $w \in V(G + H)\backslash S$. Consider the following cases:

Case 1. $w \in N_G(u)$. Then $uw \in E(G+H)$ and hence, $(S\backslash\{u\})\cup\{w\} = \{v, w\}$ is a weakly connected dominating set of $G + H$.

Case 2. $w \in V(H)\backslash S$.

If $w \in N_H(v)$, then $vw \in E(G + H)$ and hence, $(S\backslash\{v\}) \cup \{w\} = \{u, w\}$ is a weakly connected dominating set of $G + H$. Suppose that $w \in V(H)\backslash N_H[v]$. Since $uw \in E(G + H)$, $(S\backslash\{u\}) \cup \{w\} = \{v, w\}$. Since $\langle V(H)\backslash N_H[v]\rangle$ is complete, $\{w\}$ is a weakly connected dominating set of $\langle V(H)\backslash N_H[v]\rangle$. Also, $\{v\}$ is a weakly connected dominating set of $\langle N_{G+H}[v]\rangle$. Hence, S is a weakly connected dominating set of $G + H$. Thus, S is a secure weakly connected dominating set of $G + H$.

Finally, suppose there exists $u \in V(G)$ and $v \in V(H)$ such that $\langle V(G)\backslash N_G[u]\rangle$ and $\langle V(H)\backslash N_H[v]\rangle$ are complete subgraphs of G and H, respectively. Let $S = \{u, v\}$. Then S is a weakly connected dominating set of $G + H$. Let $z \in V(G + H)\backslash S$. Then either $z \in V(G)\backslash\{u\}$ or $z \in V(H)\backslash\{v\}$. Suppose $z \in V(G)\backslash\{u\}$. If $z \in N_G(u)$, then $(S\backslash\{u\}) \cup \{z\} = \{z, v\}$ is a weakly connected dominating set of $G + H$. If $z \in V(G)\backslash N_G[u]$, then $(S\backslash\{v\}) \cup \{z\} = \{u, z\}$. Since $\langle V(G)\backslash N_G[u]\rangle$ is complete, $\{z\}$ is a weakly connected dominating set of $\langle V(G)\backslash N_G[u]\rangle$. Thus, S is a weakly connected dominating set of $G + H$. Similarly, if $z \in V(H)\backslash\{v\}$, then S S is a weakly connected dominating set of $G + H$. Hence, S is a secure weakly connected dominating set of $G + H$. Since $G + H$ is non-complete, $\gamma_{sw}(G + H) = |S| = 2$. □

Theorem 2.5 *Let G and H be non-complete graphs and suppose that $\gamma_{sw}(G+H) \neq 2$. Then $\gamma_{sw}(G + H) = 3$ if and only if one of the following holds:*

(*i*) *$\gamma_s(G) = 3$ or $\gamma_s(H) = 3$.*

(*ii*) *$\gamma(G) = 2$ or $\gamma(H) = 2$.*

(*iii*) *There exists $D_1 \subseteq V(G)$ with $|D_1| = 2$ such that $\langle V(G) \backslash N_G[D_1] \rangle$ is complete or there exists $D_2 \subseteq V(H)$ with $|D_2| = 2$ such that $\langle V(H) \backslash N_H[D_2] \rangle$ is complete.*

Proof: Suppose $\gamma_{sw}(G + H) = 3$. Let $S = \{u, v, w\}$ be a secure weakly connected dominating set of $G + H$. Consider the following cases:

Case 1. $S \subseteq V(G)$ or $S \subseteq V(H)$.

Suppose $S \subseteq V(G)$. Clearly S is a secure dominating set of G. Thus, $\gamma_s(G) \leq |S| = 3$. Since $\gamma_{sw}(G + H) \neq 2$, $\gamma_s(G + H) \neq 2$ by Theorem 2.4. Hence, $\gamma_s(G) = 3$. Similarly, $\gamma_s(H) = 3$.

Case 2. $|S \cap V(G)| = 2$ or $|S \cap V(H)| = 2$.

Assume that $|S \cap V(G)| = 2$, say $u, v \in V(G)$. Then $w \in V(H)$. Let $D_1 = \{u, v\}$. If D_1 is a dominating set of G, then $\gamma(G) = 2$ and (*ii*) holds. If D_1 is not a dominating set of G, then $V(G) \backslash N_G[D_1] \neq \varnothing$. Suppose $\langle V(G) \backslash N_G[D_1] \rangle$ is not complete. Then there exists $x, y \in V(G) \backslash N_G[D_1]$ such that $xy \notin E(G + H)$. Since $x, y \in H_{G+H}(w)$, $(S \backslash \{w\}) \cup \{x\} = \{u, v, x\}$ is not a dominating set of $G + H$. This contradicts the assumption that S is a secure weakly connected dominating set of $G + H$. Hence, $\langle V(G) \backslash N_G[D_1] \rangle$ is complete and (*iii*) holds.

For the converse, suppose first that (*i*) holds, say $\gamma_s(G) = 3$. Let $S = \{a, b, c\}$ be a secure dominating set of G. Clearly, S is a weakly connected dominating set of $G + H$. Let $z \in V(G + H) \backslash S$. If $z \in V(G) \backslash S$, since S is a secure dominating set of G, there exist, say $a \in S$, such that $az \in E(G)$ and $(S \backslash \{a\}) \cup \{z\} = \{z, b, c\}$ is a dominating set of G. Hence, $(S \backslash \{a\}) \cup \{z\} = \{z, b, c\}$ is a secure weakly connected dominating set of $G + H$. Thus, $\gamma_{sw}(G + H) \leq |S| = 3$. Since $\gamma_{sw}(G + H) \neq 2$, $\gamma_{sw}(G + H) = 3$.

Next, suppose that (*ii*) holds, say $\gamma(G) = 2$. Let $\{x, y\}$ be a dominating set of G. Choose $z \in V(H)$ and let $S = \{x, y, z\}$. Then S is a secure weakly connected dominating set of $G + H$. Thus, $\gamma_{sw}(G + H) = 3$.

Finally, suppose there exists $D_1 \subseteq V(G)$ with $|D_1| = 2$ such that $\langle V(G) \backslash N_G[D_1] \rangle$ is complete. Let $D_1 = \{u, v\}$. Pick $w \in V(H)$ and let $S = \{u, v, w\}$. Then S is a weakly connected dominating set of $G + H$. Let $z \in V(G + H) \backslash S$. Consider the following cases:

Case 1. $z \in V(H) \backslash \{w\}$.

Then $uz \in E(G + H)$ and $(S \backslash \{u\}) \cup \{z\} = \{z, v, w\}$ is a weakly connected dominating set of $G + H$.

Case 2. $z \in N_G(S)$.

Then there exists, say $u \in S$, such that $uz \in E(G+H)$ and $(S \backslash \{u\}) \cup \{z\} = \{z, v, w\}$ is a weakly connected dominating set of $G + H$.

Case 3. $z \in V(G)\backslash N_G[D_1]$.

Then $wz \in E(G+H$ and $(S\backslash\{w\})\cup\{z\} = \{u,v,z\}$. Since $\langle V(G)\backslash N_G[D_1]\rangle$ is complete, $\{z\}$ is a weakly connected dominating set of $\langle V(G)\backslash N_G[D_1]\rangle$. Also, $D_1 = \{u,v\}$ is a weakly connected dominating set of $N_{G+H}[D_1]$. Thus, $(S\backslash\{w\}) \cup \{z\} = \{u,v,z\}$ is a weakly connected dominating set of $G+H$. Hence, S is a secure weakly connected dominating set of $G+H$.

Therefore, $\gamma_{sw}(G+H) = 3$. □

Remark 2.6 *Let $m \geq 4$ and $n \geq 4$ be integers. Then $\gamma_{sw}(K_{m,n}) = 4$.*

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