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Defining the Entropy of Hierarchical Organizations

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A measure of the order within a hierarchical organizational structure is proposed based on an analogy with the thermodynamic entropy of physical sciences. Organizational entropy is defined in a manner that readily allows for practical calculations. This calculation could be used to relate the order or entropy of an organization to its functional role. Additionally, it could be used to monitor the change in entropy over time and provide an impetus for periodic restructuring. This theory gives two general contributions to the entropy of an organizational structure: horizontal entropy due to changes within a level of the hierarchy and vertical entropy due to changes between levels along reporting lines within the organization. In addition to employing the thermodynamic entropy analog, identical theoretical results are obtained when calculating the Kolmogorov entropy or algorithmic complexity of an organizational hierarchy and establishes the generality of the approach. Computer simulations on model hierarchical structures show the boundaries of the vertical and horizontal contribution to entropy. It is postulated that each organization will have a specific entropy that optimizes organizational functions.

Keywords: organizational structure, entropy.

1. Why Define Organizational Entropy and How It Can be Useful

Many fields will claim that "form follows function". This is true in architecture, engineering design, biology, and also in organizational management. The structure of an organization is often tied to the function [Carley, 1995]. Organizations that require flexibility and rapid decision making will have "flat" structures with considerable authority at lower levels of the organization. Strongly hierarchical organizations will exist where high risk, strategic decision making is required. One can argue that each organization should customize its structure to suit its mission and goals.

Organizations are not static and their structure and very mission can change over time. The functions of the organization may change in response to changes in the industry. Alternatively, the organizational structure may "degrade" over time due to loss

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of personnel, reassignments or merging of functional units. So there may be a tendency for an optimal organizational structure to evolve over time to a less effective structure. In these cases, a restructuring of the institution is called for to bring the organization back in alignment with its optimal structure.

The development of computational and mathematical organization theory over the last several decades has led to a wide array of quantitative metrics that may be used to characterize the structure, dynamics and function of organizations. These tools have benefited theorists interested in evaluating formal organizational models and managers who seek practical assessments to improve efficiency and innovation of their organizational units. Often, quantitative metrics used in organizational management are borrowed from other fields. Graph theory has been used to develop a range of measures to quantify organizational structures and social network connections (see, for example, Hummon [1995], Krackhardt et al. [1994], McGrath et al. [2003]). Metrics and computational organizational models have also been inspired by the natural sciences, particularly physics (see Fabac [2008], Stepanic et al. [2000]), and by information theory [Schlick et al., 2013]. The physics-based studies implicitly ask to what extent social systems of interacting "agents" mimic physical systems of interacting particles. Thermodynamics offers one of the most promising avenues for relating the two fields. Attempts to find correlates between physical thermodynamic quantities and social or organizational quantities have been made by Kasac [2004] and Stepanic et al. [2000]. These studies are largely theoretical in nature and model broad, generic social characteristics. Our paper has a narrower focus. We present a simple, quantitative metric based on thermodynamics that is tailored to measure a specific property (the entropy) of hierarchical organizations.

This paper proposes a measure, the organizational entropy, to provide a quantitative measure of the disorder of the reporting structure of an organization. A highly ordered, low entropy organization would have a very hierarchical structure with strict lines of reporting. A federation or consortium would be a low order, high entropy structure with comparable authority in each member, a comparatively weaker administrative unit and very loose lines of communication. High order may be good in one instance and not desirable in another. A quantitative measure of order or organizational entropy makes no judgment of the benefit or disincentive to the order of an organization. It just provides a rigorous way of measuring that order.

The term "entropy" appears in a number of different contexts with different meanings. In this work, organizational entropy is defined by analogy to thermodynamic entropy as used in physics and chemistry. Entropy is a measure of disorder of a system. A perfect, flawless crystal at the zero of temperature has zero entropy. There is perfect order and, at the zero of temperature, there are no internal motions to disturb that order. A gas, on the other hand, has much higher entropy as its particles are free to move around in any direction, colliding with each other and moving off into new directions. For an organization, low entropy would correspond to highly uniform reporting structures. Figure 1 and Figure 2 shows organizational charts where the nodes represent individuals within the organization and the lines represent oversight or reporting relationships. Comparatively

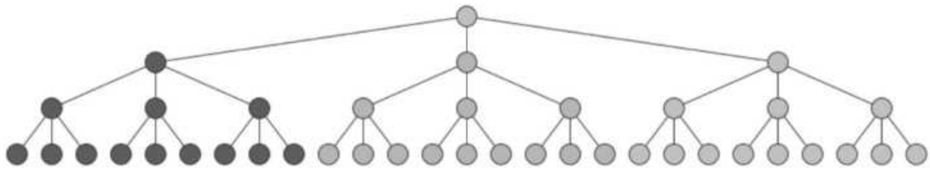

Figure 1. A highly structured organization with 4 hierarchical levels and 40 employees. Every manager directly oversees three subordinates. The numeric string encodes the reporting structure of the organization. To generate the coded string, the tree is scanned level-by-level to record the number of subordinates under each node.

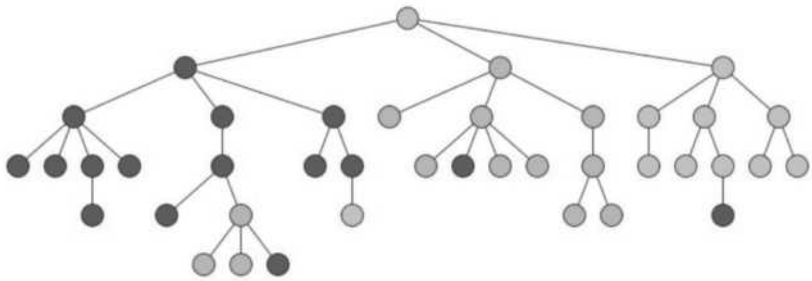

Figure 2. A disordered organization with 6 hierarchical levels and 40 employees. The numeric string encodes the reporting structure of the organization.

speaking, Figure 1 would be a low entropy organization because the structure is very orderly with a perfect ordering of reports. Different areas of expertise, represented as red, orange and blue, are each contained within the three main organizational units. Figure 2 on the other hand shows an organization of higher entropy because the regularity of the structure is disrupted by variability of the reporting structure and the areas of expertise are mixed across units. Both organizations have the same number of individuals but clearly have different structures. The breaking of order in the second graph gives rise to a high entropy.

In this paper, we propose how to measure the organizational entropy. This begs the question: Why would one want to measure this entropy? If "form follows function", there may be a value to measuring the entropy of each form. A given organizational structure can be assigned an organizational entropy value. We anticipate for a specific function there will be common structural elements that may have a common range of entropy values. It is then possible to compare or rank organizations according to their entropy and compare these rankings to some measure of the effectiveness of the organization. Such a ranking may lead to insights about the factors that lead to an effective organization. For instance, a rigid hierarchical organization may not be very innovative and this could affect overall performance. Similarly, a randomly connected line of reporting would also lead to a dysfunctional organization. So highly ordered (low entropy) or highly disordered (high

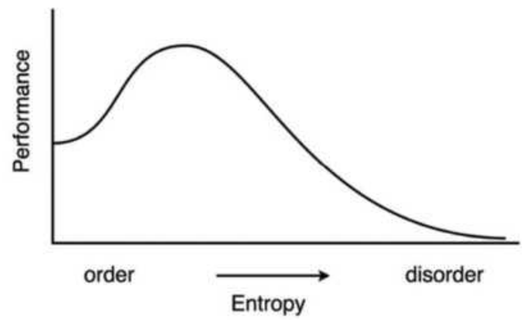

Figure 3. Hypothesized functional dependence of an organizations performance and its entropy.

entropy) organizations are both unlikely to be effective performers. There may well be an optimal organizational entropy in between complete order and complete disorder that affords the best structure for a given performance criterion (see Figure 3).

A second application would be to measure the entropy of an organization over time and see how it "evolves". The Second Law of Thermodynamics says that physical systems will naturally evolve from high order to low order. That is the entropy of a system will increase over time. While we are not positing that organizational structures necessarily follow the Second Law in this manner, it is a common observation that organizations need to restructure from time to time. Such restructuring can be a result of an accumulation of internal personnel and reporting line decisions that result in organizational dysfunction or inefficiencies. Alternatively, the external environment of the organization may change and require that a new set of functions and expertise be brought into the organization. Both effects could move the organizational structure and its associated entropy away from the optimum in Figure 3. So a potential use of organizational entropy is to be a bell weather to observe how organizations drift away from their optimal structure.

2. How to Measure Organizational Entropy

There is an arsenal of techniques for defining organizational entropy. Thermodynamic entropy has the longest history and relates to our everyday physical world. A drinking glass dropped on the floor is more likely to break into hundreds of shards than for hundreds of shards to spontaneously assembly into a drinking glass. Nature spontaneously moves from low entropy (order) to high entropy (disorder). Over time, a number of definitions of entropy have arisen from various fields- Shannon information entropy from communication theory, Kolmogorov entropy or computational complexity from theoretical computer science and maximum entropy techniques from Bayesian statistics. While all of these definitions are motivated by problems in their specific fields, they have deep connections with each other and demonstrating the relationship between them is an active field of research. To demonstrate the generality of our definition of organizational entropy, we derive identical sets of equations for calculating organizational entropy from both a thermodynamic entropy and from a Kolmogorov entropy approach. These two approaches are briefly described below:

Thermodynamic entropy - this model is based on the thermodynamic concept of entropy used in numerous examples in physical chemistry. The two primary calculations used in this paper are the entropy of mixing (to apply to mixing of expertise within a functional unit) and the entropy of expansion of an ideal gas (to apply to expansion or contraction of staffing in an organizational unit).

Kolmogorov entropy - this model arises from Kolmogorov's concept of computational complexity used in theoretical computer science. The organizational complexity is given by the shortest description in binary code required to describe the organizational structure. Complicated structures require long descriptions while simple structures can be easily described. Thus, encoding the description of a complicated structure will require more bits than that of a simple structure.

3. The Calculation of Organizational Entropy

This section develops the mathematics behind the calculation of organizational entropy. The first subsection focuses on analogies with simple examples from physical chemistry and shows how these can be used to describe entropy changes due to "personnel shuffling" and also due to the expansion or contraction of organizational units. The intent of this section is to be heuristic and to provide a number of practical examples of calculations. The following subsection is more mathematical in nature and describes the effects of the alteration of reporting lines and consequently the shuffling of entire units within the organization. A formal mathematical development of personnel shuffling would follow along the same lines as that of unit shuffling, but is excluded here for brevity. So all the effects of both personnel and unit shuffling can be obtained using the approach presented in this section and should be inclusive of all entropic effects of the model.

Before proceeding, a definition of terms and the specifics of the model are first required. We define a hierarchical model as one in which a single individual (referred to as the CEO) is the convergent point for all reporting lines of the whole organization. Parenthetically, this requirement means that all the graphical organizational structures generated by this model will have the property of a "rooted tree" in graph theoretical terms. The direct reports to the CEO represent the first tier in the hierarchy. A unit in the hierarchy is a functional department that reports to a unit above it in the hierarchy. For instance, the VP for Operations may have budget, marketing and human resource (among others) reporting to him or her. These are then the units in the next tier. Each tier of the hierarchy has units that only report to an individual in the tier above it. Note that this representation may not be strictly accurate. For instance, a CEO will typically receive reports from a number of VPs as well as administrative assistants. Under this model, all these individuals are identified on the same tier even though their authority within the organization might be quite different. All direct reports to the CEO are in the first tier. All units that report to a first-tier unit reside in the second tier, etc. Borrowing a term from genealogy, a "lineage" of a given unit within the hierarchy is the collection of subordinate units whose reporting lines trace back to the unit in question. Note that not all lineages will necessarily have the same depth. Just like family trees that have lineages that terminate at different generations, organizational trees may have lineages that terminate at different tiers.

In a general derivation of organizational entropy as in Section 3.2, mathematical terms are identified that relate to the entropy contribution from within tiers and a separate term for the entropy contribution between tiers. The sum of the entropy contribution within tiers is referred to as the "horizontal entropy". The entropy contribution produced by variation across tiers is referred to as the "vertical entropy" and may be thought of as being a "generationally-dependent" measure of the lineages within the hierarchy. The total entropy is equal to the sum of the vertical entropy and horizontal entropy. These two different contributions mean that hierarchies can look quite different and still have the same total organizational entropy. In general, the horizontal entropy is maximal when units within each tier are populated with the same number of individuals. Also, the vertical entropy is maximal when the population at each generation or tier along a linage remains the same.

3.1. Thermodynamic Analogies for the Entropy of Personnel Shuffling

Invariably over the history of an organization, people get moved from one unit to the next. These moves are often motivated by personnel or human resource issues and are not necessarily made to make the organization stronger, more efficient or more innovative. This phenomenon is referred to here as personnel shuffling. Personnel shuffling is like shuffling a new deck of cards where everything is ordered by suit. After shuffling the suits are mixed and the deck is overall in greater disorder or higher entropy. In a similar fashion, the organizational entropy increases with personnel shuffling. However, as we will see personnel shuffling can be more complicated than just shuffling a deck of cards. In addition to changing personnel, there can be an expansion or contraction of a unit and also there can be a shuffling of supervisors of a unit.

Two equations are needed to calculate the change in organizational entropy resulting from these personnel changes. The first equation is the entropy of mixing, a well-known equation in physical chemistry that can also be derived from information theory (see below). When two pure components are mixed the change in entropy that results from this mixing is given by (cf. Atkins [1997]):

$$\Delta S_{mixing} = -\sum_{i=1}^{l} f_i \log_2 f_i \tag{1}$$

where the sum is over the l components that are mixed and f_i is the fraction of the i^{th} component. In thermodynamics, Eq. 1 would have a physical constant multiplier on the right hand side to give entropy units. In information theory, there is no need for this constant because the units are in "bits", e.g. binary digits. In information theory, the logarithm is taken as Base 2 to give units in bits. In our approach, the information theory notation is used. The total entropy change of a hierarchical unit is given by the sum of the entropy changes across each level:

$$\Delta S_{mixing} = -\sum_{j=1}^{N}\sum_{i=1}^{l} f_{j,i}\log_2 f_{j,i} \tag{2}$$

where N is the total number of organizational units, and $f_{j,i}$ is the fraction of the i^{th} component in the j^{th} unit.

The second equation needed is the equation that accounts for expansion or contraction of units. This equation also has an analog in physical chemistry with the expansion of an ideal gas (cf. Atkins [1997]). The equation for unit expansion or contraction is:

$$\Delta S_{exp/cont} = \log_2\left(\frac{n_{after}}{n_{before}}\right) \tag{3}$$

where n_{after} and n_{before} represent the number of staff in the unit after and before the change. Note that when $n_{after} > n_{before}$, the entropy change is positive, and when $n_{after} < n_{before}$, the entropy change is negative indicating a decrease in entropy. The total entropy change of the organization is given by the sum of the entropy changes of each unit:

$$\Delta S_{exp/cont} = \sum_{i}^{N}\log_2\left(\frac{n_{i,after}}{n_{i,before}}\right) \tag{4}$$

When the organization as a whole loses positions, the entropy is reduced and when it gains positions the entropy increases.

3.1.1. Sample Calculation of Expansion and Contraction

When an organization adds positions, the entropy of expansion increases ($\Delta S_{exp} > 0$) and when it reduces positions $\Delta S_{exp} < 0$. For example, if a unit expands from 3 to 5 personnel, we find $\Delta S_{exp} = \log 2(5/3) = 0.737$ (see Figure 4a). If the unit loses the two new employees later on, then $\Delta S_{exp} = \log 2(3/5) = 0.737$ (see Figure 4b). The entropy change depends on how units are defined. Suppose a large firm that has 60 employees in the marketing division hires a new employee. If one considers this division as a single unit then the entropy change is modest: $\Delta S_{exp} = \log 2(61/60) = 0.0238$. However, if the marketing division is composed of twelve sub-units, each with 5 employees then the entropy increase of the division will be more than ten times larger: $\Delta S_{exp} = \sum_{i=1}^{12} \log_2(n_{i,after}/5) = 11\log_2$

Figure 4. Example entropy change due to (a) expansion or (b) contraction of a single unit. The squares represent personnel in each unit.

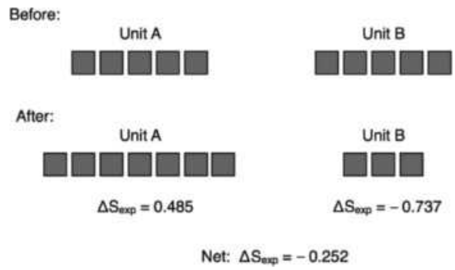

Figure 5. Example entropy change when two personnel are transferred from one unit (unit B) to another (unit A). In this example, the initial equal distribution of personnel across the units corresponds to the maximum entropy state (for a fixed number of employees). The net entropy is reduced when personnel shuffling leads to a more unequal distribution of staff across units.

5/5 + $\log_2(6/5) = 0.263$. In other words, the entropy of expansion is sensitive to coarse graining effects: it depends on how finely units are defined within the organization. When applying this method, a standardized method must be used when defining the units across the organization.

If personnel are transferred from one unit (unit B) to another (unit A), then unit B will experience a reduction in S_{exp} while S_{exp} will increase for unit A (see Figure 5). The net entropy of expansion S_{exp} will increase if the reorganization acts to even out the number of staff in each unit and will decrease if it leads to a larger disparity in staffing between the units. For a fixed number of staff, the entropy of expansion is maximized when each unit in the organization has equal numbers of employees. For example if units A and B each have 5 staff initially, and two staff from B are transferred to A (resulting in 7 and 3 staff for A and B respectively), then $S_{exp} = \log_2(3/5) + \log_2(7/5) = -0.252$. In this example, the net entropy of expansion is reduced as the distribution of staff across units becomes more heterogeneous.

3.1.2. Sample Calculation of Mixing With Expansion and Contraction

The entropy of mixing S_{mixing} is nonzero when an organizational unit houses staff with expertise from different areas. It is maximized when a unit has equal numbers of staff from multiple areas. Suppose, for example, an HR department replaces one of its 10 staff members with someone from another department (e.g. sales). The change in mixing entropy would be $\Delta S_{mixing} = (1/10)\log_2(1/10) + (9/10)\log_2(9/10) = 0.469$. The positive value indicates the mixture of skills within the unit increased. Figure 6 shows an example where a staff member from one unit (say sales) is transferred to another unit (say HR). If the sales team originally had 5 staff members and the HR department had 3 staff, the net entropy change would be $\Delta S_{mixing} + \Delta S_{exp} = 2.415 + 0.093 = 2.51$. In this scenario the transfer of a staff member to a unit outside his/her expertise leads to a net increase in the

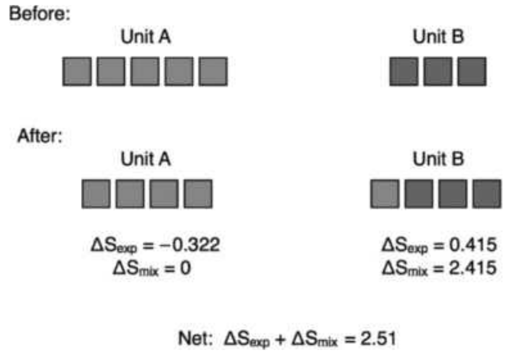

Figure 6. Units A and B are initially staffed by personnel with different expertise (color coded red and blue). The mixing of expertise in unit B leads to an increase in the mixing entropy. Because the net entropy of expansion is also positive, the net entropy change is positive.

entropy of the organization. However, net negative entropy changes are also possible if the entropy of expansion is sufficiently negative. These examples suggest that both measures ΔS_{mixing} and ΔS_{exp} could be important statistics to consider when evaluating the reason for a net increase or decrease in the overall entropy of an organization.

3.1.3. A More Complicated Sample Calculation

A specific example is now considered that illustrates the various types of personnel shuffling- shuffling between units, contraction/expansion of a unit, and replacement of a unit head. These are shown in Figure 7 where the organization before and after shuffling is shown. There are three units in the organizational structure consisting of 7 employees in Marketing, 6 in Sales and 6 in Human Resources. The color represents the expertise within the unit and of individuals. Historically, the Director has always come from Sales and so that person is coded blue as well. Now consider the following changes to the organization (illustrated in Figure 5):

- The Director retires and is replaced by a promising employee in Marketing.
- An employee in Sales is let go. One of the Human Resource staff, although not technically qualified for the job, applies and gets the job.
- Human Resources has lost an important employee and is viewed as need an immediate replacement. Marketing is generally viewed as being over staffed, so in addition to losing an employee to the Directors position, a second employee is shifted over to Human Resources.

As a result of these personnel changes, the organizational entropy has changed in a number of different ways. First, Sales and Human Resources are no longer "pure" units because they have staff coming in from the outside that do not necessarily have direct expertise in this area. This shuffling results in an increase in entropy. Although the Director has changed, it is

Figure 7. Hypothetical organization before personnel shuffling.

Figure 8. Hypothetical organization after personnel shuffling.

still a pure unit and, therefore, there is no entropy associated with this. Parenthetically, there may be a slight information entropy associated with this change depending on the descriptors of that position and this will be discussed in the next section. Finally, there was a contraction in Marketing and this results in a decrease of entropy or increase in order. To conceptualize this a unit with a hundred individuals, even if they are all "pure" in their expertise, is more complicated than a unit with one individual. So the smaller the unit, the more order.

We now calculate the total entropy change in the example introduced in Figure 7. The number of personnel in marketing, sales and human resources changes from $(n_M, n_S, n_{hr}) = (7, 6, 5)$ to $(5, 6, 5)$ respectively. The resulting entropy of expansion is $\Delta S_{exp} = -0.48$, which reflects the overall reduction in staffing of the organization. The mixing ratios for the three units are: marketing $f_{m,i} = (1, 0, 0)$, sales $f_{s,i} = (0, 5/6, 1/6)$, and human resources $f_{hr}, i = (1/5, 0, 4/5)$ giving mixing entropies for each of the units to be $\Delta S_m = 0$, $\Delta S_s = 0.65$, $\Delta S_{hr} = 0.72$. The total mixing entropy of the organization is $\Delta S_{mix} = 0 + 0.65 + 0.72 = 1.37$. The total entropy change for the organization is the sum of the expansion and mixing entropies: $\Delta S = -0.48 + 1.37 = 0.89$. We see that the overall entropy of the organization increased due to personnel shuffling despite the overall downsizing.

3.2. The Entropy of Organizational Unit Shuffling

Organizational entropy can arise from mechanisms other than personnel shuffling. Personnel shuffling represent the movement of nodes within a hierarchical structure. However, we can also consider what happens when the connections or lines are moved. This corresponds to changing the reporting structure while keeping the units intact. A uniform unit is defined as a cohesive unit in which all members have expertise in the functionality of the unit. For instance, a marketing department consisting entirely of marketers is uniform. A marketing department with marketers plus a financial analysis and an accountant has higher disorder because it contains individuals normally not assigned to it and that are outside of its function. Organizational disorder can also arise from a reporting relationship that is not aligned with the function. For instance, if the head of marketing is reassigned to report to the Vice President for Human Resources, this represents a disorder in the hierarchy because of a functionally improper line of reporting. Such alterations are treated as disorders in the hierarchy of an organization.

In this section, the organizational entropy for a hierarchical structure is calculated. We consider a hierarchy that has L different levels and amongst those L levels are M departments or organizational units. The number of units at the l^{th} level of the hierarchy is given by M^l, so that

$$M = \sum_{i=1}^{L} M^l \tag{5}$$

The hierarchy is organized in such a manner that each unit at the l^{th} level reports to a specific unit at the $l + 1$ level. The number of units reporting to the u^{th} office at the $l + 1$ level is given by: M_u^l, and assigning all units to a unit above them gives:

$$M^l = \sum_u M_u^l \tag{6}$$

We now ask the question, how many possible structures would fit this condition and this number is designated as Ω_{tree} and the entropy of the organization by analogy to the statistical mechanics of physical systems is defined as:

$$S_{tree} = \ln \Omega_{tree} \tag{7}$$

To calculate this entropy, we first calculate how many ways that M units can be distributed over L levels. This is a combinatorial problem and is given by: $M!\prod_{i=1}^{L} M^l!$. We then ask the question on how the M^l units on a level can be distributed across the different reporting structures in the units above them. This is given by: $\mathrm{M}^l!\prod_{i=1} M_u^l!$. The total number of possible organization structures under these conditions is:

$$\Omega_{tree} = \left[\frac{M!}{\prod_{i=1}^{L} M^l!}\right] \prod_{l=1}^{L} \frac{M^l!}{\prod_{i=1} M_u^l!} \tag{8}$$

Following some manipulations and the use of Stirling's approximation (note: Stirling's approximation will probably not hold), the entropy of the hierarchical tree is:

$$\begin{aligned} S_{tree} &= -\sum_{i=1}^{L} M^l \ln\left(\frac{M^l}{M}\right) - \sum_{i=1}^{L} \sum_u M_u^l \ln\left(\frac{M_u^l}{M^l}\right) \\ &= \sum_{l=1}^{L} M^l \ln P^l - \sum_{l=1}^{L} \sum_u M_u^l \ln P_u^l \end{aligned} \tag{9}$$

where $P^l = (M^l/M)$ and $P_u^l = (M_u^l/M^l)$. The first term on the right had side represents the entropy of mixing of all the units across the levels of the hierarchical tree. We define this term as the horizontal mixing entropy:

$$S_H = -\sum_{l=1}^{L}\sum_{u} M_u^l \ \ln\ P_u^l \tag{10}$$

The second sum is the sum over all levels of the entropy of mixing of the units within a level. We define it as the vertical mixing entropy:

$$S_V = -\sum_{l=1}^{L} M^l \ \ln\ P^l \tag{11}$$

3.2.1. Sample Calculations Illustrating Vertical and Horizontal Entropy Effects

As an example, consider an organization with 15 employees. We assume that the organization is headed by a single individual (producing what graph theorists refer to as a rooted tree). Numerical simulations reveal that there are 2,674,440 possible reporting structures, five examples of which are shown in Figure 9.

Figure 10 shows the vertical and horizontal mixing entropies for the $N = 15$ tree family. The diagonal lines indicate lines of constant total entropy $S_V + S_H$. The flat reporting structure (tree A) has the minimum total entropy and the purely vertical structure (tree C) has the maximum total entropy. The hierarchical binary tree (B) has a net entropy that is intermediate between these two bounds. Perhaps surprising is the fact that it has a maximum horizontal entropy for the class of trees with the same level populations. The horizontal mixing entropy measures the degree to which the units are spread evenly across the organization. Consider the physics analogy of a gas: the entropy is maximized if the

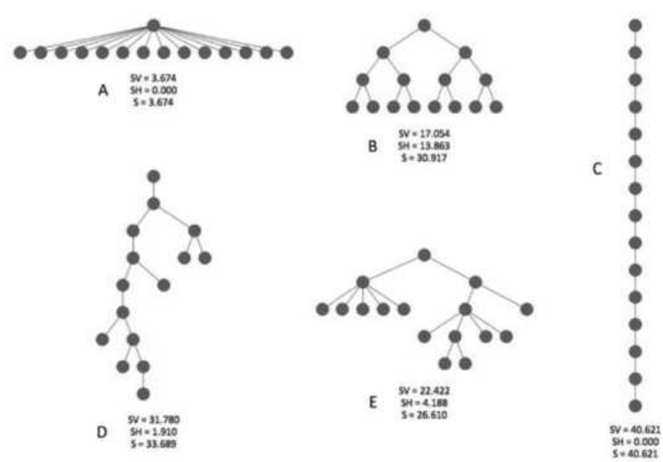

Figure 9. Example reporting structures with 15 nodes. Each structure is labeled by the horizontal, vertical and total mixing entropies and a lexicographic string that encodes the reporting structure.

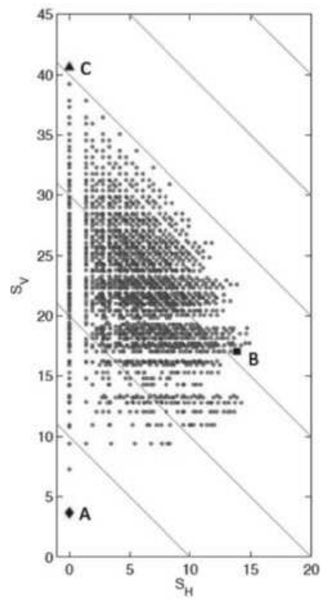

Figure 10. Structural entropy measures S_v vs. S_H for all possible trees with N = 15 nodes. The diagonal lines indicate lines of constant total entropy $S_v + S_H$. The locations of organizational trees A, B and C are marked.

gas uniformly fills its container. The entropy would be reduced if the gas molecules were confined to a smaller region within the container. Thus, the horizontal entropy is maximized when subordinates are distributed evenly across a given level of the organization.

As an organization evolves its entropy will evolve as well. One can imagine following the trajectory of an organization in the S_V vs. S_H "phase space" shown in Figure 10. We constructed a simple simulation to follow the organizational entropy over time assuming two simple rules: (1) the total number of nodes in the organization remained constant, (2) single nodes were chosen at random and the reporting line was randomized.

The aim of this simulation is not to model the dynamics of real-world organizations, but rather to gain insight into what characteristics the entropy measures are sensitive to. Figure 8 shows the evolution of the vertical, horizontal and net entropies as a function of the iteration number. Figure 9 shows the organization at three times during the simulation. The horizontal and vertical entropies appear to be roughly anti-correlated: as the vertical entropy rises, the horizontal entropy declines. This behavior is due to the conservation of nodes. Removing a node from a highly populated level and adding it to a less populated level will reduce the horizontal entropy while increasing the vertical entropy. This anticorrelated behavior means that the variance of the total entropy is significantly less that the individual entropies. As a result of this effect, the phase-space trajectory will tend to fluctuate along the diagonals of constant total entropy.

Preliminary Fourier analyses of the time series indicate that the statistical fluctuations of the entropies appear to roughly follow a $|f(k)| \propto k^{-3/2}$ power law, which we speculate arises from the branching pattern of the trees. It is interesting to note that this power

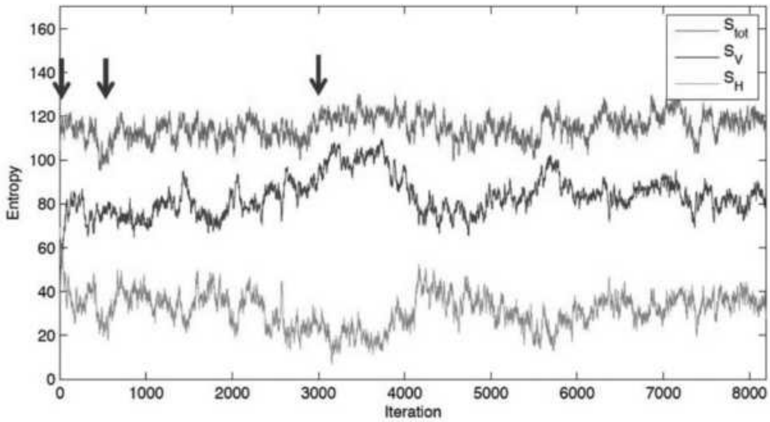

Figure 11. Evolution of the vertical, horizontal and total mixing entropies for a ternary hierarchical tree with 4 levels and 40 nodes. The iteration number counts the number of personnel changes over time. The three arrows correspond to the three reporting structures shown in Figure 12.

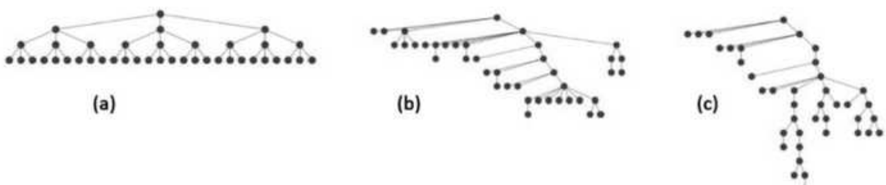

Figure 12. Evolution of a hierarchical tree given the dynamical rules described in the text. The trees correspond to iteration numbers 0, 500 and 3000 in Figure 11.

law is intermediate between $1/f$ noise associated with critical phenomena and random walk processes on Euclidean spaces.

4. Connection with Computational or Kolmogorov Entropy

The Kolmogorov entropy or computational complexity of an object is defined as the shortest possible description in bits of that object (cf. Cover and Thomas [2006]). While there are a number of theoretical challenges in defining this quantity, as a practical matter there are simple estimates for computational complexity that can be quite useful. We take the simple case of descripting the individuals within a level of a hierarchical organization. One way to estimate the computational complexity is to use the "lexicographic trick" [Cover and Thomas, 2006]. If we are to describe the assignments of individual workers within a level of the hierarchy, a very short description of this would be the following instructions: "Make a lexicon of all the possible working assignments according to a prescribed rule. The desired description can be found on line n^* of the lexicon". The encoding of the instructions will actually be quite small and not make a large contribution to the complexity. In general, the largest contribution will come from the number n^*. The encoding of a number will take $\log_2 n^*$ bits. In general, we will have λ so the encoding of this number, $\log_2 \lambda$ represents an upper limit of the complexity. If there are N individuals in the level of the hierarchy and each unit had n_i employees, then the number of possible combinations is:

$$\lambda = \frac{N!}{\prod_i n_i} \tag{12}$$

The computational complexity is then bounded by:

$$\log_2 \lambda = N \log_2 N - \sum_i n_i \log_2 n_i = -\sum_i n_i \log_2 f_i \tag{13}$$

where Stirling's approximation to the logarithm of a factorial has been used and f_i is defined as in Eq.1. The complexity per number of employees, $\log_2 \lambda/N$ then gives Eq. 1. Thus, the estimate of the computational complexity of the level in the hierarchy is identical to the entropy of mixing as in Eq. 1. This shows that both the physical entropy approach and the computational complexity approach yield the same result for the organizational entropy. This argument is readily extended to the other calculations described by the physical (i.e. thermodynamic) entropy approach.

5. Conclusion

In this work, a mathematical definition of organizational entropy motivated by analogies with both thermodynamic and Kolmogorov entropy is presented. As demonstrated in the text, this definition allows for the entropy to be readily calculated for any organization given the personnel distribution along reporting lines. The organizational entropy has two contributions: the horizontal entropy that measures the disorder within tiers and the vertical entropy that measure the disorder along the lineages or between tiers. As described earlier, there are a number of potential applications of organizational entropy. It can be us be used as an empirical parameter to monitor an organization over time and indicate when a restructuring might be appropriate. Similarly, it might be used to classify structures designed for specific functions and indicate when an optimal structure has been achieved. The mathematical formalism presented here is a starting point for exploration of optimal organizational structures. Using maximum entropy approaches (cf. Cover and Thomas [2006]), the statistical properties of organizational structures that maximize the entropy under different constraints can be determined. For instance, a "cost function" may be introduced that distinguishes the cost or value of personnel between tiers or between lineages. Maximizing entropy under fixed costs would lead to a mathematical description of the optimal structure given these costs. Again, exploiting the analogy with thermodynamics, the total cost could be related to the analogue of the energy the system and relationships between entropy and costs could be derived that are analogous to thermodynamic relationships between energy and entropy. Such extensions of the present theory are under investigation.

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