Lifting Integer Variables in Minimal Inequalities Corresponding To Lattice-Free Triangles*

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Abstract. Recently, Andersen et al. [1] and Borozan and Cornuéjols [3] characterized the minimal inequalities of a system of two rows with two free integer variables and nonnegative continuous variables. These inequalities are either split cuts or intersection cuts derived using maximal lattice-free convex sets. In order to use these minimal inequalities to obtain cuts from two rows of a general simplex tableau, it is necessary to extend the system to include integer variables (giving the two-dimensional mixed integer infinite group problem), and to develop lifting functions giving the coefficients of the integer variables in the corresponding inequalities. In this paper, we analyze the lifting of minimal inequalities derived from lattice-free triangles.

Maximal lattice-free triangles in $\mathbb{R}^2$ can be classified into three categories: those with multiple integral points in the relative interior of one of its sides, those with integral vertices and one integral point in the relative interior of each side, and those with non integral vertices and one integral point in the relative interior of each side. We prove that the lifting functions are unique for each of the first two categories such that the resultant inequality is minimal for the mixed integer infinite group problem, and characterize them. We show that the lifting function is not necessarily unique in the third category. For this category we show that a fill-in inequality (Johnson [11]) yields minimal inequalities for mixed integer infinite group problem under certain sufficiency conditions. Finally, we present conditions for the fill-in inequality to be extreme.

1 Introduction

Recently, Andersen et al. [1], Borozan and Cornuéjols [3], and Cornuéjols and Margot [5] have developed methods to analyze a system of two rows with two free integer variables and nonnegative continuous variables. They show that facets of system

$$f + \sum_{i=1}^{n} w^i y_i \in \mathbb{Z}^2 \quad y_i \in \mathbb{R}_+, \quad i \in \{1, ..., n\} \tag{1}$$

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are either split cuts or intersection cuts (Balas [2]). These new families of cutting planes are important since some of them cannot be obtained using a finite number of Gomory mixed integer cuts (GMIC); see Cook et al. [4]. We state the following theorem, modified from Borozan and Cornuéjols [3] (See also Theorem 1 in Andersen et al. [1]).

Theorem 1. *For the system*

$$f + \sum_{w \in \mathbb{Q}^2} w y(w) \in \mathbb{Z}^2, \quad y(w) \geq 0 \tag{2}$$

where y has a finite support, an inequality of the form $\sum_{w \in \mathbb{Q}^2} \pi(w) y(w) \geq 1$ is minimal if the closure of the set

$$P(\pi) = \{w \in \mathbb{Q}^2 | \pi(w - f) \leq 1\} \tag{3}$$

is a maximal lattice-free convex set in $\mathbb{R}^2$. Moreover, given a maximal lattice-free convex set $P(\pi)$ such that $f \in interior(P(\pi))$, the function

$$\pi(w) = \begin{cases} 0 & \textit{if } w \in \textit{recession cone of } P(\pi) \\ \lambda & \textit{if } f + \frac{w}{\lambda} \in \textit{Boundary}(P(\pi)) \end{cases} \tag{4}$$

is a minimal valid inequality for (2). □

One way to obtain valid cutting planes for two rows of a simplex tableau using the minimal inequalities for (2) is to relax the non-basic integer variables to be continuous variables. In order to strengthen such a cutting plane we need to derive minimal inequalities for a system like (2) which also has integer variables. Thus, the goal of this study is to lift integer variables into the minimal inequalities for (2). This requires characterizing and analyzing valid lifting functions and obtaining the strongest possible coefficients for the integer variables. (See Nemhauser and Wolsey [12] for an overview on lifting.)

In Sect. 2, we present the relationship between this lifting problem and minimal inequalities for the mixed integer infinite-group problem. In Sect. 3 we study the fill-in procedure of Johnson [11] and create a framework for analyzing the strength of these inequalities. In Sect(s). 4 and 5 we prove that if $P(\pi)$ is a triangle with multiple integral points in the relative interior of one side or if $P(\pi)$ is a triangle with integral vertices and one integral point in the relative interior of each side, then there exists a unique lifting function ϕ such that (ϕ, π) is minimal for the mixed integer group problem. In Sect. 6, we prove that if $P(\pi)$ is a triangle with non-integral vertices with exactly one integral point in the relative interior of each side then there may not exist a unique function ϕ such that (ϕ, π) is minimal. We then present sufficient conditions for the fill-in procedure to generate minimal functions in this case. We conclude in Sect. 7.

2 Preliminaries

Observe that the integer variables in (2) have no sign restrictions. This corresponds to the so-called group relaxation that was discovered and studied by

Gomory [6], Gomory and Johnson [7, 8, 10, 9] and Johnson [11]. We present notation and a brief overview of mixed integer infinite group problem and establish its relationship to (2).

Let I^2 denote the infinite group of real two-dimensional vectors where addition is taken modulo 1 componentwise, i.e., $I^2 = \{(u_1, u_2) \mid 0 \leq u_i < 1 \ \forall 1 \leq i \leq m\}$. Let S^2 represent the set of real two-dimensional vectors $w = (w_1, w_2)$ that satisfy $max_{1 \leq i \leq 2}|w_i| = 1$. For an element $u \in \mathbb{R}^2$, we use the symbol $\mathbb{P}(u)$ to denote the element in I^2 whose i^{th} entry is $u_i(mod\ 1)$. We use the symbol $\bar{0}$ to represent the zero vector in $\mathbb{R}^2$ and I^2.

The mixed integer infinite group problem is defined next.

Definition 1 (Johnson [11]). *Let U be a subgroup of I^2 and W be any subset of S^2. Then the mixed integer infinite group problem, denoted here as $MI(U, W, r)$, is defined as the set of pairs of functions $x : U \rightarrow \mathbb{Z}_+$ and $y : W \rightarrow \mathbb{R}_+$ that satisfy*

1. $\sum_{u \in U} ux(u) + \mathbb{P}(\sum_{w \in W} wy(w)) = r$, $r \in I^2$,
2. *x and y have finite supports.* □

If all the $x(u)$'s are fixed to zero in $MI(I^2, S^2, r)$, the problem would reduce to that presented in (2) where $r \equiv \mathbb{P}(-f)$.[1] Thus, we need to lift the integer variables into the inequality π to obtain a pair of functions (ϕ, π) corresponding to valid inequalities for $MI(I^2, S^2, r)$. We next define these valid inequalities for $MI(I^2, S^2, r)$ more precisely.

Definition 2 (Johnson [11]). *A valid function for $MI(I^2, S^2, r)$ is defined as a pair of functions, $\phi : I^2 \rightarrow \mathbb{R}_+$ and $\mu_\phi : S^2 \rightarrow \mathbb{R}_+$, such that $\sum_{u \in I^2} \phi(u)x(u) + \sum_{w \in S^2} \mu_\phi(w)y(w) \geq 1$, $\forall (x, y) \in MI(I^2, S^2, r)$, where $\phi(\bar{0}) = 0$ and $\phi(r) = 1$.* □

[We will use the terms valid inequality and valid function interchangeably]. Note here that the relationship between the functions π of Theorem 1 and $\mu_\phi : S^2 \rightarrow \mathbb{R}_+$ in Definition 2 is straight forward. Since π is positively homogenous, we can construct μ_ϕ in a well-defined fashion by restricting the domain of π to S^2. Conversely, given μ_ϕ, π is the gauge function which is the homogenous extension of μ_ϕ. Because of this close relationship, we will use the same symbol for both the functions. See Gomory and Johnson [9] for a presentation of how these inequalities can be used to generate valid cutting planes for two rows of a simplex tableau. Next we define the notion of minimal inequalities.

Definition 3 (Johnson [11]). *A valid function (ϕ, π) is minimal for $MI(U, W, r)$ if there does not exist a valid function (ϕ^*, π^*) for $MI(U, W, r)$ different from (ϕ, π) such that $\phi^*(u) \leq \phi(u)\ \forall u \in U$ and $\pi^*(w) \leq \pi(w)\ \forall w \in W$.* □

[1] Note here that columns corresponding to the continuous variables are assumed to be rational in (2). However, we will assume that $W = S^2$. The function π can be computed for irrational values using Theorem 1. Therefore, this assumption does not pose any difficulties.

Therefore, given the function $\pi : \mathbb{R}^2 \to \mathbb{R}_+$, the goal of this study is to derive a function $\phi : I^2 \to \mathbb{R}_+$ so that the pair (ϕ, π) forms minimal inequality for $MI(I^2, W, r)$. We next present a proposition (that uses Lemma 5 of Andersen et al. [1] for its proof) to classify maximal lattice-free triangles in $\mathbb{R}^2$. Each category is separately analyzed in Sect. 4-6.

Proposition 1. *If P is a maximal lattice-free triangle in $\mathbb{R}^2$, then exactly one of the following is true:*

1. *One side of P contains more than one integral point in its relative interior.*
2. *All the vertices are integral and each side contains one integral point in its relative interior.*
3. *The vertices are non-integral and each side contains one integral point in its relative interior.* □

3 Coefficient for Integer Variables: Fill-in Procedure

We begin this Sect. with a presentation of the fill-in procedure developed by Gomory and Johnson [8] and Johnson [11] that is used to generate valid inequalities for the infinite group problem. We then present some techniques to analyze the minimality of the inequalities that are constructed using the fill-in procedure. Finally, we present conditions under which the fill-in inequalities are extreme.

Definition 4 is adapted from the original definition of fill-in procedure.

Definition 4 (Fill-in procedure). *Let $P(\pi)$ be a bounded lattice-free convex set. Let G be a subset of I^2 such that the subgroup of I^2 generated by G is finite. Let $V : G \to \mathbb{R}_+$ be a function such that*

$$\sum_{u \in G} x(u)V(u) + \sum_{w \in S^2} y(w)\pi(w) \geq 1 \tag{5}$$

is satisfied for every $(x, y) \in MI(I^2, S^2, r)$ such that $x(u) = 0 \ \forall u \notin G$. Define the fill-in function $\phi^{G,V} : I^2 \to \mathbb{R}_+$ as follows:

$$\phi^{G,V}(u) = min_{n(v) \in \mathbb{Z}_+} \left\{ \sum_{v \in G} n(v)V(v) + \pi(w) \mid \sum_{v \in G} n(v)v + w \equiv u \right\}. \tag{6}$$

□

It can be verified that the construction of $\phi^{G,V}$ is equivalent to the original fill-in procedure of Johnson [11] in which we start with the subgroup S^G of I^2 generated by G and the valid function $\tilde{\phi} : S^G \to \mathbb{R}_+$ defined as $\tilde{\phi}(v) = \min_{n(u) \in \mathbb{Z}_+} \{\sum_{u \in G} n(u)V(u) | \sum_{u \in G} n(u)u = v\}$. It is easily verified that the pair $(\phi^{G,V}, \pi)$ forms a subadditive valid inequality for $MI(I^2, S^2, r)$.

The fill-in procedure may be interpreted as a two-step lifting scheme. In the first step we obtain the inequality (5) by lifting integer variables corresponding to columns in the set G. The lifting coefficients, (i.e., V) may depend on the order of lifting of these variables, i.e., for a given set G there may exist two different functions V_1 and V_2 such that both functions eventually yield strong cutting planes for $MI(I^2, S^2, r)$. Once the integer variables corresponding to columns in the set G are lifted, the lifting in the second step (of the rest of the integer variables) is completely defined by the choice of G and V.

It can be verified that the function $\phi^{G,V}$ can be evaluated in finite time for each $u \in I^2$. We next study conditions under which $\phi^{G,V}$ is a minimal function. We begin with some results regarding π and $\phi^{\{\bar{0}\},\{0\}}$ (We denote $\phi^{\{\bar{0}\},\{0\}}$ by $\phi^{\bar{0}}$).

Proposition 2. *If π satisfies at least one point at equality for (2), then $\phi^{\bar{0}}(r) = 1$.* □

We next define a subset of I^2 for which we are guaranteed 'good' coefficients even if the set G only contains the element $\bar{0}$. This result helps in proving that under certain conditions the lifting function is unique.

Definition 5. *Let d^1 , d^2, ... be the extreme rays of $P(\pi)$, i.e., $d^i + f$ are the extreme points of $P(\pi)$. Let s_i be the line segment between vertices $d^i + f$ and $d^{i+1} + f$ (where $d^4 := d^1$ when $P(\pi)$ is triangle). Let p^i be the set of integer points in the relative interior of s_i. For an integral point $X^j \in p^i$, let $\delta^{ij} d^i + (1 - \delta^{ij}) d^{i+1} + f = X^j$ where $0 < \delta^{ij} < 1$. Define the $D_{ij}(\pi) = \{\rho d^i + \gamma d^{i+1} | 0 \leq \rho \leq \delta^{ij}, 0 \leq \gamma \leq 1 - \delta^{ij}\}$. Let $D(\pi) = \cup_{i,j} D_{ij}(\pi)$.* □

See Fig(s). 2 and 3 for illustration of $D(\pi)$ (represented as the shaded region within the triangle).

Proposition 3. *Let $P(\pi)$ be a bounded maximal lattice free convex set. For any $v \in D(\pi)$ the following are true:*

1. *$\sum_{u \in I^2} ux(u) + \sum_{w \in \mathbb{R}^2} wy(w) + f \in \mathbb{Z}^2$ has a solution $(\bar{x}, \bar{y})$ with $\bar{x}(\mathbb{P}(v)) > 0$ which satisfies the cutting plane $(\phi^{\bar{0}}, \pi)$ at equality.*
2. *$\phi^{\bar{0}}(\mathbb{P}(v)) = \pi(v)$.*
3. *$\phi^{\bar{0}}(\mathbb{P}(v)) + \phi^{\bar{0}}(\mathbb{P}(r - v)) = 1$.*
4. *If $(\bar{\phi}, \pi)$ is any valid inequality for $MI(I^2, S^2, r)$, then $\bar{\phi}(\mathbb{P}(v)) \geq \phi^{\bar{0}}(\mathbb{P}(v))$.* □

Corollary 1. *Let $P(\pi)$ be a bounded maximal lattice free convex set. Then $lim_{h \to 0^+} \frac{\phi^{\bar{0}}(\mathbb{P}(wh))}{h} = \pi(w) \; \forall \; w \in \mathbb{R}^2$.* □

Similar to the proof of Proposition 3 the following result can be proven.

Proposition 4. *Let $P(\pi)$ be a maximal lattice-free triangle. If $u^* \notin D(\pi)$ and $\phi^{\bar{0}}(\mathbb{P}(u^*)) = \pi(u^*)$, then $\phi^{\bar{0}}(\mathbb{P}(u^*)) + \phi^{\bar{0}}(\mathbb{P}(r - u^*)) > 1$.* □

Using a characterization for minimal inequalities for $MI(I^2, S^2, r)$ from Johnson [11] (Theorem 6.1), Proposition 2, Corollary 1, Proposition 4, and the fact that $\phi^{G,V} \leq \phi^{\bar{0}}$ we can now verify the following result.

Corollary 2. *If π is a valid and minimal function for (2) and $\phi^{G,V}(u) + \phi^{G,V}(r - u) = 1 \; \forall u \in I^2$, then $(\phi^{G,V}, \pi)$ is minimal for $MI(I^2, S^2, r)$. Moreover, $(\phi^{\bar{0}}, \pi)$ is minimal for $MI(I^2, S^2, r)$ iff $\mathbb{P}(D(\pi)) = I^2$.* □

If the function $\phi^{G,V}$ is minimal, then we show that it must be the unique minimal function. The proof uses the fact that minimal functions must be subadditive. This result allows us to construct extreme inequalities for the infinite group problem using the modified fill-in procedure.

Theorem 2. *Let $(\phi^{G,V}, \pi)$ be minimal for $MI(I^2, S^2, r)$. If (ϕ', π) is any valid minimal function for $MI(I^2, S^2, r)$ such that $\phi'(u) = V(u)\ \forall u \in G$, then $\phi'(v) = \phi^{G,V}(v)\ \forall v \in I^2$.* □

We say that (V, π) is an extreme inequality if (V_1, π_1) and (V_2, π_2) are valid inequalities like (5) and $V = \frac{1}{2}V_1 + \frac{1}{2}V_2$, $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ imply that $V_1 = V_2 = V$, $\pi_1 = \pi_2 = \pi$.

Corollary 3. *If (V, π) is extreme and $(\phi^{G,V}, \pi)$ is minimal for $MI(I^2, S^2, r)$, then $(\phi^{G,V}, \pi)$ is an extreme valid inequality for $MI(I^2, S^2, r)$.* □

4 Multiple Integral Points in the Relative Interior of One Side

We begin with a construction that has properties similar to that of a maximal lattice-free triangle with multiple integral points in the relative interior of one side. We denote the length of a line segment uv as $|uv|$. An illustration of Construction 1 is given in Fig. 1.

Construction 1. Let abc be any nontrivial triangle, i.e., a, b, and c are distinct points. Let d be any point in the interior of the triangle. Let ew be a line segment parallel to the side bc where e belongs to the relative interior of line segment ab and w belongs to the relative interior of the line segment ac. Let ep and wm be line segments parallel to ad, meeting bd and cd at p and m respectively. This is a well-defined step, since bda and cda form non trivial triangles.

Let q be the point at which the line passing through ep meets bc. Let gh be a line segment on bc such that it has the same length as that of ew. (This is well-defined since e and w belong to the relative interior of lines ab and ac. Therefore $|gh| = |ew| < |bc|$). WLOG of generality, we assume that g lies to the left of q. (The proof will be similar for the case when g is to the right of q).

Let ei be a line segment parallel to bd meeting ad at i. Let wj be a line segment parallel to cd meeting ad at j. Let kg be a line parallel to dc meeting bd at k. Let gr be a line segment parallel to bd meeting dc at r. Let t be the point at which the line passing through ep meets gr. Let hn be a line segment parallel to bd meeting dc at n. Let hs be a line segment parallel to dc meeting bd at s. (It can be verified that these construction steps are well-defined). Let l be the point of intersection of gr and hs. [This is well-defined: Since d is in the strict interior of the triangle abc, bd and cd are not parallel. Therefore the lines passing through the segments gr and hs are not parallel and must intersect.] □

Proposition 5. *For Construction 1,*

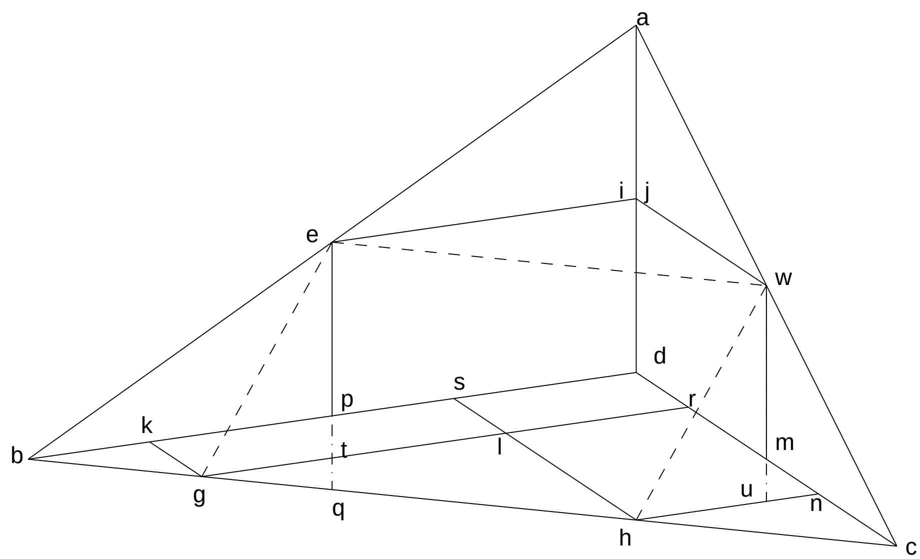

Fig. 1. An example of Construction 1.

1. *i and j are the same point. There exists a point u such that wm extended to wu intersects hn.*
2. *Triangle glh is symmetric to triangle eiw. [Two triangles are symmetric if the length of their three sides are equal]*
3. *Triangle wuh is symmetric to etg.* □

Using Proposition 5, we can prove the main result of this Sect., that is presented next.

Theorem 3. *If $P(\pi)$ is a maximal lattice-free triangle with multiple integral points in the relative interior of one side, then (ϕ, π) is minimal for $MI(I^2, S^2, r)$ iff $\phi = \phi^{\bar{0}}$.* □

This result is similar to the result in one dimension where the unique lifting function yields GMIC. We sketch the main steps in the proof of Theorem 3:

1. Any maximal lattice-free triangle with multiple integral points in the relative interior of one side can be represented by the triangle abc in Construction 1, where the point d represents f (this is the fractional vector in (2)) and the points e, w, g, h represent the integral points in the relative interior of each side.
2. Using (2.) and (3.) of Proposition 5, it can be verified that $\mathbb{P}(D(\pi)) = \mathbb{P}(ewgh)$. Finally, it can be shown that $\mathbb{P}(ewgh) = I^2$ since ew and gh must be parallel and e, w, g, h represent integral points. The result now follows from Corollary 2.

Figure 2 shows a maximal lattice-free triangle with 3 integral points in the relative interior of one of its sides ($D(\pi)$ is represented as the shaded region) and the function $\phi^{\bar{0}}$.

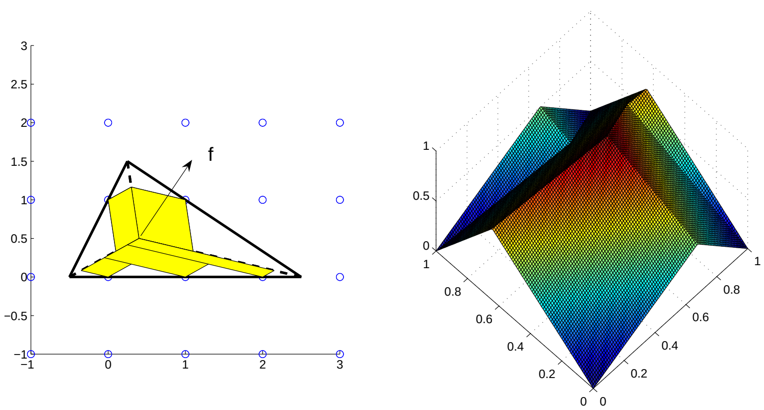

Fig. 2. Lattice-free Triangle with more than one integral point in the relative interior of one side.

5 Single Integral Point in the Relative Interior of Each Side and Integral Vertices

We begin this Sect. with the observation that since (2) is a modular equation, without loss of generality, we may replace f with $f + (n_1, n_2)$, where $n_1, n_2 \in \mathbb{Z}$ are such that one of the integral points in the relative interior of the side of the triangle in $(0, 0)$. The next proposition shows that a study of a specific subclass of triangles allows us to generalize results to any triangle in this category.

Proposition 6. *Let $P(\pi)$ be a maximal lattice-free triangle with $f \in interior P(\pi)$. Let M be a two-by-two unimodular matrix. Let $MP(\pi) = \{(x_1, x_2) | M^{-1}(x_1, x_2) \in P(\pi)\}$. Define the functions $M\phi : I^2 \rightarrow \mathbb{R}_+$ and $M\pi : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ as $M\phi(u) = \phi(\mathbb{P}(M^{-1}u))$ and $M\pi(w) = \pi(M^{-1}w)$. (Note that $M\pi$ is the function corresponding to the maximal lattice free triangle $MP(\pi)$ and $Mf \in MP(\pi)$.) Then (ϕ, π) is a minimal inequality for $MI(I^2, S^2, r)$ iff $(M\phi, M\pi)$ is a minimal inequality for $MI(I^2, S^2, \mathbb{P}(Mr))$.* □

We have assumed that one of the integral points in the relative interior of the side of the triangle in $(0,0)$. Now by applying a suitable unimodular matrix transformation, we can assume without loss of generality that the other two integral point in the relative interior of the other two sides of the triangle are $(1,0)$, and $(0,1)$. Now using a proof similar to that of Theorem 3 and using Proposition 6 we can prove the following result.

Theorem 4. *If $P(\pi)$ is a maximal lattice-free triangle with integral vertices and one integral point in the relative interior of each side, then (ϕ,π) is minimal for $MI(I^2,S^2,r)$ iff $\phi=\phi^{\bar{0}}$.* $\square$

Figure 3 show an example of a lattice-free triangle and $\phi^{\bar{0}}$ is this case.

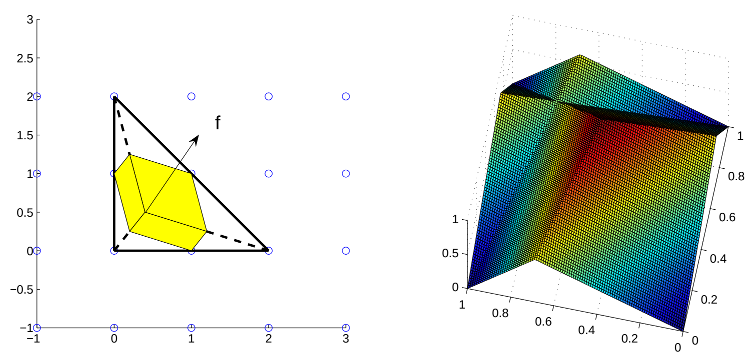

Fig. 3. Lattice-free triangle with integral vertices and one integral point in the relative interior of each side.

6 Single Integral Point in the Relative Interior of Each Side and Non-integral Vertices

In this Sect., we first show that in contrast to the previous cases, if $P(\pi)$ is a triangle with single integral point in the relative interior of each side and non-integral vertices, then $(\phi^{\bar{0}},\pi)$ is not minimal. We then present some sufficient conditions for generating a minimal inequality using the fill-in procedure.

By Proposition 6, we need to only analyze triangles whose integral points in the relative interior of its sides are: $(0,0)$, $(1,0)$, and $(0,1)$. Let s_1, s_2 and s_3 be the sides of $P(\pi)$ passing through $(1,0)$, $(0,1)$, and $(0,0)$ respectively. It can be verified that either the slope of s_1 is positive (and s_1 is not vertical) and the slope of s_2 is negative or vice-verse. Henceforth we assume WLOG that slope of s_1 is negative and the slope of s_2 is positive (and s_1 is not vertical).

To prove that $(\phi^{\bar{0}}, \pi)$ is not minimal, we show that $\mathbb{P}(D(\pi))$ is a proper subset of I^2 and then use Corollary 2. This is achieved by verifying that the area of $D(\pi)$ is less than 1 in this case.

Proposition 7. *The area of $D(\pi)$ is maximized if f is one of the vertices of $P(\pi)$.* □

Now using Propositions 7 and 6 and checking for the maximum possible area of $D(\pi)$ we obtain the following result.

Theorem 5. *If $P(\pi)$ is a lattice-free triangle with single integral point in the relative interior of each side and non-integral vertices, then $(\phi^{\bar{0}}, \pi)$ is not minimal for $MI(I^2, S^2, r)$.* □

Example 1. Let $P(\pi)$ be the triangle with vertices $(0.25, 1.25)$, $(-0.75, 0.25)$, and $(1.25, -5/12)$ and let $f = (0.5, 0.5)$. Then it can be verified that $P(\pi)$ is a lattice-free triangle with only one integer point in the relative interior of each of its sides and non-integral vertices. $\phi^{\bar{0}}(0.1, 0.2) = 1.1$ and $\phi^{\bar{0}}$ is not minimal. Also there are two distinct functions ϕ^{v_0} (see notation/result of Theorem 6) and ϕ_2 such that both (ϕ^{v_0}, π) and (ϕ_2, π) are minimal. See Fig. 4. □

Therefore, starting from different sets G and corresponding functions V, it may be possible to construct different functions $\phi^{G,V}$ that are all minimal. We first concentrate on the case when G is a single non-zero element set. In this specific case, we use the notation ϕ^u to denote $\phi^{\{u\},V(u)}$ where $V(u) = \max_{n \in \mathbb{Z}, n \geq 1} \left\{ \frac{1-\pi(w)}{n} | w \equiv r - nu \right\}$.

The main result on this Sect. is presented next. (This result can be used for any lattice-free maximal triangle with non-integral vertices and only one integral point in the relative interior of each side by using Proposition 6).

Theorem 6. *Let $P(\pi)$ be a triangle whose integral points in the relative interior of its sides are: $(0,0)$, $(1,0)$, and $(0,1)$. Let the slope of s_1 be negative (and not vertical) and the slope of s_2 be positive. Define $0 < \delta^{12}, \delta^{23}, \delta^{31} < 1$ such that $\delta^{12} d^1 + (1 - \delta^{12}) d^2 + f = (1,0)$, $\delta^{23} d^2 + (1 - \delta^{23}) d^3 + f = (0,1)$, and $\delta^{31} d^3 + (1 - \delta^{31}) d^1 + f = (0,0)$. Let $\hat{u} = \left(\frac{1 - \delta^{12} + \delta^{23}}{2} \right) d^2 + (\delta^{12} + \delta^{31} - 1) d^1$. Denote $\mathbb{P}(r - \hat{u})$ by v_0. If $V(v_0) = 1 - \pi(\hat{u})$, then (ϕ^{v_0}, π) is minimal for $MI(I^2, S^2, r)$.*
□

We sketch the main steps in the proof of Theorem 6:

1. Construct a set $T(\pi)$ defined as follows: (Refer to Fig. 5)

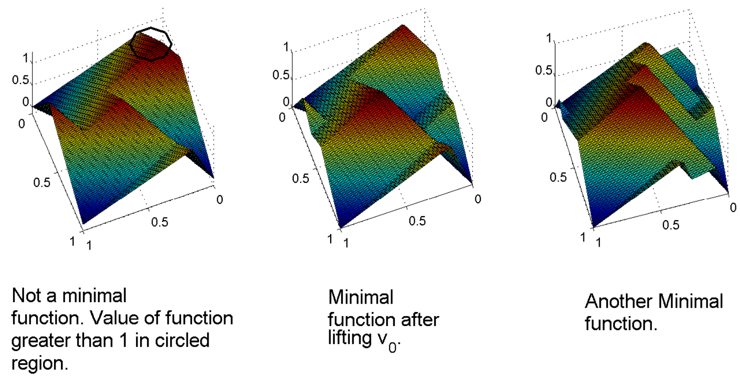

Fig. 4. Example where the function $\phi^{\bar{0}}$ is not minimal. $\phi^{v_0, V_{v_0}}$ is minimal. There exist distinct functions ϕ^{v_0} and ϕ_2 such that (ϕ^{v_0}, π) and (ϕ_2, π) are minimal.

(a) Let Q^1 be the quadrilateral whose vertices are: $f + (1 - \delta^{12})d^2$ (represented by j), $f + \delta^{23}d^2$ (represented by i), $f + \delta^{23}d^2 + (\delta^{12} - 1 + \delta^{31})d^1$ (represented by o), $f + (1 - \delta^{12})d^2 + (\delta^{12} - 1 + \delta^{31})d^1$ (represented by p).

(b) Let Q^2 be the quadrilateral whose vertices are: $f + \delta^{31}d^3$ (represented by k) , $f + \delta^{31}d^3 + (\delta^{23} - 1 + \delta^{12})d^2$ (represented by q), $f + (1 - \delta^{23})d^3 + (\delta^{23} - 1 + \delta^{12})d^2$ (represented by z), $f + (1 - \delta^{23})d^3$ (represented by l).

(c) Let Q^3 be the quadrilateral whose vertices are: $f + \delta^{12}d^1$ (represented by m), $f + (1 - \delta^{31})d^1$ (represented by n), $f + (1 - \delta^{31})d^1 + (\delta^{31} - 1 + \delta^{23})d^3$ (represented by t), $f + \delta^{12}d^1 + (\delta^{31} - 1 + \delta^{23})d^3$ (represented by s).

(d) Let $T(\pi) = (D(\pi) + \{f\}) \cup Q^1 \cup Q^2 \cup Q^3$. (Note $D(\pi) + \{f\}$ is represented by $dihl$, $dken$, and $dmgj$).

2. Prove that $\mathbb{P}(T(\pi)) = I^2$. This involves applying a sequence of lattice-preserving operations to subsets of $T(\pi)$ as shown in Fig. 6.

3. Refer to fig. 5. The point $\hat{u} + f$ which is represented by u_0, is the center of the line segment op. Let $u1_0$ be the center of the line segment qk. Let α be the center of ij, i.e., $\alpha = f + \frac{1}{2}(1 - \delta^{12} + \delta^{23})d^2$. Let β be the center of zl, i.e., $\beta = f + (1 - \delta^{23})d^3 + \frac{1}{2}(1 - \delta^{12})d^2$. We use the symbols Q^{11}, Q^{12}, Q^{21}, and Q^{22} to represent the quadrilateral $\{\alpha, j, p, u_0\}$, $\{\alpha, i, o, u_0\}$, $\{k, l, \beta, u1_0\}$, and $\{\beta, z, q, u1_0\}$ respectively. Construct the function $\phi_1 : T(\pi) \to \mathbb{R}_+$ as follows.

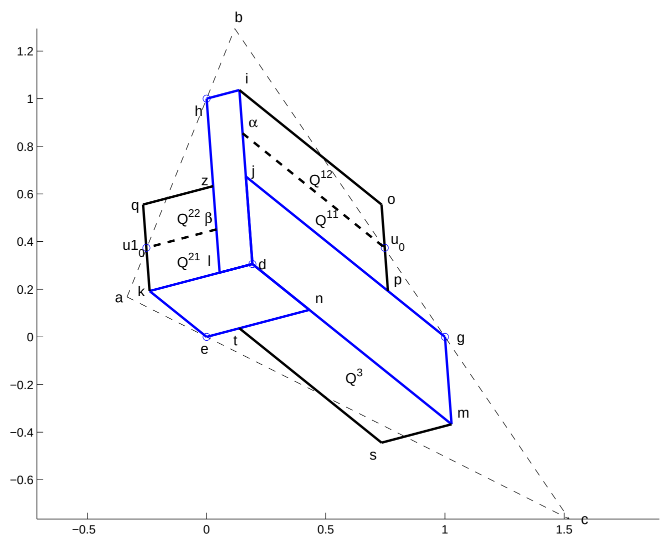

Fig. 5. Figure illustrating u_0, α, and β

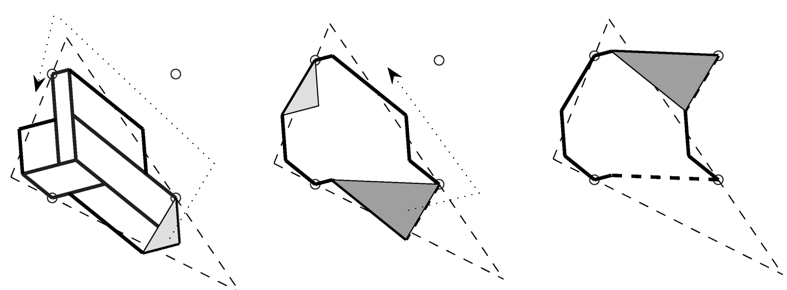

Fig. 6. Sequence of operations to prove that $\mathbb{P}(T(\pi)) = I^2$

$$\phi_1(u) = \begin{cases} V_{v_0} + \pi(u - f - (1-\delta^{31})d^1 + \frac{1}{2}(-1+\delta^{12}+\delta^{23})d^2 - (0,1)) & \text{if } u \in Q^{12} \\ V_{v_0} + \pi(u - f - (1-\delta^{31})d^1 + \frac{1}{2}(-1+\delta^{12}+\delta^{23})d^2 - (-1,1)) & \text{if } u \in Q^{22} \\ V_{v_0} + \pi(u - f - (1-\delta^{31})d^1 + \frac{1}{2}(-1+\delta^{12}+\delta^{23})d^2) & \text{if } u \in Q^3 \\ \pi(u-f) & \text{otherwise} \end{cases}$$

4. Construct the function $\tilde{\phi}(u) = \min\{\phi_1(w) | w \in T(\pi), \mathbb{P}(w - f) = u\}$. Prove that $\tilde{\phi} \geq \phi^{v_0}$.
5. Prove that $\tilde{\phi}(u) + \tilde{\phi}(r - u) = 1 \ \forall u \in I^2$.

6. Now use the following theorem from Johnson [11]: If $\phi : I^2 \to \mathbb{R}_+$ is a valid function for $MI(I^2, \emptyset, r)$ and if $\phi(u) + \phi(r-u) \leq 1 \ \forall u \in I^2$, then ϕ is subadditive. This shows that $(\tilde{\phi}, \pi)$ is minimal and that $\tilde{\phi} = \phi^{v_0}$.

7 Conclusion

In this paper, we analyzed lifting functions for nonnegative integer variables when starting from minimal valid inequalities for a system of two rows with two free integer variables and nonnegative continuous variables. We proved that unique lifting functions exist in the case when the original inequality for the continuous variables corresponds to either a maximal lattice-free triangle with multiple integral points in the relative interior of one of its sides or a maximal lattice-free triangle with integral vertices and one integral point in the relative interior of each side. The resulting lifted inequality is minimal for $MI(I^2, S^2, r)$. In Theorem 6, we showed that under suitable conditions, starting with a specific cyclic subgroup of I^2 and using the fill-in procedure leads to minimal inequalities for $MI(I^2, S^2, r)$ when the inequality π corresponds to a lattice-free triangle with non-integral vertices and one integral point in the relative interior of each side.

We hope that these new families of minimal inequalities for $MI(I^2, S^2, r)$ may perform well computationally, since the coefficient for continuous variables in these inequalities is not dominated by any other inequality for the two-dimensional infinite group problem. These inequalities may also prove to be a source for numerically stable high-rank cuts.

Future research directions include analysis of maximal lattice-free quadrilaterals and extensive computational experiments.

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