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IDENTITIES CONCERNING BERNOULLI AND EULER POLYNOMIALS

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ABSTRACT. We establish two general identities for Bernoulli and Euler polynomials, these identities of a new type have many consequences. The most striking result in this paper is as follows: If n is a positive integer, $r+s+t=n$ and $x+y+z=1$, then we have

$$r\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n + s\begin{bmatrix} t & r \\ y & z \end{bmatrix}_n + t\begin{bmatrix} r & s \\ z & x \end{bmatrix}_n = 0$$

where

$$\begin{bmatrix} s & t \\ x & y \end{bmatrix}_n := \sum_{k=0}^{n}(-1)^k\binom{s}{k}\binom{t}{n-k}B_{n-k}(x)B_k(y).$$

This symmetric identity implies the curious ones of Miki and Matiyasevich as well as some new identities for Bernoulli polynomials such as

$$\sum_{k=0}^{n}\binom{n}{k}^2 B_k(x)B_{n-k}(x) = 2\sum_{\substack{k=0\\k\neq n-1}}^{n}\binom{n}{k}\binom{n+k-1}{k}B_k(x)B_{n-k}.$$

1. INTRODUCTION

Bernoulli numbers B_n ($n\in\mathbb{N}=\{0,1,2,\ldots\}$) are rational numbers defined by

$$B_0=1 \quad\text{and}\quad \sum_{k=0}^{n}\binom{n+1}{k}B_k=0 \quad (n\in\mathbb{Z}^+=\{1,2,3,\ldots\}).$$

Similarly, Euler numbers E_n ($n\in\mathbb{N}$) are integers given by

$$E_0=1 \quad\text{and}\quad \sum_{\substack{k=0\\2\mid n-k}}^{n}\binom{n}{k}E_k=0 \quad (n\in\mathbb{Z}^+).$$

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For $n \in \mathbb{N}$ the Bernoulli polynomial $B_n(x)$ and the Euler polynomial $E_n(x)$ are as follows:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad \text{and} \quad E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}.$$

Clearly $B_n(0) = B_n$ and $E_n(1/2) = E_n/2^n$. Here are some basic properties of Bernoulli and Euler polynomials we will need later.

$$B_n(1-x) = (-1)^n B_n(x),\ \Delta(B_n(x)) = nx^{n-1};$$
$$E_n(1-x) = (-1)^n E_n(x),\ \Delta^*(E_n(x)) = 2x^n.$$

(The operators Δ and Δ^* are defined by $\Delta(f(x)) = f(x+1) - f(x)$ and $\Delta^*(f(x)) = f(x+1) + f(x)$.) Also, $B'_{n+1}(x) = (n+1)B_n(x)$ and $E'_{n+1}(x) = (n+1)E_n(x)$.

For a sequence $\{a_n\}_{n\in\mathbb{N}}$ of complex numbers, its dual sequence $\{a_n\}_{n\in\mathbb{N}}$ are given by $a_n^* = \sum_{k=0}^{n} \binom{n}{k}(-1)^k a_k$ $(n \in \mathbb{N})$. It is well known that $a_n^{**} = a_n$. In 2003 Z. W. Sun [S2] deduced some combinatorial identities in dual sequences. The sequences $\{(-1)^n B_n\}_{n\in\mathbb{N}}$ and $\{(-1)^n E_n(0)\}_{n\in\mathbb{N}}$ are both self-dual sequences (see [S2]), later we will make use of this fact.

In 1978 H. Miki [Mi] discovered the following curious identity:

$$\sum_{k=2}^{n-2} \frac{B_k B_{n-k}}{k(n-k)} - \sum_{k=2}^{n-2} \binom{n}{k} \frac{B_k B_{n-k}}{k(n-k)} = 2H_n \frac{B_n}{n}$$

for every $n = 4, 5, \ldots$, where $H_n = 1 + 1/2 + \cdots + 1/n$. In 1997 Y. Matiyasevich [Ma] found another identity of this type:

$$(n+2) \sum_{k=2}^{n-2} B_k B_{n-k} - 2 \sum_{k=2}^{n-2} \binom{n+2}{k} B_k B_{n-k} = n(n+1)B_n$$

for any $n = 4, 5, \ldots$. These two identities are of a deep nature. In fact, all known proofs of these identities by others are complicated (cf. [Mi], [Ge] and [DS]); for example, the approach of G. V. Dunne and C. Schubert [DS] was even motivated by quantum field theory and string theory.

Recently the authors ([PS] and [SP]) presented a new method to handle such identities. Though their approach only involves differences and derivatives of polynomials, they were able to use the powerful method to extend Miki's and Matiyasevich's identities to identities concerning the sums $\sum_{k=0}^{n} B_k(x)B_{n-k}(y)$ and

$$\sum_{k=1}^{n-1} \frac{B_k(x)}{k} \cdot \frac{B_{n-k}(y)}{n-k} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{B_k(x)}{k} B_{n-k}(y) + \frac{1}{n} \sum_{l=1}^{n-1} \frac{B_l(y)}{l} B_{n-l}(x)$$

(where n is a positive integer). They also handled similar sums related to Euler polynomials.

Let n be any positive integer. Observe that

$$\sum_{k=0}^{n} B_k(x)B_{n-k}(y) = \sum_{k=0}^{n}(-1)^k\binom{-1}{k}B_k(x)B_{n-k}(y)$$

and

$$\begin{aligned}-\sum_{k=1}^{n}\frac{B_k(x)}{k}B_{n-k}(y) &= \sum_{k=1}^{n}(-1)^k\binom{-1}{k-1}\frac{B_k(x)}{k}B_{n-k}(y)\\ &= \lim_{t\to 0}\frac{1}{t}\sum_{k=1}^{n}(-1)^k\binom{t}{k}B_k(x)B_{n-k}(y).\end{aligned}$$

Inspired by this observation, here we investigate relations among the sums

$$\sum_{k=0}^{n}(-1)^k\binom{s}{k}\binom{t}{n-k}P_k(x)Q_{n-k}(y)$$

with $P, Q \in \{B, E\}$. Our main result is as follows.

Theorem 1.1. *Let $n \in \mathbb{Z}^+$ and $x + y + z = 1$.*

(i) *If $r + s + t = n - 1$, then*

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}B_k(x)E_{n-k}(z)\\ &-(-1)^n\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{t}{n-k}B_k(y)E_{n-k}(z)\\ =&\frac{r}{2}\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t}{n-1-l}E_l(y)E_{n-1-l}(x).\end{aligned}\tag{1.1}$$

(ii) *If $r + s + t = n$, then we have the symmetric identity*

$$r\begin{bmatrix} s & t\\ x & y\end{bmatrix}_n + s\begin{bmatrix} t & r\\ y & z\end{bmatrix}_n + t\begin{bmatrix} r & s\\ z & x\end{bmatrix}_n = 0 \tag{1.2}$$

where

$$\begin{bmatrix} s & t\\ x & y\end{bmatrix}_n := \sum_{k=0}^{n}(-1)^k\binom{s}{k}\binom{t}{n-k}B_{n-k}(x)B_k(y). \tag{1.3}$$

Corollary 1.1. *Let n be any positive integer. Then*

$$\begin{aligned}&\frac{n+1}{2}\sum_{k=0}^{n-1}E_k(x)E_{n-1-k}(y)\\ =&\sum_{k=0}^{n}\binom{n+1}{k}\left((-1)^{n-k}B_k(x)-B_k(y)\right)E_{n-k}(x-y)\end{aligned}\tag{1.4}$$

and

$$\begin{aligned}&(n+2)\sum_{k=0}^{n}B_k(x)B_{n-k}(y)\\ =&\sum_{k=0}^{n}\binom{n+2}{k}\left((-1)^{n-k}B_k(x)+B_k(y)\right)B_{n-k}(x-y).\end{aligned}\tag{1.5}$$

Proof. Clearly $(1-x)+y+(x-y)=1$. Applying Theorem 1.1(i) with $s=t=-1$ we then get (1.4). By Theorem 1.1(ii),

$$(n+2)\begin{bmatrix}-1 & -1\\ 1-x & y\end{bmatrix}_n=\begin{bmatrix}-1 & n+2\\ y & x-y\end{bmatrix}_n+\begin{bmatrix}n+2 & -1\\ x-y & 1-x\end{bmatrix}_n.$$

This is an equivalent version of (1.5). □

Remark 1.1. (1.5) in the case $x=y=0$ yields Matiyasevich's identity since $B_{2l+1}=0$ for $l=1,2,3,\ldots$.

Corollary 1.2. *Let $n\in\mathbb{Z}^+$. Then*

$$\begin{aligned}&\frac{n+1}{2}\sum_{k=0}^{n-1}\binom{n}{k}\binom{n}{k+1}E_k(x)E_{n-1-k}(y)\\ =&\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{n}((-1)^{n-k}B_k(x)-B_k(y))E_{n-k}(x-y);\end{aligned}\tag{1.6}$$

in particular,

$$\begin{aligned}&\sum_{k=0}^{n-1}\binom{n}{k}\binom{n}{k+1}E_k(x)E_{n-1-k}(x)\\ =&\frac{8}{(n+1)^2}\sum_{k=0}^{n-1}\binom{n+1}{k}\binom{n+k}{n}B_k(x)\left(2^{n-k+1}-1\right)B_{n-k+1}.\end{aligned}\tag{1.7}$$

We also have

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}^2B_k(x)B_{n-k}(y)\\ =&\sum_{k=0}^{n}\binom{n}{k}\binom{n+k-1}{k}((-1)^{n-k}B_k(x)+B_k(y))B_{n-k}(x-y);\end{aligned}\tag{1.8}$$

in particular,

$$\sum_{k=0}^{n}\binom{n}{k}^2 B_k(x)B_{n-k}(x)=2\sum_{\substack{k=0\\k\neq n-1}}^{n}\binom{n}{k}\binom{n+k-1}{k}B_k(x)B_{n-k}. \quad (1.9)$$

Proof. As $(-n-1)+n+n=n-1$ and $(1-x)+y+(x-y)=1$, by Theorem 1.1(i) we have

$$\sum_{k=0}^{n}(-1)^k\binom{-n-1}{k}\binom{n}{k}\left(B_k(1-x)-(-1)^nB_k(y)\right)E_{n-k}(x-y)$$
$$=\frac{-n-1}{2}\sum_{l=0}^{n-1}(-1)^l\binom{n}{l}\binom{n}{n-1-l}E_l(y)E_{n-1-l}(1-x)$$

which can be reduced to (1.6). (1.7) follows from (1.6) in the case $y=x$ since $((-1)^l-1)E_l(0)=4(2^{l+1}-1)B_{l+1}/(l+1)$ for $l=1,2,3,\ldots$. (It is known that $(l+1)E_l(x)=2(B_{l+1}(x)-2^{l+1}B_{l+1}(x/2))$ (cf. [S1]).)

In light of Theorem 1.1(ii),

$$\begin{bmatrix} n & n\\ y & 1-x\end{bmatrix}_n=\begin{bmatrix} n & -n\\ 1-x & x-y\end{bmatrix}_n+\begin{bmatrix} -n & n\\ x-y & y\end{bmatrix}_n.$$

This is equivalent to (1.8). In the case $y=x$, (1.8) gives (1.9) because $((-1)^l+1)B_l=2B_l$ for $l=0,2,3,\ldots$. □

Remark 1.2. Putting $x=1/2$ in (1.7) and noting that $B_k(1/2)=(2^{1-k}-1)B_k$ (see, e.g., [S1]), we then get the following identity:

$$\frac{(n+1)^2}{8}\sum_{k=0}^{n-1}\binom{n}{k}\binom{n}{k+1}E_kE_{n-1-k}$$
$$=-\sum_{k=0}^{n-1}\binom{n+1}{k}\binom{n+k}{n}2^{n-k}(2^{k-1}-1)(2^{n-k+1}-1)B_kB_{n-k+1}$$

for any $n\in\mathbb{Z}^+$. Similarly, (1.9) in the case $x=0$ yields the following new identity:

$$\sum_{k=2}^{n-2}\binom{n}{k}^2B_kB_{n-k}-2\sum_{k=2}^{n-2}\binom{n}{k}\binom{n+k-1}{k}B_kB_{n-k}=2\binom{2n-1}{n-1}B_n$$

for every $n=4,5,\ldots$.

From Theorem 1.1 we can also deduce the following result.

Theorem 1.2. *Let $n \in \mathbb{Z}^+$, and let t, x, y, z be parameters with $x+y+z = 1$. Then we have*

$$\begin{aligned}&\frac{(-1)^n}{2}\sum_{k=0}^{n-1}\binom{t}{k}E_k(x)E_{n-1-k}(y)\\=&\frac{1}{n-t}\sum_{k=0}^{n}\binom{n-t}{k}B_k(x)E_{n-k}(z)+\binom{t}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{E_k(z)}{t-k}B_{n-k}(y)\end{aligned}\tag{1.10}$$

and

$$\begin{aligned}&\frac{n}{2}\binom{t}{n}\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{E_k(x)}{t-k}E_{n-1-k}(y)-(-1)^nE_n(z)\binom{t}{n}\sum_{k=0}^{n-1}\frac{1}{t-k}\\=&(-1)^n\sum_{k=1}^{n}\binom{t}{n-k}\frac{B_k(y)}{k}E_{n-k}(z)-\sum_{k=1}^{n}\binom{n-1-t}{n-k}\frac{B_k(x)}{k}E_{n-k}(z).\end{aligned}\tag{1.11}$$

Also,

$$\begin{aligned}&\frac{(-1)^{n-1}}{n}\binom{t-1}{n-1}\sum_{k=0}^{n}\binom{n}{k}\frac{B_k(x)}{t-k}B_{n-k}(y)-\frac{B_n(z)}{n}\binom{t-1}{n-1}\sum_{k=1}^{n-1}\frac{1}{t-k}\\&=\frac{1}{t}\sum_{k=1}^{n}\binom{t}{n-k}\frac{B_k(y)}{k}B_{n-k}(z)+\frac{(-1)^n}{n-t}\sum_{k=1}^{n}\binom{n-t}{n-k}\frac{B_k(x)}{k}B_{n-k}(z).\end{aligned}\tag{1.12}$$

Corollary 1.3. *Let $n \in \mathbb{Z}^+$ and $x + y + z = 1$. Then*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n+1}{k}\left((-1)^nB_k(x)-B_k(y)\right)E_{n-k}(z)\\&=\frac{n+1}{2}\sum_{l=0}^{n-1}(-1)^lE_l(x)E_{n-1-l}(y),\end{aligned}\tag{1.13}$$

$$\begin{aligned}&\sum_{k=1}^{n}\binom{n}{k}\frac{B_k(x)}{k}E_{n-k}(z)-\sum_{k=1}^{n}(-1)^k\frac{B_k(y)}{k}E_{n-k}(z)\\&=\frac{(-1)^n}{2}\sum_{l=0}^{n-1}\binom{n}{l}E_l(y)E_{n-1-l}(x)-H_nE_n(z)\end{aligned}\tag{1.14}$$

and

$$\begin{aligned}&(-1)^n\sum_{k=0}^{n}\binom{n+1}{k}B_{n-k}(x)B_k(y)+\sum_{k=0}^{n-1}\binom{n+1}{k}\frac{B_{n-k}(x)}{n-k}B_k(z)\\&=(n+1)\sum_{k=1}^{n}(-1)^k\frac{B_k(y)}{k}B_{n-k}(z)+(1-H_n)(n+1)B_n(z).\end{aligned}\tag{1.15}$$

We also have

$$\begin{aligned}&\frac{(-1)^n}{2}\left\lfloor\frac{n+1}{2}\right\rfloor\sum_{k=0}^{\lfloor n/2\rfloor}\binom{\lfloor n/2\rfloor}{k}E_k(x)E_{n-1-k}(y)\\ =&\sum_{k=0}^{\lfloor (n+1)/2\rfloor}\binom{\lfloor (n+1)/2\rfloor}{k}B_k(x)E_{n-k}(z)\\ &+(-1)^{\lfloor (n-1)/2\rfloor}B_{\lfloor (n+1)/2\rfloor}(y)E_{\lfloor n/2\rfloor}(z),\end{aligned}\tag{1.16}$$

where $\lfloor\alpha\rfloor$ *denotes the integral part of a real number* α.

Proof. Taking $t=-1$ in Theorem 1.2 we immediately get (1.13)–(1.15). (1.16) follows from (1.10) by letting $t\to\lfloor n/2\rfloor$ and noting that

$$\lim_{t\to\lfloor n/2\rfloor}\frac{\binom{t}{n}}{t-\lfloor n/2\rfloor}=(-1)^{\lfloor (n-1)/2\rfloor}\frac{\lfloor (n-1)/2\rfloor!\lfloor n/2\rfloor!}{n!}.$$

We are done. □

Corollary 1.4. *Let* $n\in\mathbb{Z}^+$ *and* $x+y+z=1$. *Then*

$$\begin{aligned}&\frac{1}{2}\sum_{k=1}^{n-1}(-1)^k\frac{E_k(x)}{k}E_{n-1-k}(y)-\frac{1}{2}H_{n-1}E_{n-1}(y)\\ =&-\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}\frac{E_k(z)}{k}B_{n-k}(y)-\frac{(-1)^n}{n}\sum_{k=1}^{n}\binom{n}{k}H_kE_k(z)B_{n-k}(x)\end{aligned}\tag{1.17}$$

and

$$\begin{aligned}&\frac{(-1)^{n-1}}{2}\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{E_k(x)}{k}E_{n-1-k}(y)+H_{n-1}\frac{E_n(z)+(-1)^nB_n(y)}{n}\\ =&\sum_{k=1}^{n-1}(-1)^k\frac{B_k(y)}{k}\cdot\frac{E_{n-k}(z)}{n-k}+\sum_{k=1}^{n}\binom{n-1}{k-1}H_{k-1}\frac{B_k(x)}{k}E_{n-k}(z).\end{aligned}\tag{1.18}$$

We also have

$$\begin{aligned}&\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{B_k(x)}{k^2}\left(B_{n-k}(y)+(-1)^nB_{n-k}(z)\right)\\ =&\sum_{k=1}^{n-1}(-1)^{n-k}\frac{B_k(y)}{k}\cdot\frac{B_{n-k}(z)}{n-k}-H_{n-1}\frac{B_n(y)+(-1)^nB_n(z)}{n}\end{aligned}\tag{1.19}$$

Remark 1.3. In the case $x=y=0$ and $z=1$, (1.19) yields Miki's identity.

The next section is devoted to a proof of Theorem 1.1. Theorem 1.2 and Corollary 1.4 will be proved in Section 3.

2. Proof of Theorem 1.1

Lemma 2.1 ([PS, Lemma 2.1]). *Let $P(x), Q(x) \in \mathbb{C}[x]$ where $\mathbb{C}$ is the field of complex numbers.*

(i) *We have*

$$\Delta(P(x)Q(x)) = P(x+1)\Delta(Q(x)) + \Delta(P(x))Q(x) \tag{2.1}$$

and

$$\Delta^*(P(x)Q(x)) = P(x+1)\Delta^*(Q(x)) - \Delta(P(x))Q(x). \tag{2.2}$$

(ii) *If $\Delta(P(x)) = \Delta(Q(x))$, then $P'(x) = Q'(x)$. If $\Delta^*(P(x)) = \Delta^*(Q(x))$, then $P(x) = Q(x)$.*

The following lemma has the same flavor with Theorem 1.1 of Sun [S2].

Lemma 2.2. *Let $\{a_l\}_{l=0}^\infty$ be a sequence of complex numbers, and $\{a_l^*\}_{l=0}^\infty$ be its dual sequence. Set*

$$A_k(t) = \sum_{l=0}^k \binom{k}{l}(-1)^l a_l t^{k-l} \text{ and } A_k^*(t) = \sum_{l=0}^k \binom{k}{l}(-1)^l a_l^* t^{k-l} \tag{2.3}$$

for $k = 0, 1, 2, \ldots$. Let $n \in \mathbb{Z}^+$, $r + s + t = n - 1$ and $x + y + z = 1$. Then

$$\sum_{k=0}^n (-1)^k \binom{r}{k} x^{n-k} \left(\binom{s}{n-k} A_k(y) - (-1)^n \binom{t}{n-k} A_k^*(z) \right) = 0. \tag{2.4}$$

Proof. By Remark 1.1 of Sun [S2],

$$(-1)^k A_k^*(z) = A_k(x+y) = \sum_{l=0}^k \binom{k}{l} x^{k-l} A_l(y).$$

Therefore

$$\begin{aligned}
&\sum_{k=0}^n (-1)^k \binom{r}{k}\binom{s}{n-k} x^{n-k} A_k * (z) \\
=&\sum_{k=0}^n \binom{r}{k}\binom{s}{n-k} x^{n-k} \sum_{l=0}^k \binom{k}{l} x^{k-l} A_l(y) \\
=&\sum_{l=0}^n x^{n-l} A_l(y) \sum_{k=l}^n \binom{r}{l}\binom{r-l}{k-l}\binom{s}{n-k} \\
=&\sum_{l=0}^n \binom{r}{l} x^{n-l} A_l(y) c_l
\end{aligned}$$

where

$$\begin{aligned}c_l =&\sum_{k=l}^{n}\binom{r-l}{k-l}\binom{s}{n-k} = \binom{r+s-l}{n-l} \text{ (by Vandermonde's identity)}\\ =&(-1)^{n-l}\binom{l-r-s+n-l-1}{n-l} = (-1)^{n-l}\binom{t}{n-l}.\end{aligned}$$

Thus (2.4) follows. □

Remark 2.1. If we let $a_l = (-1)^l B_l$ for $l = 0, 1, 2, \ldots$, then $A_k(t) = A_k^*(t) = B_k(t)$. Also, $A_k(t) = A_k^*(t) = E_k(t)$ if $a_l = (-1)^l E_l(0)$ for $l = 0, 1, 2, \ldots$.

`Proof of Theorem 1.1`. We fix y and view $z = 1 - x - y$ as a function in x.

(i) Set

$$P(x) = \sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}B_k(x)E_{n-k}(z).$$

Then, by Lemma 2.1, $\Delta^*(P(x))$ coincides with

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}\Delta^*(B_k(x)E_{n-k}(z))\\ =&\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}\left(B_k(x+1)2(z-1)^{n-k} - kx^{k-1}E_{n-k}(z)\right)\\ =&2\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}(z-1)^{n-k}B_k(x+1) + r\Sigma\end{aligned}$$

where

$$\begin{aligned}\Sigma =&\sum_{k=1}^{n}(-1)^{k-1}\binom{r-1}{k-1}\binom{s}{n-k}x^{k-1}E_{n-k}(z).\\ =&(-1)^{n-1}\sum_{l=0}^{n-1}(-1)^l\binom{r-1}{n-1-l}\binom{s}{l}x^{n-1-l}E_l(z).\end{aligned}$$

Applying Lemma 2.2 and Remark 2.1 we obtain that

$$\begin{aligned}\Delta^*(P(x)) =&2(-1)^n\sum_{k=0}^{n}(-1)^k\binom{r}{k}\binom{t}{n-k}(z-1)^{n-k}B_k(y)\\ &+ r\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t}{n-1-l}x^{n-1-l}E_l(y).\end{aligned}$$

It follows that $\Delta^*(P(x))=\Delta^*(Q(x))$ where

$$\begin{aligned}Q(x)=&(-1)^n\sum_{k=0}^n(-1)^k\binom{r}{k}\binom{t}{n-k}B_k(y)E_{n-k}(z)\\&+\frac{r}{2}\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t}{n-1-l}E_l(y)E_{n-1-l}(x).\end{aligned}$$

Thus $P(x)=Q(x)$ by Lemma 2.1. This is equivalent to the desired (1.1). □

(ii) Set

$$P_n(x)=\begin{bmatrix}r & s\\ z & x\end{bmatrix}_n=\sum_{k=0}^n(-1)^k\binom{r}{k}\binom{s}{n-k}B_k(x)B_{n-k}(z).$$

By Lemma 2.1,

$$\begin{aligned}\Delta(B_k(x)B_{n-k}(z))=&\Delta(B_k(x))B_{n-k}(z)+B_k(x+1)\Delta(B_{n-k}(z)))\\=&kx^{k-1}B_{n-k}(z)-(n-k)B_k(x+1)(z-1)^{n-k-1}\end{aligned}$$

for every $k=0,1,\ldots,n$. Thus

$$\Delta(P_n(x))=rR(x)-s\sum_{k=0}^{n-1}(-1)^k\binom{r}{k}\binom{s-1}{n-k-1}B_k(x+1)(z-1)^{n-k-1}$$

where

$$\begin{aligned}R(x)=&\sum_{k=1}^n(-1)^k\binom{r-1}{k-1}\binom{s}{n-k}x^{k-1}B_{n-k}(z)\\=&(-1)^n\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{r-1}{n-1-l}x^{n-1-l}B_l(z).\end{aligned}$$

Applying Lemma 2.2 and Remark 2.1 we obtain that

$$\begin{aligned}\Delta(P_n(x))=&-r\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t-1}{n-1-l}x^{n-1-l}B_l(y)\\&-s(-1)^{n-1}\sum_{l=0}^{n-1}(-1)^l\binom{r}{l}\binom{t-1}{n-1-l}(z-1)^{n-1-l}B_l(y)\end{aligned}$$

It follows that $\Delta(P_n(x)) = \Delta(Q_n(x))$ where

$$\begin{aligned}Q_n(x) =& -\frac{r}{t}\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t}{n-l}B_{n-l}(x)B_l(y)\\ &- (-1)^n\frac{s}{t}\sum_{l=0}^{n-1}(-1)^l\binom{r}{l}\binom{t}{n-l}B_{n-l}(z)B_l(y).\\ =& -\frac{r}{t}\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t}{n-l}B_{n-l}(x)B_l(y)\\ &- \frac{s}{t}\sum_{k=1}^{n}(-1)^k\binom{t}{k}\binom{r}{n-k}B_k(z)B_{n-k}(y).\end{aligned}$$

Thus $P_n'(x) = Q_n'(x)$ by Lemma 2.1.

Observe that $P_n'(x)$ coincides with

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^k\binom{r}{k}\binom{s}{n-k}kB_{k-1}(x)B_{n-k}(z)\\ &-\sum_{k=0}^{n-1}(-1)^k\binom{r}{k}\binom{s}{n-k}(n-k)B_k(x)B_{n-k-1}(z)\\ =&\sum_{k=0}^{n-1}(-1)^{k+1}\binom{r}{k+1}\binom{s}{n-1-k}(k+1)B_k(x)B_{n-1-k}(z)\\ &-\sum_{k=0}^{n-1}(-1)^k\binom{r}{k}\binom{s}{n-k}(n-k)B_k(x)B_{n-1-k}(z)\\ =&\sum_{k=0}^{n-1}(-1)^{k-1}\binom{r}{k}\binom{s}{n-1-k}(r-k+(s-n+k+1))B_k(x)B_{n-1-k}(z)\\ =&(t-1)\begin{bmatrix} r & s\\ z & x\end{bmatrix}_{n-1}\end{aligned}$$

and

$$\begin{aligned}Q_n'(x) =& -r\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t-1}{n-l-1}B_{n-l-1}(x)B_l(y)\\ &+ s\sum_{k=1}^{n}(-1)^k\binom{t-1}{k-1}\binom{r}{n-k}B_{k-1}(z)B_{n-k}(y)\\ =& -r\begin{bmatrix} s & t-1\\ x & y\end{bmatrix}_{n-1} - s\begin{bmatrix} t-1 & r\\ y & z\end{bmatrix}_{n-1}.\end{aligned}$$

Thus the equality $P_n'(x)=Q_n'(x)$ gives that

$$r\begin{bmatrix} s & t' \\ x & y\end{bmatrix}_{n-1}+s\begin{bmatrix} t' & r \\ y & z\end{bmatrix}_{n-1}+t'\begin{bmatrix} r & s \\ z & x\end{bmatrix}_{n-1}=0$$

where $t'=t-1=n-1-(r+s)$. Replacing $n-1$ by n we then obtain the required identity (1.2). This concludes the proof. $\square$

3. Proofs of Theorem 1.2 and Corollary 1.4

Lemma 3.1. *Let n be a nonnegative integer and s be a parameter. Then*

$$\lim_{t\to 0}\frac{1}{t}\left(\binom{s+t}{n}-\binom{s}{n}\right)=\binom{s}{n}\sum_{0\leqslant l<n}\frac{1}{s-l}. \tag{3.1}$$

In particular,

$$\lim_{t\to 0}\frac{1}{t}\left(\binom{t-1}{n}-(-1)^n\right)=(-1)^{n-1}H_n. \tag{3.2}$$

Proof. Observe that

$$\binom{s+t}{n}=\binom{s}{n}\prod_{0\leqslant l<n}\frac{s+t-l}{s-l}=\binom{s}{n}\prod_{0\leqslant l<n}\left(1+\frac{t}{s-l}\right).$$

So (3.1) follows. In the case $s=-1$, (3.1) turns out to be (3.2). $\square$

`Proof of Theorem 1.2`. (1.1) in the case $s=-1$ yields that

$$\begin{aligned}&(-1)^n\sum_{k=0}^{n}\binom{n-t}{k}B_k(x)E_{n-k}(z)\\ &-(-1)^n\sum_{k=0}^{n}(-1)^k\binom{n-t}{k}\binom{t}{n-k}B_k(y)E_{n-k}(z)\\ =&\frac{n-t}{2}\sum_{l=0}^{n-1}\binom{t}{n-1-l}E_{n-1-l}(x)E_l(y).\end{aligned}$$

For each $k=0,1,\ldots,n$ we clearly have

$$\begin{aligned}\binom{n-t}{k}\binom{t}{n-k}=&\binom{n}{k}\binom{t}{n}\frac{(n-t)(n-t-1)\cdots(n-t-k+1)}{(t-n+k)\cdots(t-n+1)}\\ =&(-1)^k\binom{n}{k}\binom{t}{n}\frac{t-n}{t-n+k}.\end{aligned}$$

Therefore

$$\frac{(-1)^n}{2}\sum_{k=0}^{n-1}\binom{t}{k}E_k(x)E_{n-1-k}(y)-\frac{1}{n-t}\sum_{k=0}^{n}\binom{n-t}{k}B_k(x)E_{n-k}(z)$$
$$=\binom{t}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{B_k(y)}{t+k-n}E_{n-k}(z)=\binom{t}{n}\sum_{l=0}^{n}\binom{n}{l}\frac{E_l(z)}{t-l}B_{n-l}(y).$$

This proves (1.10).

Now we come to prove (1.11) and view $s=n-1-r-t$ as a function in r. In light of (1.1),

$$\frac{1}{2}\sum_{l=0}^{n-1}(-1)^l\binom{s}{l}\binom{t}{n-1-l}E_l(y)E_{n-1-l}(x)$$
$$=\frac{1}{r}\sum_{k=0}^{n}(-1)^k\binom{r}{k}E_{n-k}(z)\left(\binom{s}{n-k}B_k(x)-(-1)^n\binom{t}{n-k}B_k(y)\right)$$
$$=\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{r-1}{k-1}E_{n-k}(z)\left(\binom{s}{n-k}B_k(x)-(-1)^n\binom{t}{n-k}B_k(y)\right)$$
$$+(-1)^nE_n(z)\frac{(-1)^n\binom{s}{n}-\binom{t}{n}}{r}.$$

By Lemma 3.1,

$$\lim_{r\to 0}\frac{1}{r}\left((-1)^n\binom{s}{n}-\binom{t}{n}\right)=\lim_{r\to 0}\frac{1}{r}\left(\binom{r+t}{n}-\binom{t}{n}\right)=\binom{t}{n}\sum_{l=0}^{n-1}\frac{1}{t-l}.$$

As in the proof of (1.10), we also have

$$(-1)^l\binom{n-1-t}{l}\binom{t}{n-1-l}=\binom{n-1}{l}\binom{t}{n-1}\frac{t-(n-1)}{t-(n-1)+l}$$
$$=\frac{n}{t+l-(n-1)}\binom{t}{n}\binom{n-1}{l}$$

for every $l=0,1,\ldots,n-1$. Thus, by letting $r\to 0$ we get from the above that

$$\frac{n}{2}\binom{t}{n}\sum_{l=0}^{n-1}\binom{n-1}{l}\frac{E_l(y)E_{n-1-l}(x)}{t+l-n+1}-(-1)^nE_n(z)\binom{t}{n}\sum_{l=0}^{n-1}\frac{1}{t-l}$$
$$=-\sum_{k=1}^{n}E_{n-k}(z)\left(\binom{n-1-t}{n-k}\frac{B_k(x)}{k}-(-1)^n\binom{t}{n-k}\frac{B_k(y)}{k}\right),$$

which is equivalent to (1.11).

Now we turn to prove (1.12). Let us view $s=n-r-t$ as a function in r. Then

$$\begin{aligned}\lim_{r\to0}\begin{bmatrix}s&t\\x&y\end{bmatrix}_n=&\begin{bmatrix}n-t&t\\x&y\end{bmatrix}_n=\sum_{k=0}^n\binom nk\binom tn\frac{t-n}{t-n+k}B_{n-k}(x)B_k(y)\\=&(t-n)\binom tn\sum_{l=0}^n\frac{B_l(x)}{t-l}B_{n-l}(y).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\lim_{r\to0}\frac1r\left(s\begin{bmatrix}t&r\\y&z\end{bmatrix}_n+t\begin{bmatrix}r&s\\z&x\end{bmatrix}_n\right)\\=&(n-t)(-1)^{n-1}\sum_{k=0}^{n-1}\binom tk\frac{B_{n-k}(y)}{n-k}B_k(z)\\&-t\sum_{k=1}^n\binom{n-t}{n-k}\frac{B_k(x)}{k}B_{n-k}(z)+(-1)^nB_n(z)R\end{aligned}$$

where

$$\begin{aligned}R=&\lim_{r\to0}\frac1r\left((n-t-r)\binom tn+(-1)^nt\binom{n-t-r}{n}\right)\\=&\lim_{r\to0}\frac1r\left(t\binom{r+t-1}{n}-(t-n)\binom tn\right)-\binom tn\\=&\lim_{r\to0}\frac tr\left(\binom{r+t-1}{n}-\binom{t-1}{n}\right)-\binom tn\\=&t\binom{t-1}{n}\sum_{l=0}^{n-1}\frac{1}{t-1-l}-\binom tn=t\binom{t-1}{n}\sum_{k=1}^{n-1}\frac1{t-k}.\end{aligned}$$

Applying (1.2) we then get (1.12) from the above.

The proof of Theorem 1.2 is now complete. □

`Proof of Corollary 1.4`. (1.10) can be rewritten in the form

$$\begin{aligned}&\frac{(-1)^n}{2}t\sum_{k=1}^{n-1}\binom{t-1}{k-1}\frac{E_k(x)}{k}E_{n-1-k}(y)+\frac{(-1)^n}{2}E_{n-1}(y)\\=&\sum_{k=0}^n\left(\frac{\binom{n-t}{k}}{n-t}-\frac{\binom nk}{n}\right)B_k(x)E_{n-k}(z)+\frac1n\sum_{k=0}^n\binom nkB_k(x)E_{n-k}(z)\\&+\frac tn\binom{t-1}{n-1}\left(\frac{B_n(y)}{t}+\sum_{k=1}^n\binom nk\frac{E_k(z)}{t-k}B_{n-k}(y)\right).\end{aligned}$$

Letting $t \to 0$ we get that

$$\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}B_k(x)E_{n-k}(z)+(-1)^{n-1}\frac{B_n(y)}{n}=\frac{(-1)^n}{2}E_{n-1}(y). \qquad (3.3)$$

Thus

$$\begin{aligned}
&\frac{(-1)^n}{2}\sum_{k=1}^{n-1}\binom{t-1}{k-1}\frac{E_k(x)}{k}E_{n-1-k}(y)\\
=&\sum_{k=0}^{n}\frac{1}{t}\left(\frac{\binom{n-t}{k}}{n-t}-\frac{\binom{n}{k}}{n}\right)B_k(x)E_{n-k}(z)+\frac{B_n(y)}{nt}\left(\binom{t-1}{n-1}-(-1)^{n-1}\right)\\
&+\frac{1}{n}\binom{t-1}{n-1}\sum_{k=1}^{n}\binom{n}{k}\frac{E_k(z)}{t-k}B_{n-k}(y).
\end{aligned}$$

Letting $t \to 0$ we then have

$$\begin{aligned}
&\frac{(-1)^n}{2}\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k}E_k(x)E_{n-1-k}(y)+\frac{(-1)^{n-1}}{n}\sum_{k=1}^{n}\binom{n}{k}\frac{E_k(z)}{k}B_{n-k}(y)\\
=&\sum_{k=0}^{n}\lim_{t\to 0}\frac{n\binom{n-t}{k}-(n-t)\binom{n}{k}}{tn(n-t)}B_k(x)E_{n-k}(z)+\frac{B_n(y)}{n}(-1)^nH_{n-1}.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\lim_{t\to 0}\frac{n\binom{n-t}{k}-(n-t)\binom{n}{k}}{t(n-t)}=\lim_{t\to 0}\left(\frac{\binom{n}{k}}{n-t}-\frac{n}{n-t}\cdot\frac{\binom{n-t}{k}-\binom{n}{k}}{-t}\right)\\
=&\frac{1}{n}\binom{n}{k}-\binom{n}{k}\sum_{l=0}^{k-1}\frac{1}{n-l}=-\binom{n}{k}\sum_{0<l<k}\frac{1}{n-l}=\binom{n}{k}(H_{n-k}-H_{n-1}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{(-1)^{n-1}}{2}\sum_{k=1}^{n-1}\frac{(-1)^k}{k}E_k(x)E_{n-1-k}(y)+\frac{(-1)^{n-1}}{n}\sum_{k=1}^{n}\binom{n}{k}\frac{E_k(z)}{k}B_{n-k}(y)\\
=&(-1)^nH_{n-1}\frac{B_n(y)}{n}+\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}(H_{n-k}-H_{n-1})B_k(x)E_{n-k}(z)\\
=&(-1)^nH_{n-1}\frac{B_n(y)}{n}-\frac{H_{n-1}}{n}\sum_{k=0}^{n}\binom{n}{k}B_k(x)E_{n-k}(z)\\
&+\frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}H_lE_l(z)B_{n-l}(x)\\
=&-H_{n-1}\frac{(-1)^n}{2}E_{n-1}(y)+\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}H_kE_k(z)B_{n-k}(x).
\end{aligned}$$

This proves (1.17).

We can reformulate (1.11) as follows:

$$\begin{aligned}&\frac{t}{2}\binom{t-1}{n-1}\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{E_k(x)}{t-k}E_{n-1-k}(y)+\frac{1}{2}\binom{t-1}{n-1}E_{n-1}(y)\\&-(-1)^nE_n(z)\frac{t}{n}\binom{t-1}{n-1}\sum_{k=1}^{n-1}\frac{1}{t-k}-(-1)^n\frac{E_n(z)}{n}\binom{t-1}{n-1}\\=&(-1)^nt\sum_{k=1}^{n-1}\binom{t-1}{n-k-1}\frac{B_k(y)}{k}\cdot\frac{E_{n-k}(z)}{n-k}+(-1)^n\frac{B_n(y)}{n}\\&-\sum_{k=1}^{n}\left(\binom{n-1-t}{n-k}-\binom{n-1}{n-k}\right)\frac{B_k(x)}{k}E_{n-k}(z)\\&-\sum_{k=1}^{n}\binom{n-1}{n-k}\frac{B_k(x)}{k}E_{n-k}(z).\end{aligned}$$

Letting $t\to 0$ we find that

$$\frac{(-1)^{n-1}}{2}E_{n-1}(y)+\frac{E_n(z)}{n}=(-1)^n\frac{B_n(y)}{n}-\sum_{k=1}^{n}\binom{n-1}{n-k}\frac{B_k(x)}{k}E_{n-k}(z),$$

i.e.,

$$\sum_{k=0}^{n}\binom{n}{k}B_k(x)E_{n-k}(z)=(-1)^n\left(B_n(y)+\frac{n}{2}E_{n-1}(y)\right).\tag{3.4}$$

Therefore

$$\begin{aligned}&\frac{1}{2}\binom{t-1}{n-1}\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{E_k(x)}{t-k}E_{n-1-k}(y)\\&-(-1)^n\frac{E_n(z)}{n}\binom{t-1}{n-1}\sum_{k=1}^{n-1}\frac{1}{t-k}\\=&(-1)^n\sum_{k=1}^{n-1}\binom{t-1}{n-k-1}\frac{B_k(y)}{k}\cdot\frac{E_{n-k}(z)}{n-k}\\&+\sum_{k=1}^{n-1}\frac{\binom{n-1-t}{n-k}-\binom{n-1}{n-k}}{-t}\cdot\frac{B_k(x)}{k}E_{n-k}(z)\\&+\frac{E_{n-1}(y)}{2}\cdot\frac{(-1)^{n-1}-\binom{t-1}{n-1}}{t}+\frac{E_n(z)}{n}\cdot\frac{1-(-1)^{n-1}\binom{t-1}{n-1}}{t}.\end{aligned}$$

Letting $t \to 0$ we obtain that

$$\begin{aligned}
&\frac{(-1)^n}{2}\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{E_k(x)}{k}E_{n-1-k}(y)-\frac{E_n(z)}{n}H_{n-1}\\
=&\sum_{k=1}^{n-1}(-1)^{k-1}\frac{B_k(y)}{k}\cdot\frac{E_{n-k}(z)}{n-k}\\
&+\sum_{k=1}^{n-1}\binom{n-1}{n-k}\left(\sum_{l=0}^{n-k-1}\frac{1}{n-1-l}\right)\frac{B_k(x)}{k}E_{n-k}(z)\\
&+H_{n-1}\left(\frac{(-1)^{n-1}}{2}E_{n-1}(y)+\frac{E_n(z)}{n}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{(-1)^n}{2}\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{E_k(x)}{k}E_{n-1-k}(y)+\sum_{k=1}^{n-1}(-1)^k\frac{B_k(y)}{k}\cdot\frac{E_{n-k}(z)}{n-k}\\
=&\sum_{k=1}^{n}\binom{n-1}{n-k}(H_{n-1}-H_{k-1})\frac{B_k(x)}{k}E_{n-k}(z)\\
&+H_{n-1}\left(\frac{(-1)^{n-1}}{2}E_{n-1}(y)+2\frac{E_n(z)}{n}\right)\\
=&-\sum_{k=1}^{n}\binom{n-1}{k-1}H_{k-1}\frac{B_k(x)}{k}E_{n-k}(z)\\
&+H_{n-1}\left(\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{B_k(x)}{k}E_{n-k}(z)+\frac{(-1)^{n-1}}{2}E_{n-1}(y)+\frac{2}{n}E_n(z)\right)\\
=&-\sum_{k=1}^{n}\binom{n-1}{k-1}H_{k-1}\frac{B_k(x)}{k}E_{n-k}(z)+H_{n-1}\frac{E_n(z)+(-1)^nB_n(y)}{n}.
\end{aligned}$$

This proves (1.18).

We can easily get (1.19) by calculating the limitation of the the left hand side of (1.12) minus the right hand side as t tends to 0. □

References

[DS] G. V. Dunne and C. Schubert, *Bernoulli number identities from quantum field theory*, preprint, `arXiv:math.NT/0406610`, 2004.

[Ge] I. M. Gessel, *On Miki's identity for Bernouli numbers*, J. Number Theory, to appear.

[Ma] Y. Matiyasevich, *Identities with Bernoulli numbers*, `http://logic.pdmi.ras.ru/~yumat/Journal/Bernoulli/bernulli.htm`, 1997.

[Mi] H. Miki, *A relation between Bernoulli numbers*, J. Number Theory **10** (1978), 297–302.

[PS] H. Pan and Z. W. Sun, *New identities involving Bernoulli and Euler polynomials*, submitted, 2004, `arXiv:math.NT/0407363`.

[S1] Z. W. Sun, *Introduction to Bernoulli and Euler polynomials*, a talk given at Taiwan, 2002, `http://pweb.nju.edu.cn/zwsun/BerE.pdf`.

[S2] Z. W. Sun, *Combinatorial identities in dual sequences*, European J. Combin. **24** (2003), 709–718.

[SP] Z. W. Sun and H. Pan, *New identities involving Bernoulli and Euler polynomials. II*, submitted, 2004, `arXiv:math.NT/0408223`.