Stable Ramsey’s theorem and measure

Damir D. Dzhafarov
University of Chicago

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Stable Ramsey’s theorem

Definition

Let $X \subseteq \omega$ be an infinite set and let $n, k \in \omega$.

- $[X]^n := \{ Y \subseteq X : |Y| = n \}$.
- A coloring on $X$ is a function $f : [X]^n \rightarrow k = \{0, \ldots, k-1\}$.
- A set $H \subseteq X$ is homogeneous for $f$ if $f \upharpoonright [H]^n$ is constant.
- A coloring $f : [X]^2 \rightarrow k$ is stable if for all $x \in X$, $\lim_{y \in X} f(x, y)$ exists.
(RT\(_n^k\)) Every \( f : [\omega]^n \to k \) has an infinite homogeneous set.

(SRT\(_2^k\)) Every stable \( f : [\omega]^2 \to k \) has an infinite homogeneous set.

We deal only with stable colorings, and only with \( k = 2 \).
Stable Ramsey’s theorem

Fact

- For every computable stable $f : [\omega]^2 \rightarrow 2$, there is a $\Delta^0_2$ set every infinite subset or cosubset of which computes an infinite homogeneous set for $f$.

- For every $\Delta^0_2$ set $A$, there is a computable stable $f : [\omega]^2 \rightarrow 2$ such that every infinite homogeneous set of $f$ is a subset or cosubset of $A$. 
Theorem (Hirschfeldt, 2006)

Every $\Delta^0_2$ set has an infinite subset or cosubset $H \lt_T \emptyset'$. 
Stable Ramsey’s theorem

Theorem (Hirschfeldt, 2006)

Every $\Delta^0_2$ set has an infinite subset or cosubset $H <_T \emptyset'$. 

Theorem (Downey, Hirschfeldt, Lempp, and Solomon, 2001)

There is a $\Delta^0_2$ set with no low infinite subset or cosubset.
Stable Ramsey’s theorem

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Stable Ramsey’s theorem

Definition (Mileti, 2005)

Let \( d \) be a degree.

- Let \( \mathcal{C}_d \) be the class of all \( \Delta^0_2 \) sets with an infinite subset or cosubset of degree \( \leq d \).
- Say \( d \) is s-Ramsey if \( \mathcal{C}_d = \Delta^0_2 \).
Stable Ramsey’s theorem

**Definition (Mileti, 2005)**

Let $d$ be a degree.

- Let $C_d$ be the class of all $\Delta^0_2$ sets with an infinite subset or cosubset of degree $\leq d$.
- Say $d$ is *s-Ramsey* if $C_d = \Delta^0_2$.

**Theorem (Mileti, 2005)**

- *There is no s-Ramsey degree* $d < 0'$.
- *There is no low$_2$ s-Ramsey degree.*
A martingale is a map $M : 2^{<\omega} \to \mathbb{Q}^{\geq 0}$ such that for all $\sigma \in 2^{<\omega}$,

$$M(\sigma) = \frac{M(\sigma 0) + M(\sigma 1)}{2}$$

A martingale succeeds on a set $X$ if $\limsup_{n \to \infty} M(X \upharpoonright n) = \infty$.

A class $\mathcal{C}$ of $\Delta^0_2$ sets is $\Delta^0_2$ null if there is a martingale $M \leq_T \emptyset'$ that succeeds on every $X \in \mathcal{C}$.
Measure

**Theorem (Hirschfeldt and Terwijn, 2008)**

*The class of low sets is not \( \Delta^0_2 \) null.*

**Corollary**

*The class of \( \Delta^0_2 \) sets having a low infinite subset or cosubset is not \( \Delta^0_2 \) null.*
Definition

A degree $d$ is *almost s-Ramsey* if $C_d$ is not $\Delta^0_2$ null.
Definition

A degree $d$ is *almost s-Ramsey* if $C_d$ is not $\Delta^0_2$ null.

Theorem (Dzhafarov)

*There is no almost s-Ramsey degree $d < 0'$.***
Theorem (Dzhafarov)

There is an almost $s$-Ramsey degree that is not $s$-Ramsey.

Proof idea.

Fix $A \in \Delta^0_2$ with no low infinite subset or cosubset.

Let $M_0, M_1, \ldots$ list all $\emptyset'$-computable martingales.

For all $i$, fix $L_i$ on which $M_i$ does not succeed and $\bigoplus_{j \leq i} L_i$ is low.

Let $D[0] = L_0$.

If $(\exists x \in A)(\exists F \text{ finite})[F[0] \restriction \max F = D[0] \restriction \max F \land \Phi_F(x) \downarrow = 1]$:

let $r_1 = \phi_F(x)$, make $F \subset D$, and let $D[1] = F[1] \cup \{x \in L_1 : x > r_1\}$.

If not, let $D[1] = L_1$.

Continue.
Theorem (Dzhafarov)

There is an almost $s$-Ramsey degree that is not $s$-Ramsey.

Proof idea.

- Fix $A \in \Delta^0_2$ with no low infinite subset or cosubset.
- Let $M_0, M_1, \ldots$ list all $\emptyset'$-computable martingales.
- For all $i$, fix $L_i$ on which $M_i$ does not succeed and $\bigoplus_{j \leq i} L_i$ is low.
- Let $D[0] = L_0$.
- If $(\exists x \notin A)(\exists F \text{ finite})[F[0] \upharpoonright \max F = D[0] \upharpoonright \max F \land \Phi^F_0(x) \downarrow = 1]$:
  - let $r_1 = \varphi^F_0(x)$, make $F \subseteq D$, and let $D[1] = F[1] \cup \{x \in L_1 : x > r_1\}$.
- If not, let $D[1] = L_1$.
- Continue.
Measure

Theorem (Dzhafarov)

There is an almost $s$-Ramsey degree $d < 0''$ which is not $s$-Ramsey.
Theorem (Dzhafarov)

There is an almost s-Ramsey degree $d < 0''$ which is not s-Ramsey.

Question

Is there a low$_2$ almost s-Ramsey degree?
Recall the following principles:

(COH) For every family $\langle X_i : i \in \mathbb{N} \rangle$ there is a set $C$ such that for all $i$, $C \subseteq^* X_i$ or $C \subseteq^* \overline{X_i}$.

(DNR) For every set $X$ there is an $f$ such that for all $e$, $\Phi^X_e(e) \neq f(e)$.

Theorem (Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman, 2006) Over RCA$_0$, $\text{SRT}^2_2 = \Rightarrow \text{DNR}$.

Question Over RCA$_0$, does $\text{SRT}^2_2 = \Rightarrow \text{WKL}_0$ or $\text{SRT}^2_2 = \Rightarrow \text{COH}$?
Recall the following principles:

(COH) For every family \( \langle X_i : i \in \mathbb{N} \rangle \) there is a set \( C \) such that for all \( i \), \( C \subseteq^* X_i \) or \( C \subseteq^* \overline{X_i} \).

(DNR) For every set \( X \) there is an \( f \) such that for all \( e \), \( \Phi_e^X(e) \neq f(e) \).

**Theorem (Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman, 2006)**

Over \( \text{RCA}_0 \), \( \text{SRT}^2_2 \implies \text{DNR} \).

**Question**

Over \( \text{RCA}_0 \), does \( \text{SRT}^2_2 \implies \text{WKL}_0 \) or \( \text{SRT}^2_2 \implies \text{COH} \)?
Define the following principles:

(SRAM) For every set $X$ there is a set $Y$ such that every stable coloring $f \leq_T X$ has an infinite homogeneous set $H \leq_T Y$.

(ASRAM) For every set $X$ there is a set $Y$ such that for every $X$-computable approximation to a martingale $M$ there is a stable coloring $f \leq_T X$ on which $M$ does not succeed and which has an infinite homogeneous set $H \leq_T Y$.

(ASRT$^2_2$) For every approximation $M_s$ to a martingale $M$ there is a stable coloring $f \leq_T M_s$ on which $M$ does not succeed and which has an infinite homogeneous set.
Theorem (Dzhafarov)

Over RCA₀, the following implications hold:

\[ \text{ACA₀} \rightarrow \text{SRAM} \rightarrow \text{SRT}^2 \rightarrow \text{ASRT}^2 \rightarrow \text{DNR} \rightarrow \text{RCA₀} \]

(Double arrows are not reversible in RCA₀.)
Thank you for your attention.