Performance Bounds on State-Feedback Controllers with Network Delay

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***Abstract*— This paper describes a convex optimization approach to bound the performance degradation of control systems subject to network loop delay. We consider a linear time-invariant continuous-time plant connected over a communication network to a remote controller. The network introduces bounded but time-varying delays between the controller and plant. We establish an upper bound on the worst-case performance degradation due to the network delays as a function of the delay bound, which can be used as a design parameter for the networked implementation. Numerical simulation results illustrate the degree of conservativeness of the bounds.**

I. INTRODUCTION

A networked control system (NCS) consists of spatially distributed sensors, actuators, and controllers that exchange information through a communication network. NCSs have been finding application in a broad range of areas, including from mobile sensor networks [8], [12], unmanned aerial vehicles [7], collaborative haptics systems [6], and medical applications such as surgery over the Internet [1]. The main effects of introducing a network are time-varying delays between plant and controller, band-limited channel communication, and lost information packets. Such effects can affect the dynamic behavior of controlled system significantly [10]. As a result, existing analysis and design techniques have to be modified to take the communication properties into account. This is in contrast to traditional control systems in which delays in signal transmission are negligible.

A variety of problems for NCSs have been formulated and studied in recent years. Problems arising due to timing variations in control systems are outlined in [16] and analyzed in [11]. Zhang and Branicky present a Lyapunov-based bounding approach and propose a randomized algorithm to find the largest value of the transmission delay for which stability can be guaranteed [18]. Hespanha *et al.* address the effects of channel limitations in terms of packet rates, sampling, network delay, and packet dropouts on estimation and control synthesis [5]. Sinopoli *et al.* propose a mathematical framework to optimally design networked control systems using the common UDP and TCP protocols over lossy physical layer links [14], [13]. A linear matrix inequality (LMI) approach to NCSs with delays and losses is presented in Yu *et al.* [17]. Simulation tools aimed at evaluating the effects of network delays are described in [2] and [9].

In contrast to the work described above, this paper takes the approach suggested in [15], which focuses on performance degradation when a nominal controller is implemented under non-ideal conditions. Skaf and Boyd [15] consider performance degradation of a state-feedback optimal linear-quadratic regulater (LQR) design when there is jitter in the sampling period. Here we adopt this framework to study performance degradation caused by time-varying network delays. Knowing the upper bound on the worst-case performance degradation as a function of the bound on the time-varying network delay can have practical implications. Given the network that will be used for the implantation, the bound on the worst-case performance can be used to determine if the control design is adequate. Alternatively, a specification for the maximum performance degradation that can be tolerated can be translated into a design specification for the maximum delay for the network design.

The following section introduces the network delay problem considered in this paper and reviews briefly the relationship of this problem to the problem considered in [15]. Section III develops the mathematical problem formulation. Section IV details the convex optimization approach used to derive bounds on controller performance degradation. Experimental results are presented in Section V. Section VI summarizes the contribution of this paper and discusses directions for future research.

II. CONTROL WITH NETWORK DELAYS

We consider control systems in which the plant is connected to a remote controller over a communication network, as shown in Fig. 1. Sensors sample the measured plant variables at times t_i^s, $i = 0, 1, \ldots$, generating the sampled output sequence y_i. This vector of values is sent over a network to the controller with a delay δ_y. The controller computes the control input u_i which is sent back to the plant over a network with a delay δ_u, and this control value is applied to the plant when it is received. Thus, the control input u_i based on the i^{th} measurement y_i is received at the plant at time $t_i^r = t_i^s + \delta_y + \delta_u$. We call the total delay between the sampling of the plant output and the receipt of the corresponding control value the *network loop delay*, denoted by δ. The network loop delay can also include the computation delay in the controller if it is not negligible. The network loop delay can be different for each sample, so we denote the delay for the i^{th} sample by δ_i.

The convex optimization approach taken in this paper to bound the performance degradation due to network delay is similar to the approach taken by Skaf and Boyd in [15] for the problem of timing variations in the sampling instants t_i^s. They consider the effect of sampling jitter on the performance of an optimal LQR state-feedback controller

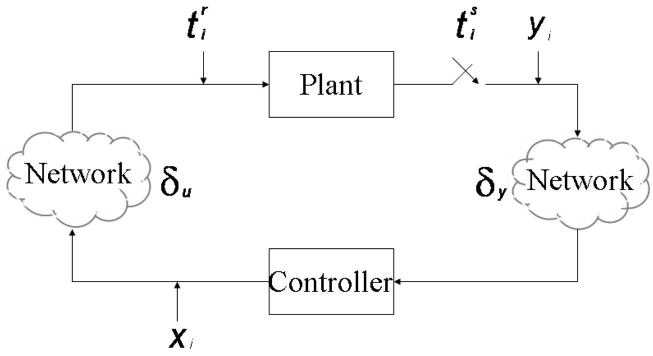

Fig. 1. Plant and controller with network delays.

designed for the nominal case of a constant sampling period, T. The samplings at time instants t_i^s have jitter bounded by

$$T - \overline{\delta_s} \leq t_{i+1}^s - t_i^s \leq T + \overline{\delta_s} \qquad \forall\, i \geq 0 \tag{1}$$

where the parameter $\overline{\delta_s}$ is the worst-case sampling jitter bounded by T. The controller calculates the piecewise-constant control signal $\mathbf{u}_i$, which is then received instantly at the plant input at $t_i^r = t_i^s$. This timing model allows for arbitrary drift in the sampling instants t_i^s from the nominal time instants *i*T, since the constraint (1) is enforced on the relative time difference between the system sampling instants.

In contrast, our scenario assumes perfect sampling times at the plant ($t_i^s = iT$) and instead studies the effect of variable time delay (δ_y, δ_u) between the plant and controller, introduced by a communication network. Our aim is to put a bound on the relative performance degradation caused by the worst-case sequence of such network delays.

III. PROBLEM FORMULATION

We consider a linear time-invariant (LTI) continuous-time plant with dynamics given by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{2}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state of the system, $\mathbf{u}(t) \in \mathbb{R}^m$ is the input to the system, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are the system and input matrices, respectively, and $\mathbf{x}_0$ is a given initial plant state. We assume the plant state is sampled with a sample period of T seconds and the network loop delay δ_i for each sample is bounded by a constant maximum delay $\overline{\delta}$. We assume the maximum delay is less than one sampling period. Therefore, for each $i = 0, 1, \ldots$,

$$0 \leq \delta_i \leq \overline{\delta} < T. \tag{3}$$

We denote a complete sequence of network loop delays by $\Delta = (\delta_0, \delta_1, \ldots)$.

For the NCS shown in Fig. 1, the control input signal applied to the plant is a piecewise constant signal given by

$$\mathbf{u}(t) = \mathbf{u}_i, \qquad t_i^r \leq t < t_{i+1}^r \qquad i = 0, 1, \ldots, \tag{4}$$

When $\delta_0 > 0$, $t_0^r = \delta_0 > 0$, in which case (4) does not determine the input signal for $0 \leq t < t_0^r$. On this initial time interval, we assume without loss of generality that $\mathbf{u}(t) = \mathbf{u}_{-1} \equiv \mathbf{0}$.

Given the input signal (4) and a network loop delay sequence Δ, the LTI plant state at the sampling instants $t_i^s = iT$, $i = 0, 1, \ldots$ is given by

$$\mathbf{x}_{i+1} = A_d\mathbf{x}_i + B_0(\delta_i)\mathbf{u}_i + B_1(\delta_i)\mathbf{u}_{i-1}, \tag{5}$$

where $\mathbf{x}_i = \mathbf{x}(iT)$, and the system matrices are given by

$$A_d = e^{AT}, \quad B_0(\delta) = \int_0^{T-\delta} e^{A(T-\tau)}\,\mathrm{d}\tau B$$

and

$$B_1(\delta) = \int_0^{\delta} e^{A(T-\tau)}\,\mathrm{d}\tau B$$

To represent the discrete-time system (5) in standard form, we define an augmented state vector

$$\mathbf{z}_i = \begin{bmatrix} \mathbf{x}_i \\ \mathbf{u}_{i-1} \end{bmatrix}$$

and write the state equation in terms of $\mathbf{z}_i$ as

$$\mathbf{z}_{i+1} = F(\delta_i)\mathbf{z}_i + G(\delta_i)\mathbf{u}_i, \quad \mathbf{z}(0) = \mathbf{z}_0 \tag{6}$$

where $\mathbf{z}_0^T = [\mathbf{x}_0^T \ \mathbf{0}^T]$ and

$$F(\delta) = \begin{bmatrix} A_d & B_1(\delta) \\ 0 & 0 \end{bmatrix} \qquad G(\delta) = \begin{bmatrix} B_0(\delta) \\ I \end{bmatrix}$$

The *nominal system* is defined as the discrete-time system that results when there is no network delay ($\delta_i = 0\ \forall i \geq 0$), and $(F_{nom}, G_{nom}) = (F(0), G(0))$. We are interested in bounding the worst-case degradation of the performance of the control system designed for the nominal system due to bounded time-varying network delay given in (3).

As a performance measure, we use linear-quadratic discrete-time cost given by

$$J(\mathbf{z}_0, \mathbf{u}, \Delta) = \sum_{i=0}^{\infty} \mathbf{z}_i{}^T Q\mathbf{z}_i + \mathbf{u}_i{}^T R\mathbf{u}_i, \tag{7}$$

where $\mathbf{u}$ denotes the control sequence $\mathbf{u}_0, \mathbf{u}_1, \ldots$, $Q \in \mathbb{R}^{(n+m)\times(n+m)}$ is symmetric positive semi-definite, and $R \in \mathbb{R}^{m\times m}$ is symmetric positive definite. We assume $(F_{nom}, Q^{1/2})$ is observable. In general, the cost $J(\mathbf{z}_0, \mathbf{u}, \Delta)$, which depends on the network delay sequence Δ as well as the initial state $\mathbf{z}_0$ and input sequence $\mathbf{u}$, can be infinite.

Separating the initial state $\mathbf{z}_0$ from the summation, we rewrite (7) as

$$J(\mathbf{z}_0, \mathbf{u}, \Delta) = \sum_{i=0}^{\infty} \begin{bmatrix} \mathbf{z}_i \\ \mathbf{u}_i \end{bmatrix}^T \Gamma(\delta_i) \begin{bmatrix} \mathbf{z}_i \\ \mathbf{u}_i \end{bmatrix} + \mathbf{z}_0{}^T Q\mathbf{z}_0, \tag{8}$$

where

$$\Gamma(\delta_i) = \begin{bmatrix} Q^d(\delta_i) & S^d(\delta_i) \\ S^d(\delta_i)^T & R^d(\delta_i) \end{bmatrix}$$

with the matrix entries given by

$$Q^d(\delta_i) = F(\delta_i)^T QF(\delta_i), \qquad S^d(\delta_i) = F(\delta_i)^T QG(\delta_i)$$

and

$$R^d(\delta_i) = G(\delta_i)^T QG(\delta_i) + R.$$

The matrix $\Gamma(\delta)$ is positive semi-definite and for a given network delay sequence Δ, $J(\mathbf{z}_0, \mathbf{u}, \Delta)$ is convex in $\mathbf{z}_0$ and the control sequence $\mathbf{u}$.

IV. BOUNDING PERFORMANCE DEGRADATION

We consider a nominal state-feedback controller given by

$$\mathbf{u}_i = \begin{bmatrix} K_1 \\ 0 \end{bmatrix}^T \begin{bmatrix} \mathbf{x}_i \\ \mathbf{u}_{i-1} \end{bmatrix} = K\mathbf{z}_i \quad (9)$$

where $K \in \mathbb{R}^{m\times(n+m)}$ is the (constant) feedback gain designed to minimize the nominal cost $J_{nom}(\mathbf{z}_0) = J(\mathbf{z}_0, \mathbf{u}, \mathbf{0})$, where $\Delta = \mathbf{0}$ denotes the sequence of zero network loop delays.

With state feedback, (6) now becomes

$$\mathbf{z}_{i+1} = (F(\delta_i) + G(\delta_i)K)\mathbf{z}_i \quad (10)$$

We assume the nominal system is stable, that is the eigenvalues of the system matrix $(F_{nom} + G_{nom}K)$ all have magnitudes less than one.

The cost (8), now only a function of the initial state and the network delay sequence, becomes

$$J(\mathbf{z}_0, \Delta) = \sum_{i=0}^{\infty} \mathbf{z}_i{}^T \begin{bmatrix} I \\ K \end{bmatrix}^T \Gamma(\delta_i) \begin{bmatrix} I \\ K \end{bmatrix} \mathbf{z}_i + \mathbf{z}_0{}^T Q\mathbf{z}_0 \quad (11)$$

For a given network delay sequence Δ, the cost J($\mathbf{z}_0, \Delta$) is a convex quadratic function in the initial state $\mathbf{z}_0$.

The original cost function (7) with linear state feedback incorporated becomes

$$J(\mathbf{z}_0, \Delta) = \sum_{i=0}^{\infty} \mathbf{z}_i{}^T (Q + K^T RK)\mathbf{z}_i \quad (12)$$

For the nominal system, the cost (12) is $J_{nom}(\mathbf{z}_0) = \mathbf{z}_0{}^T P_{nom}\mathbf{z}_0$, where P_{nom} is the (unique) solution of the Ricatti equation

$$(F_{nom} + G_{nom}K)^T P(F_{nom} + G_{nom}K) - P + Q + K^T RK = 0$$

Defining the *worst-case cost* with respect to network loop delays as

$$J_{wc}(\mathbf{z}_0) = sup_\Delta J(\mathbf{z}_0, \Delta), \quad (13)$$

we define the relative performance degradation compared to the nominal system as the relative increase in the LQR cost due to network delay for a specific initial state $\mathbf{z}_0$, which is given by

$$\frac{J_{wc}(\mathbf{z}_0) - J_{nom}(\mathbf{z}_0)}{J_{nom}(\mathbf{z}_0)} \qquad \mathbf{z}_0 \neq 0.$$

The worst-case relative performance degradation over all initial states is then given by

$$\eta = sup_{\mathbf{z}_0 \neq 0} \frac{J_{wc}(\mathbf{z}_0) - J_{nom}(\mathbf{z}_0)}{J_{nom}(\mathbf{z}_0)} \quad (14)$$

This number η is always nonnegative, since $J_{wc}(\mathbf{z}_0) \geq J_{nom}(\mathbf{z}_0)$ for any $\mathbf{z}_0$.

To bound the performance degradation, we find a quadratic function of the form $V(\mathbf{z}) = \mathbf{z}^T P\mathbf{z}$ such that $V(\mathbf{z}) \geq J(\mathbf{z}, \Delta)$ for all $\mathbf{z}$ and all network delay sequences Δ that satisfy our delay bound (3). A sufficient condition for such a $V(\mathbf{z})$ is given by the following Lemma.

Lemma 1: Let $V(\mathbf{z}) : \mathbb{R}^{n+m} \to \mathbb{R}$ be a quadratic function defined as $V(\mathbf{z}) = \mathbf{z}^T P\mathbf{z}$. If $P \succeq 0$ and

$$\forall\, 0 \leq \delta \leq \bar{\delta} \qquad \begin{bmatrix} I \\ K \end{bmatrix}^T \begin{bmatrix} \Psi(P, \delta) + \Gamma(\delta) \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} \preceq 0, \quad (15)$$

where

$$\Psi(P, \delta) = \begin{bmatrix} F(\delta)^T PF(\delta) - P & F(\delta)^T PG(\delta) \\ G(\delta)^T PF(\delta) & G(\delta)^T PG(\delta) \end{bmatrix}, \quad (16)$$

then

$$J(\mathbf{z}_0, \Delta) \leq V(\mathbf{z}_0) + \mathbf{z}_0^T Q\mathbf{z}_0$$

for all $\mathbf{z}_0$ and all Δ satisfying (3).

Proof: $V(\mathbf{z}_k) - V(\mathbf{z}_0) = \sum_{i=0}^{k-1} V(\mathbf{z}_{i+1}) - V(\mathbf{z}_i)$
$= \sum_{i=0}^{k-1} \mathbf{z}_{i+1}^T P\mathbf{z}_{i+1} - \mathbf{z}_i^T P\mathbf{z}_i$
$= \sum_{i=0}^{k-1} \mathbf{z}_i^T (F(\delta_i) + G(\delta_i)K)^T P(F(\delta_i) + G(\delta_i)K)\mathbf{z}_i - \mathbf{z}_i^T P\mathbf{z}_i$
$= \sum_{i=0}^{k-1} \begin{bmatrix} \mathbf{z}_i \\ K\mathbf{z}_i \end{bmatrix}^T \begin{bmatrix} F(\delta_i)^T PF(\delta_i) - P & F(\delta_i)^T PG(\delta_i) \\ G(\delta_i)^T PF(\delta_i) & G(\delta_i)^T PG(\delta_i) \end{bmatrix} \begin{bmatrix} \mathbf{z}_i \\ K\mathbf{z}_i \end{bmatrix}$
$= \sum_{i=0}^{k-1} \mathbf{z}_i^T \begin{bmatrix} I \\ K \end{bmatrix}^T \begin{bmatrix} \Psi(P, \delta_i) \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} \mathbf{z}_i$

Using (15),

$$V(\mathbf{z}_k) - V(\mathbf{z}_0) \leq -\sum_{i=0}^{k-1} \mathbf{z}_i^T \begin{bmatrix} I \\ K \end{bmatrix}^T \Gamma(\delta_i) \begin{bmatrix} I \\ K \end{bmatrix} \mathbf{z}_i$$

By changing the inequality sign we obtain

$$\sum_{i=0}^{k-1} \mathbf{z}_i^T \begin{bmatrix} I \\ K \end{bmatrix}^T \Gamma(\delta_i) \begin{bmatrix} I \\ K \end{bmatrix} \mathbf{z}_i \leq V(\mathbf{z}_0) - V(\mathbf{z}_k) \leq V(\mathbf{z}_0).$$

Adding $\mathbf{z}_0^T Q\mathbf{z}_0$ to both sides of the inequality gives

$$\mathbf{z}_0^T Q\mathbf{z}_0 + \sum_{i=0}^{k-1} \mathbf{z}_i^T \begin{bmatrix} I \\ K \end{bmatrix}^T \Gamma(\delta_i) \begin{bmatrix} I \\ K \end{bmatrix} \mathbf{z}_i \leq V(\mathbf{z}_0) + \mathbf{z}_0^T Q\mathbf{z}_0$$

Letting the index of summation $k \to \infty$ we obtain

$$J(\mathbf{z}_0, \Delta) \leq V(\mathbf{z}_0) + \mathbf{z}_0^T Q\mathbf{z}_0.$$

An upper bound on $J(\mathbf{z}_0, \Delta)$ is thus given by $V(\mathbf{z}_0) + \mathbf{z}_0^T Q\mathbf{z}_0$ for all $\mathbf{z}_0 \in \mathbb{R}^{(n+m)}$ and for all network delay sequences Δ. ■

It follows from Lemma 1 that $J_{wc}(\mathbf{z}_0) \leq V(\mathbf{z}_0) + \mathbf{z}_0^T Q\mathbf{z}_0$ for all $\mathbf{z}_0$. Thus,

$$\frac{J_{wc}(\mathbf{z}_0) - J_{nom}(\mathbf{z}_0)}{J_{nom}(\mathbf{z}_0)} \leq \frac{V(\mathbf{z}_0) + \mathbf{z}_0^T Q\mathbf{z}_0 - J_{nom}(\mathbf{z}_0)}{J_{nom}(\mathbf{z}_0)}, \quad (17)$$

which can be expanded as

$$\frac{J_{wc}(\mathbf{z}_0) - J_{nom}(\mathbf{z}_0)}{J_{nom}(\mathbf{z}_0)} \leq \frac{\mathbf{z}_0^T P\mathbf{z_0} + \mathbf{z}_0^T Q\mathbf{z}_0}{J_{nom}(\mathbf{z}_0)} - \frac{J_{nom}(\mathbf{z}_0)}{J_{nom}(\mathbf{z}_0)} \quad (18)$$

When the control input $\mathbf{u}_{-1} = \mathbf{0}$, (18) becomes

$$\frac{J_{wc}(\mathbf{z}_0) - J_{nom}(\mathbf{z}_0)}{J_{nom}(\mathbf{z}_0)} \leq \frac{\mathbf{x}_0^T(\bar{P} + \bar{Q})\mathbf{x}_0}{\mathbf{x}_0^T P_{nom}\mathbf{x}_0} - 1 \quad (19)$$

where $\bar{P} \in \mathbb{R}^{n\times n}$, $\bar{Q} \in \mathbb{R}^{n\times n}$ and $\bar{P}_{nom} \in \mathbb{R}^{n\times n}$ are the leading n-by-n submatrices of P, Q and P_{nom}, respectively. By (14), the supremum of the left-hand side is η. The supremum of the right-hand side is given by $\lambda_{max}(L(\bar{P} + \bar{Q})L)$, where

$L = (\bar{P}_{nom})^{-1/2}$ and $\lambda_{max}(M)$ is the largest eigenvalue of a matrix M. Hence, it follows that

$$\eta \leq \lambda_{max}(L(\bar{P}+\bar{Q})L) - 1 \tag{20}$$

Applying Lemma 1, we can choose P to obtain the smallest possible upper bound on η by solving the following convex optimization problem

$$min_P \quad \lambda_{max}(L(\bar{P}+\bar{Q})L) \tag{21}$$

subject to $P \succeq 0$ and

$$\forall\, 0 \leq \delta \leq \bar{\delta} \qquad \begin{bmatrix} I \\ K \end{bmatrix}^T \begin{bmatrix} \Psi(P,\delta)+\Gamma(\delta) \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} \preceq 0.$$

The optimization problem (21) is a semi-infinite LMI in the matrix variable P. Following the approach taken in [15], we uniformly discretize the interval $[0,\bar{\delta}]$ to obtain the delay values $\delta_i = \bar{\delta} + \frac{(i-N)\bar{\delta}}{N}$, where N is the number of discretized samples and $i = 0,1\ldots N$. Hence, the semi-infinite LMI is replaced by $N+1$ discretized LMIs.

V. EXPERIMENTAL RESULTS

In our first example, we consider a linear, time-invariant system with $A \in \mathbb{R}^{6\times 6}$ and $B \in \mathbb{R}^{6\times 2}$. The entries in A and B are chosen independently from a standard gaussian distribution. Since our discrete-time state vector $\mathbf{z}_i$ contains the current system state $\mathbf{x}_i$ and the previous input $\mathbf{u}_{i-1}$, we choose the LQR cost matrices Q and R to count the input only once. Accordingly, Q is chosen to be a diagonal matrix with unit entries for the states and 0.5 for elements corresponding to the previous input. R is chosen to be $0.5I_m$ where I_m is the m^{th} order identity matrix. The sampling period T of the controller is 100 msecs. The state-feedback controller gain K minimizes the nominal discrete-time LQR cost $J_{nom}(\mathbf{z}_0)$ for all initial states $\mathbf{z}_0$. The system matrices used are given as:

$$A = \begin{bmatrix} -0.432 & 1.189 & -0.588 & -0.095 & -0.691 & -0.399 \\ -1.665 & -0.037 & 2.183 & -0.832 & 0.858 & 0.690 \\ 0.125 & 0.327 & -0.136 & 0.294 & 1.254 & 0.815 \\ 0.287 & 0.174 & 0.113 & -1.336 & -1.593 & 0.711 \\ -1.146 & -0.186 & 1.066 & 0.714 & -1.441 & 1.290 \\ 1.190 & 0.725 & 0.059 & 1.623 & 0.571 & 0.668 \end{bmatrix}$$

$$B = \begin{bmatrix} 1.190 & -1.056 \\ -1.202 & 1.415 \\ -0.019 & -0.805 \\ -0.156 & 0.528 \\ -1.604 & 0.219 \\ 0.257 & -0.921 \end{bmatrix}$$

The eigenvalues of A are

$$\begin{matrix} 2.1493 \\ 0.2111+1.9014i \\ 0.2111-1.9014i \\ -2.1659+0.5560i \\ -2.1659-0.5560i \\ -0.9548 \end{matrix}$$

Since some eigenvalues are in the right-half plane, we are dealing with an open-loop unstable system. For each value of $\bar{\delta} \in [0, 0.5T]$ we calculate an upper bound on η (the worst-case relative performance degradation) by the methods described in Section IV. We also obtain a lower bound on η by choosing an initial state $\mathbf{z}_0$ and a 'worst-case' network delay sequence that greedily maximizes $V(\mathbf{z}_i)$ at each sample time iT. By doing this, we heuristically try to make $J(\mathbf{z}_0)$ approach $J_{wc}(\mathbf{z}_0)$ for the given initial state. The initial state $\mathbf{z}_0$ is chosen as the eigenvector corresponding to the maximum eigenvalue of P_{nom}, with the entries corresponding to $\mathbf{u}_{-1}$ made identically zero. The stopping criterion is reached when the nominal (no delay) cost due to the current state $\mathbf{z}$ is less than $\varepsilon = 10^{-10}$: when $\mathbf{z}^T P_{nom}\mathbf{z} < \varepsilon$. Figure 2 shows the curves for the upper (solid red) and lower (dashed blue) η bounds vs. the normalized delay bound $\bar{\delta}$. We also calculate the upper bound on η for the same initial state (shown in solid blue) to compare it with the upper bound over all initial states.

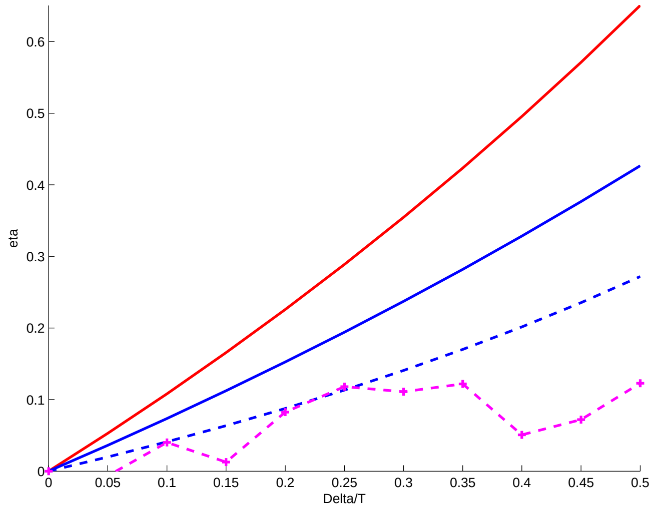

Fig. 2. *Top two curves.* Upper bound on η over all initial states (solid red) and for a specific initial state (solid blue). *Bottom two curves.* Heuristic lower bound on η (dashed blue) and Monte Carlo simulation with identical initial state (magenta). Sampling period T = 0.1 secs.

As an experimental evaluation, we carry out a Monte Carlo simulation of this system for each value of $\bar{\delta}$ in the same range. The initial state is the same as that used for the lower bound calculation in Fig. 2. For each sampling period, a value for the network delay is obtained from a uniform distribution. The next state is calculated from the system dynamics (10) and the incremental LQR cost is computed. These steps are carried out until the state vector approaches zero closely and the performance degradation for the current run is saved. Five hundred such system simulations are run for each value of $\bar{\delta}$ and the maximum value over the computed costs is taken as the relative worst-case performance η. The curve corresponding to the Monte Carlo simulation is the lowermost (magenta) in Fig. 2. The plots in Fig. 2 show that the upper bound on the worst-case relative performance degradation η is non-trivial for the chosen network delay range. For the worst-case delay bound of 0.5T, the upper bound on the degradation over the nominal controller with no delay is more than 60%. The LQR cost increases with the worst-case delay bound, which is to be expected since increasing delay adversely affects the phase margin of the nominal system. The analytical upper bound could be overly conservative, as observed from the difference between it and the heuristic lower bound on η. The upper bound for a specific initial state lies between the upper bound over all initial states and the lower bound for a specific initial state, as would be expected.

In our second example, we use the same system from the first experiment but decrease the sampling period of the controller by choosing T = 10 msecs. Intuitively, for a smaller sampling period, the effects of delay on control performance should be reduced and correspondingly, the upper and lower bounds on η should decrease as well. Figure 3 shows the corresponding plots for the upper and lower bounds, and the Monte Carlo simulation for T = 0.01 secs. As expected, each of the bounds on η has decreased compared to the previous experiment. The upper bound on the performance degradation over the nominal controller is now about 12.5% .

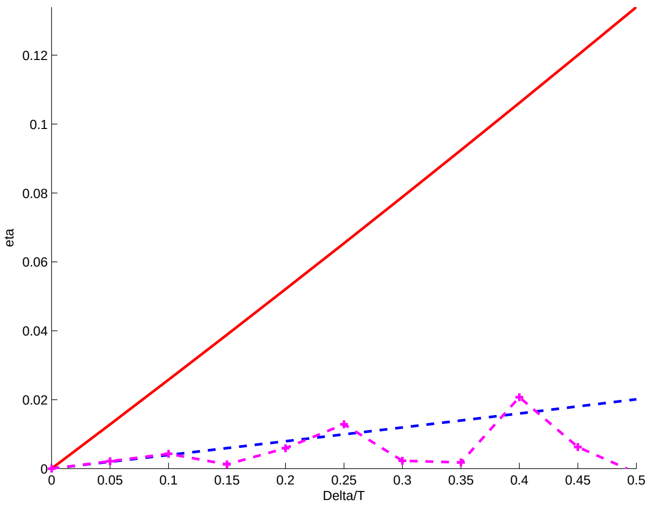

Fig. 3. *Top curve*. Upper bound on η over all initial states (solid red). *Bottom two curves*. Heuristic lower bound on η (dashed blue) and Monte Carlo simulation with identical initial state (magenta). Sampling period T = 0.01 secs.

VI. DISCUSSION AND FUTURE WORK

This paper presents a method for computing upper bounds on the worst-case degradation in the quadratic cost that will occur due to network communication delays when an LQR state-feedback controller is implemented over a network. The approach taken in this paper is similar to the approach taken in [15] to bound performance degradation caused by sampling jitter. The principal differences between the development in this paper and the results in [15] are: (i) the values of the control and state are characterized at the nominal sampling times, whereas in [15] this is not possible because the model of sampling jitter includes drift, which can lead to unbounded deviations from the nominal sampling times; and (ii) a discrete-time cost function rather than a continuous-time cost function is used for the nominal design.

There are several directions for further research. We are currently investigating methods to obtain tighter upper bounds on the worst case performance degradation. The basic approach also needs to be extended to a richer class of plant models and control laws. As a first step in this direction, we are looking at extensions of these results to output feedback with state estimation. The approach should be extended to include the performance of tracking and switching (hybrid) controllers, and systems with disturbances. The uses of performance degradation bounds for network design and implementation, as described in the introduction, are also being investigated. Finally, we are interested in extending these results to multi-node networks.

VII. ACKNOWLEDGMENTS

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