Degenerate fibres in the Stone-Čech compactification of the universal bundle of a finite group:
An application of homotopy theory to general topology

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1 Introduction

If $p : E \to B$ is a continuous surjection between completely regular spaces E and B, we may apply the Stone-Čech compactification functor β to obtain a surjection $\beta p : \beta E \to \beta B$. It is well-known that if $E = B \times F$ where F is a finite set and p is projection on the first factor, then $\beta E = \beta B \times \beta F$, and βp is again projection on the first factor. In this paper, we apply β to an n-fold covering map, that is, a local homeomorphism $p : E \to B$ such that $p^{-1}(b)$ has cardinality n for any $b \in B$. We show that the fibres of βp, while never exceeding n points, may degenerate to sets whose cardinality properly divides n (in contrast with the more usual, explosive sort of Stone-Cech "pathology").

What is particularly striking about this phenomenon is that it depends on a homotopy invariant, the sectional category, of the map p. In particular, we show that if $p : E \to B$ is an H-bundle where H is a finite group, then βp has degenerate fibres iff p has infinite sectional category. In the special case where G is a p-group and $p : EG \to BG$ is the universal G-bundle, we can show more precisely that every possible G-orbit occurs somewhere as a fibre of βp. The proof of this theorem uses a weak form of the so-called generalized Sullivan conjecture [5, 1], which is now a theorem of H. Miller. It is interesting to see the structure group G manifesting itself in this way even though it is not explicitly part of the data fed to the Stone-Čech functor.

Algebraic and general topology have grown far appart in recent years. Accordingly, we have tried to include enough detail to make the paper essentially self contained. Regarding the Stone-Čech compactification, we use few facts beyond the basic definitions. Readers unfamiliar with universal G-bundles should bear in mind the simplest non-trivial example, $G = \mathbb{Z}/2\mathbb{Z}$. The double cover of the infinite real projective space $\mathbb{RP}^\infty$ is a universal $\mathbb{Z}/2\mathbb{Z}$-bundle. No other finite group has a universal bundle which is so easily pictured; it is this case which motivated some of our terminology.

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2 Background on β

We record here some notations and facts we use regarding Stone-Čech compactifications. For further information see [8].

A space X is *completely regular* if for any point $x \in X$ and any closed set $A \subset X$ not containing x, there exists $f \in C(X)$ such that $f(x) = 0$ and $f(A) = 1$, i.e., real valued functions separate points from closed sets. A completely regular Hausdorff space is called *Tychonoff* space. It is not hard to see that that a regular space in which every point has a Tychonoff neighborhood is already a Tychonoff space. (That one needs complete regularity, as opposed merely to Hausdorff-ness, is illustrated by the upper half-plane with the Half-Disc topology [7].

If X is a topological space, we write $C(X)$ for the ring of real valued continuous functions on X. The *zero-set* of $f \in C(X)$ is the set $\mathbf{Z}(f) = \{x : f(x) = 0\}$. The family of all zero sets is denoted $\mathbf{Z}[X]$. The *cozero-set* $\mathbf{Cz}(f)$ of f is $X \setminus \mathbf{Z}(f)$. Since $\mathbf{Z}(fg) = \mathbf{Z}(f) \cup \mathbf{Z}(g)$ and $\mathbf{Z}(|f| + |g|) = \mathbf{Z}(f) \cap \mathbf{Z}(g)$, the family of all zero-sets of X is closed under finite intersections and unions. It is also useful to observe, for example, that $\{x : f(x) \geq 0\} = \mathbf{Z}(f \wedge 0)$.

A non-empty family $\mathcal{F}$ of zero-sets of X is a *z-filter* if it is closed under finite intersection, contains all zero-sets containing any one of its members, and does not contain the empty set. A maximal z-filter is called a *z-ultrafilter.* It is customary to denote ultrafilters by small letters u, v, etc. Note that two z-ultrafilters u and v differ iff there are z-sets $A \in u$ and $B \in v$ with $A \cap B = \emptyset$.

The collection of all z-ultrafilters on X is denoted βX. For a zero-set F of X, set $\overline{F} = \{u : u \in \beta X \text{ and } F \in u\}$. To topologize βX, take as a basis of closed sets the collection of sets $\overline{F}$ where F is closed in X. There is a natural map $i : X \to \beta X$ assigning to $x \in X$ the principal z-ultrafilter $i(x) = \{F : F \in \mathbf{Z}[X] \text{ and } x \in F\}$. When X is Tychonoff, the map i is a homeomorphism onto its image, and we identify X with $i(X)$. In this case, βX compactifies X: X is dense in βX, and βX is a compact Hausdorff space. Moreover, βX is universal among compactifications of X in the sense that if Y compactifies X, then there is a unique continuous map from βX to Y which is constant on X.

More generally, if X and Y are spaces and $f : X \to Y$ is continous, we can extend f to a continous map $\beta f : \beta X \to \beta Y$ by defining $\beta f(u) = \{F \in \mathbf{Z}(Y) | f^{-1}(F) \in u\}$. It is straightforward that $\beta(f \circ g) = \beta f \circ \beta g$.

Lemma 1 *Let E and B by Tychonoff. If $p : E \to B$ is surjective, then so is the map βp.*

Proof Every z-ultrafilter on B lifts to E. Since βE is compact, the image of βp is also compact, hence closed. The image of βp contains B which is dense in βB, therefore βp is surjective. ■

3 Stone-Čech Compactification of G-Bundles

For the remainder of this paper, unless otherwise stated, G denotes a non-trivial finite group of order n. By a *G-bundle*, we always mean a covering map $p : E \to B$ where E is a G-space, B is E/G and p is the canonical surjection.

We shall apply the Stone-Čech functor to such a G-bundle. As we shall see, G continues to act transitively on the fibres of $\beta p : \beta E \to \beta B$. A fibre will therefore have fewer than n points iff it contains a z-ultrafilter u fixed by some non-trivial element of G. The main results of this section (Theorems 1 and 2) give conditions under which an ultrafilter $u \in \beta E$ is fixed by a non-trivial subgroup of G.

Since the Stone-Čech functor is only well-behaved on Tychonoff spaces, we want both spaces E and B to be Tychonoff. In fact, in the context of Hausdorff spaces it is only necessary to assume that B is Tychonoff:

Lemma 2 *Let $p : E \to B$ be a local homeomorphism. Suppose B is Tychonoff and E is Hausdorff. Then E is Tychonoff.*

Proof The space E is locally Tychonoff, so it will suffice to show that E is regular. A space is regular if every point has a neighborhood basis consisting of closed sets. Let x be a point of E. Let U be an open neighborhood of x such that $p|_U$ is a homeomorphism. Let $\{N_i\}_{i\in I}$ be a neighborhood basis of closed sets for the point $p(x)$ in $p(U)$. Then $\{U \cap p^{-1}(N_i)\}_{i\in I}$ is clearly a neighborhood basis of of x.

Showing that E is regular is now just a matter of showing that each set $\tilde{N}_i = U \cap p^{-1}(N_i)$ is closed in E (and not just in U.) Take $y \in \mathrm{cl}(\tilde{N}_i)$. Since $\tilde{N}_i \subset p^{-1}(N_i)$, and $p^{-1}(N_i)$ is closed, $y \in p^{-1}(N_i))$, that is, $p(y) \in N_i$.

Thus there is a point $x \in \tilde{N}_i$ such that $p(x) = p(y)$. We claim $x = y$. Otherwise, being Hausdorff, E contains disjoint open sets V_1 and V_2 containing x and y respectively. Since $y \in \mathrm{cl}(\tilde{N}_i)$, $y \in \mathrm{cl}(V_1) \cup \mathrm{cl}(\tilde{N}_i \setminus V_1)$. But clearly $y \notin \mathrm{cl}(V_1)$. Also, as in the previous paragraph, $y \notin \mathrm{cl}(\tilde{N}_i \setminus V_1)$, since $p(y) \notin p(\tilde{N}_i \setminus V_1)$. Thus we must have $y = x \in \tilde{N}_i$, so $\tilde{N}_i$ is closed. ■

Comment To see that the hypothesis that E be Hausdorff is essential consider the real line with the origin "doubled."

Henceforth, we suppose that B is Tychonoff.

Lemma 3 *Let $p : E \to B$ be a G-bundle. Then the fibres of the map βp are* *orbits under the action of G. In particular, the map βp is at most n-to-one.*

Proof Let u, v be z-ultrafilters in βE. Suppose $v \neq gu$ for any $g \in G$. We wish to show that $(\beta p)(v) \neq (\beta p)(u)$. For each $g \in G$, choose $C_g \in v$ such that $C_g \notin g^{-1}u$. Then $C = \bigcap_{g \in G} C_g \in v$, so $p(C) \in (\beta p)(v)$. On the other hand, $gC \notin u$ for any $g \in G$. Since G acts transitively on the fibres of p, $p^{-1}(p(C)) = \bigcup_{g \in G} gC$. But as u is a z-ultrafilter not containing any of the (finitely many) sets gC, u does not contain this last set either. Thus, $p(C) \notin (\beta p)(u)$. ∎

Let H be a subgroup of G.

Definition A cozero-set $U \subset E$ is H**-antipolar** if U is disjoint from hU for some $h \in H$. A zero-set $F \subset E$ is H**-equatorial** if $F \cup hF = E$ for some $h \in H$, that is, if F is the complement of an H-antipolar set.

Theorem 1 *The following conditions are equivalent:*

1. *The z-ultrafilter u is fixed by all the elements of H.*
2. *u contains every H-equatorial set.*

Proof
$1 \Rightarrow 2$: Suppose u is fixed by H. Let F be an H-equatorial set, so $F \cup hF = E$ for some $h \in H$. Then either $F \in u$ or $hF \in u$, since u is a z-ultrafilter, But since u is fixed by H, $hF \in u$ implies $F \in u$. Hence $F \in u$.

$2 \Rightarrow 1$: Suppose that $hu \neq u$ for some $h \in H$. Hence there are disjoint zero-sets $C_1 \in u$ and $C_2 \in hu$. Then $C := C_1 \cap h^{-1}C_2$ is a zero-set in u disjoint from hC. Let $f \in C(E)$ be a non-negative function with zero-set C. Define $F = \{x : \ f(x) \geq f(hx)\} \cup \{x : f(x) \geq f(h^{-1}x)\}$. Then F is a zero-set, but $F \cap C = \emptyset$, so $F \notin u$. Nevertheless $F \cup hF = E$, so F is H-equatorial. ∎

Corollary 1 *H fixes points of βE if and only if the the class of H-equatorial sets has the finite intersection property.*

Proof If the H-equatorial sets satisfy the finite intersection property, then they generate a z-filter which may be extended to a z-ultrafilter. ■

Definition A cozero-set $U \subset E$ is H**-sectional** if U is disjoint from hU for all $h \in H$. A zero-set $F \subset E$ is H**-cosectional** if $F \cup hF = E$ for all $h \in H$, that is, if F is the complement of an H-sectional set.

Remark: Let $p_H : E \to E/H$ be the natural quotient map. A set $U \subseteq E$ is H-sectional if and only if U is the image of a section of p_H over a cozero-subset of E/H.

Theorem 2 *The following conditions are equivalent:*

1. *The z-ultrafilter u is fixed by some element of H.*
2. *u contains all H-cosectional sets.*

Proof
$1 \Rightarrow 2$: Suppose $hu = u$ for some non-trivial $h \in H$. Let F be an H-cosectional set, so $F \cup hF = E$. Then either $F \in u$ or $hF \in u$, since u is a z-ultrafilter. But since $hu = u$, $hF \in u$ implies $F \in u$. Hence $F \in u$.

$2 \Rightarrow 1$: Suppose that $hu \neq u$ for every non-trivial $h \in H$. Hence there is a disjoint family of zero-sets $G_h \in hu, h \in H$. Then $C = \bigcap_{h \in H \setminus \{e\}} h^{-1}G_h$ is a zero-set in u disjoint from every $hC, h \in H \setminus \{e\}$. Let $f \in C(E)$ be a non-negative function with zero-set C. Define $F_h = \{x : \ f(x) \geq f(hx)\}$. Define $F = \bigcup_{h \in H \setminus \{e\}} F_h$. Then F is a zero-set, but $F \cap C = \emptyset$, so $F \notin u$. Nevertheless $F \cup hF = E$, so F is H-cosectional. ■

Corollary 2 *H fails to act freely on βH if and only if the the class of H-cosectional has the finite intersection property.* ■

4 Fibre Degeneration and Sectional Category

A G-bundle $p : E \to B$ is principal iff G acts freely on E. A *universal (principal) G-bundle* is a numerable principal G-bundle $p : EG \to BG$ having the property that, for any numerable principal G-bundle $q : E \to B$ there exists a map $\phi : B \to BG$, unique up to homotopy, such that $q = \phi^* p$. This is equivalent to the condition that EG be contractible. (A locally trivial bundle is *numerable* if it is trivialized by a covering which supports a locally finite partition of unity; see [3]. In particular, all locally trival bundles over a paracompact base are numerable.)

In this section, we shall prove the following

Theorem 3 *If $p_G : EG \to BG$ is a universal principal G-bundle with BG paracompact and H is a subgroup of G, then βEG contains points fixed by a non-trivial subgroup of H.*

Our main tool here is the notion of the *sectional category* of a bundle $p : E \to B$. This is the minimum cardinality of a covering of B by open sets over each of which p has a section. The standard reference on sectional category is [6], where the term used is *genus.*[1] The term *sectional category* originates with James' survey of Lusternik-Schnirelmann category [4].

In light of Theorem 2 and its corollary, there exist points in βE fixed by a non-trivial element of $H \leq G$ iff the collection of H- sectional sets has the finite intersection property – equivalently, iff there is no covering of E by finitely many H-cosectional sets. Now, as remarked in section 3, $U \subseteq E$ is H-cosectional iff the canonical quotient map $p_H : E \to E/H$ has a section over $p_H(U) = U/H$. Since $p_H : E \to E/H$ is an H-bundle, we conclude that *the H-sectional sets generate a non-trivial filter iff p_H has infinite sectional category.*

We now gather some facts concerning sectional category which we will use in the sequel.

[1]Many arguments about fibre bundles hinge at some point on the availability a partition of unity (or something similar) and therefore require suitable hypotheses. Authors deal with this technicality in different ways. The approach of Švarc in [6] employs a nonstandard definition of *open covering* more restrictive than the usual one. Sectional category in our sense thus only bounds sectional category in his sense from below, in general. The notion of a numerable bundle serves a similar purpose. The most common solution is just to take all base spaces to be paracompact.

If $p : E \to B$ is a G-bundle, *G-category* of p is the minimum cardinality of a covering of B by open sets over each of which p is trivial as a G-bundle. Since a principal G-bundle is trivial iff it has a section, sectional category and G-category coincide for principal G-bundles.

The *Lusternik-Schnirelmann category* of a space X, (L-S category for short) is the minimal cardinality of an open cover $\{U_i\}$ such that each U_i is contractible in X. The L-S category of X is a homotopy invariant (see Prop. (1.1) of [4]).

Lemma 4 *The G-category of* any *universal principal G-bundle $p : E \to B$ coincides with the L-S category of B.*

Proof Let U be an open set in B. If U is contractible in B, p has a section over U.

If p has a section over U, then by the uniqueness aspect of the universal property of the bundle (pullbacks of numerable bundles always being numerable), the inclusion map from U to B is homotopic to a constant map. That is, U is contractible in B. ∎

Thus, all universal principal G-bundles have the same sectional category.

The n-fold *fibre join* of a bundle $p : E \to B$ is the bundle $p^{(n)} : E^{(n)} \to B$, where $E^{(n)}$ consists of points of the form $x = (t_1 e_1, \ldots, t_n e_n)$. Here $t_1, \ldots, t_n$ are non-negative real numbers (which we shall refer to as *weights*), such that $t_1 + \cdots + t_n = 1$, $e_i \in E$ with $p(e_1) = \ldots = p(e_n) := p^{(n)}(x)$, and we take $t_i e_i$ to be independent of e_i when $t_i = 0$.

There is a well-known construction due to Milnor of a universal G-bundle (for G discrete), obtained by taking for EG the infinite fibre join $G * G * \cdots$ of G (i.e., the direct limit of the fibre joins $G^{(n)}$), and for BG, the quotient EG/G.

Fact 1 (Prop. (8.1) in [4]; see also Theorem 3 in [6])
Let B be paracompact. Then the sectional category of $p : E \to B$ is $\leq n$ if and only if the n-fold fibre join $E^{(n)}$ admits a section.

For the reader's convenience we recall the short proof.

Suppose $U_1, ..., U_n$ is a covering of B by open sets over each of which p has a section σ_i. If $\{t_i\}$ is a partition of unity subordinate to $\{U_i\}$, then we obtain a well-defined section of $p^{(n)}$ by setting $\sigma(b) = (t_1(b)\sigma_1(b), \ldots, t_n(b)\sigma_n(b))$ for $b \in B$.

Conversely, if σ is a section of $p^{(n)}$, then we have open sets U_n of B where the n^{th} weight is non-zero, and over each U_i a section of p given by $\sigma_i = \pi_i \circ \sigma$. ■

Fact 2 (Prop. 50 in [6]) *If a discrete group G contains elements of finite order, then the G-category of the bundle $p : EG \to BG$ is infinite.* [2]

Proof of Theorem 3: Let $p : EG \to BG$ be a universal principal G-bundle. Let H be a non-trivial subgroup of G, possibly G itself. Since EG is contractible and the action of H on EG is free, the natural map $p_H : EG \to EG/H$ is a universal principal H-bundle. Since H is finite, by Lemma 4 and Fact 2, $p_H : EG \to EG/H$ has infinite sectional category. Thus, as remarked above, the H-sectional sets generate a non-trivial filter. By Corollary 2, there are points of βEG that are fixed by some non-trivial subgroup of H. ■

Note that in particular, if H is prime cyclic, then H itself must fix some points of βEG. Thus for each such H, βp has degenerate fibres containing no more than $[G : H]$ points.

5 H-Fixed Points and an Auxiliary Bundle

So far we have seen that the infinite sectional category of the universal H-bundle entails the finite intersection property for the family of H-cosectional sets which in turn guarantees the existence of points in βEG fixed by some subgroup of H, by Corollary 2.

[2] The proof of Prop 50 in [6] contains a misprint: The reference to Theorem 10 should be to Theorem 17.

We could apply Corollary 1 to prove the existence of points in βEG fixed by H itself if we knew that the family of H-equatorial sets had the finite intersection property, or equivalently, that EG can't be covered by finitely many H-antipolar sets. In this section, we show that this is in turn equivalent to a certain auxiliary bundle's having infinite sectional category.

For a finite H-set F, we write ΔF for the simplex generated by F. Since ΔF carries an action by H, we may speak of H-antipolar subsets of ΔF. We denote by λF the subcomplex of ΔF which is the union of all H-antipolar faces. Given an H-bundle $p : E \to B$, we may perform this construction on each fibre to obtain an H-bundle, which we denote by $\lambda p : \lambda E \to B$.

Lemma 5 *Let $p : B \times F \to B$ be projection onto B, with B normal and $F = \{v_1, \ldots, v_n\}$ discrete. Let V be an open set in $B \times F$. For $x \in B$, let $Q(x)$ denote the set $p(p^{-1}(x) \cap V) \subseteq F$. Then there exists a map $t : p(V) \to \Delta F$ such that $t(x) \in \Delta Q(x)$.*

Proof Set $U_j = \{x \in p(V) : |Q(x)| \leq j\}$. For $x \in U_1$, let $t(x)$ be the unique element of $Q(x)$. Now we proceed by induction on j. Thus we suppose we have defined t on U_j so that for every $A \subset F$ with $|A| = j$, if $x \in \mathrm{cl}(Q^{-1}(A))$ then $t(x) \in \Delta A$. We must show that it is possible to extend t continuously to U_{j+1} so that for every $A \subset F$ with $|A| = j + 1$, if $x \in \mathrm{cl}(Q^{-1}(A))$ then $t(x) \in \Delta A$.

Any function continuous on the closed set U_j and on each of the finitely many closed sets $\mathrm{cl}(Q^{-1}(A)) \cap U_j$ ($A \subset F$ with $|A| = j + 1$) is continuous on all of U_{j+1}. Thus it is sufficient to show that for each $A \subset F$ with $|A| = j + 1$, t may be extended continuously from $\mathrm{cl}(Q^{-1}(A)) \cap U_j$ to $\mathrm{cl}(Q^{-1}(A))$ with values in ΔA.

For each $v_i \in F \setminus A$, the coordinate function of t associated with v_i is identically zero on $\mathrm{cl}(Q^{-1}(A)) \cap U_j$, so we simply extend by zero.

Now we apply the Tietze extension theorem to the remaining coordinate functions, those associated to vertices in A. This yields a preliminary function $s : \mathrm{cl}(Q^{-1}(A)) \to [0, 1]^A$ agreeing with t on $\mathrm{cl}(Q^{-1}(A)) \cap U_j$. Finally, to extend t to $\mathrm{cl}(Q^{-1}(A))$, we compose s with a retraction from $[0, 1]^A$ onto ΔA. ∎

Lemma 6 *Let $p : E \to B$ be an H-bundle, with H acting transitively on the fibres and the space B normal. Let V be an H-antipolar subset of E. Then there is a partial section $t : p(V) \to \lambda E$ of $\lambda p : \lambda E \to B$ such that $t(x) \in \Delta(p^{-1}(x) \cap V)$ for all $x \in p(V)$.*

Proof The special case where E has the form $B \times F$ for a transitive H-set F follows immediately from Lemma 5. We now reduce the general case to this special case. Take E' to be another copy of E. Consider the pullback

$$\begin{array}{ccc} E' \times_B E & \xrightarrow{q_2} & E \\ \downarrow q_1 & & \downarrow p \\ E' & \xrightarrow{p'} & B \end{array} .$$

We regard the map $q_1 : E' \times_B E \to E'$ as an H-bundle by letting H act in the usual way on E, but trivially on E'. Moreover $E' \times_B E$ is a trivial bundle since it admits the diagonal map as a section. Thus there is a partial section $s : p'^{-1}(p(V)) \to \lambda E' \times_B E$ of $p : \lambda E' \times_B E \to E'$ such that $s(x) \in \Delta(q_1^{-1}(x) \cap q_2^{-1}(V))$ for all $x \in p'^{-1}p(V)$.

Letting H now act in the usual way on E', but trivially on E, we may view the map $q_1 : E' \times_B E \to E'$ as H-equivariant. Observe that the quotient of the bundle q_1 by this action is canonically isomorphic to the bundle $p : E \to B$. Thus, giving a partial section $t : p(V) \to \lambda E$ amounts to giving an H-equivariant partial section from $p^{-1}(p(V))$ to $\lambda E' \times_B E$.

While the partial section $s : p^{-1}(p(V)) \to \lambda E' \times_B E$ may not be H-equivariant, the fibrewise barycenter $\overline{s}$ of its translates hsh^{-1}, h running over H, will be. Clearly $\overline{s}(x) \in \Delta(q_1^{-1}(x) \cap q_2^{-1}(V))$. Letting $t : p(V) \to \lambda E$ be the quotient of $\overline{s}$ by H we get $t(x) \in \Delta(p^{-1}(x) \cap V)$ for all $x \in p(V)$ as desired. ■

Theorem 4 *Let $p : E \to B$ be an H-bundle, with H acting transitively on the fibres and the space B normal. Then following are equivalent:*

1. *$\lambda p : \lambda E \to B$ has infinite sectional category.*
2. *E can not be covered by finitely many H-antipolar sets.*

Proof
$1 \Rightarrow 2$: Suppose that E is covered by finitely many H-antipolar sets V_i. Then the open sets $U_i = p(V_i)$ cover B. Lemma 6 allows us to find a partial section t_i of $\lambda p : \lambda E \to B$ over each U_i.

Comment The hypothesis of normality permits the use of the Tietze extension theorem; it will play no role in the reverse implication.

$2 \Rightarrow 1$: Suppose $\lambda p : \lambda EH \to BH$ has finite sectional category. Then there is a finite open cover $\{U_1, \ldots, U_n\}$ of BH and a partial section s_i of $\lambda p : \lambda EH \to BH$ over each U_i. Each s_i determines an H-antipolar open subset V_i of EH as follows. If F_x is the fibre of EH over $x \in BH$, $V_i \cap F_x$ shall be the set of vertices of the smallest face containing $s_i(x)$. The V_i and all their H-translates cover EH. ■

Comment One might bypass Theorem 3 by working work with a different auxiliary bundle, one whose fibres are finite non-Hausdorff spaces with one point representing each H-antipolar set in $p^{-1}(x)$, the closure of that point consisting of the points that represent the various subsets.

6 p-Groups and the Sullivan Conjecture

Applied to a universal bundle $p : EH \to BH$, the results of the previous section tell us that βEH (and hence, βEG) has points fixed by H iff λp has infinite sectional category. Unfortunately, because of its complexity, the bundle $\lambda p : \lambda EH \to BH$ is difficult to work with directly; we do not yet know if these bundles have infinite sectional category for every H. On the other hand, it is sufficient for our purposes to show that a larger bundle has infinite sectional category. If H is a p-group, this proves feasible.

We construct a bundle $\Lambda EH \to BH$ as follows. For a finite H-space F, let ΛF denote the subcomplex of ΔF consisting of all but the cell of highest dimension. Being H-invariant, ΛF is itself an H-space and ΛF indeed contains λF as a subcomplex since the face ΔF is certainly not antipolar. Let $\Lambda EH \to BH$ denote the sub-bundle of $\Delta p : \Delta EH \to BH$ obtained by performing this construction on each fibre.

Now assume that BH is paracompact. We claim that Λp has infinite sectional category. In light of Fact 1 from section 4, it is sufficient to show that the n–fold fibre join of $\Lambda p : \Lambda EH \to BH$ has no section for any n.

Consider a particular fibre F of $EH \to BH$. Enumerate the points of F from 1 to $|F|$. By definition, each point of the n–fold join of ΛF may be represented by weighted n–tuple of the form

$$(t_1(s_{1,1}, \ldots, s_{|F|,1}), \ldots, t_n(s_{1,n}, \ldots, s_{|F|,n})$$

satisfying

(i) $t_j, s_{i,j} \in [0, 1]$ for $j = 1, \ldots, n$, $i = 1, \ldots, |F|$;
(ii) $\sum_{j=1}^{n} t_j = 1$;
(iii) $\sum_{i=1}^{|F|} s_{i,j} = 1$ for $j = 1, \ldots, n$;
(iv) for each j there exists an i such that $s_{i,j} = 0$;

with two such expressions identified precisely when they differ only at coordinates of weight 0.

Now to each such weighted n–tuple associate the $n \times |F|$ matrix $|a_{i,j}|$ where $a_{i,j} = t_j s_{i,j}$. One easily checks, first, that this matrix depends only on the point of $\Lambda F^{(n)}$, not on the particular representation of the point by a weighted n–tuple, and second, that the matrix determines the point. The $n \times |F|$–matrices which arise in this fashion are exactly those such that

(i) each $a_{i,j} \in [0, 1]$;
(ii) each row contains at least one entry equal to 0;
(iii) $\sum_{j=1}^{n} \sum_{i=1}^{|F|} a_{i,j} = 1$.

Of course H acts on the set of such matrices by permuting columns. This action can have no fixed points. Indeed, since the action of H on F is transitive and every row contains a 0, a fixed matrix would have to vanish identically, contrary to (iii).

Pulling $(\Lambda p)^{(n)} : (\Lambda EH)^{(n)} \to BH$ back along $p : EH \to BH$ gives the bundle $p \times_{BH} (\Lambda p)^{(n)} : EH \times_{BH} (\Lambda EH)^n \to EH$. Since EH is contractible, this bundle must be trivial, hence isomorphic to $p_1 : EH \times (\Lambda H)^n \to EH$, where the map is projection onto the first factor. In other words, $(\Lambda p)^{(n)} : (\Lambda EH)^{(n)} \to BH$ is isomorphic to $p_1/H : (EH \times (\Lambda H)^n)/H \to EH/H$. Thus a section of $(\Lambda p)^{(n)} : (\Lambda EH)^{(n)} \to BH$ is equivalent to an equivariant map $EH \to (\Lambda H)^{(n)}$.

On the other hand, an equivariant map from a point to $(\Lambda H)^{(n)}$ is the same thing as a fixed point. But we just saw that there are no fixed points for the action of H on $(\Lambda H)^{(n)}$. When H is a p-group, it is a consequence of Miller's version of the Sullivan conjecture that the homotopy fixed point space of $(\Lambda H)^{(n)}$, that is, $\mathrm{Hom}_H(EH, (\Lambda H)^{(n)})$ is homotopy equivalent to the fixed point space of $(\Lambda H)^{(n)}$ [[1], Theorem A]. Thus the homotopy fixed point space must be empty and there can be no equivariant map $EH \to (\Lambda EH)^{(n)}$. The sectional category of $\Lambda p : \Lambda EH \to BH$ is thus infinite and we have proved:

Theorem 5 *Let $EG \to BG$ be a universal G-bundle with BG a normal, paracompact space. Then for each p-subgroup H of G there are points in βEG fixed by H.* ■

A desire to extend this theorem to more general subgroups leads, by the considerations above, to the following problem:

Problem 1 *For what groups H, not p-groups, is the homotopy fixed point space* $\mathrm{Hom}_H(EH, (\lambda H)^{(n)})$ *empty for all n?*

7 Remarks on More General Groups

Passing to a larger bundle is equivalent to considering the finite intersection property for a larger family of zero-sets. In particular, call a cozero-set $U \subset EH$ H-**incomplete** if

$$\bigcap_{h \in H} hU = \emptyset \ ,$$

that is, if U misses at least one element in each fibre over BH. Call a zero-set H-**coincomplete** if its complement is H-incomplete. We have just seen that for p-groups H, the H-incomplete sets generate a non-trival filter. The H-incomplete filter is at least as large as the H-equatorial filter, so it is possible *a priori* that the former is trivial even when the latter is not. The following proposition gives a sufficient condition for these two filters to coincide.

Proposition *Assume that U is H-incomplete and misses not more than k points from any fibre that it meets, where $k^2 - k + 1 < n$. Then U can be covered with finitely many H-antipolar sets.*

Proof Let

$$\mathcal{S} = \{S \mid S \subset H \text{ and } |S| > n - k\} .$$

For each $S \in \mathcal{S}$, set $U_S = \bigcap_{h \in S} hU$. Note that if $p \in U$ and $S = \{h \mid p \in hU\}$, then $p \in U_S$. Thus, the sets U_S cover U. We now show that each U_S is H-antipolar. Fix $S \in \mathcal{S}$ and suppose that $j \in H$ is such that $jS \cup S \neq H$. Then there is some $x \in S$ with $j^{-1}x \notin S$, whence $j = x(x^{-1}j) \in (H \setminus S)(H \setminus S)^{-1}$. Since $|S| > n - k$, the set $(H \setminus S)(H \setminus S)^{-1}$ has no more than $k^2 - k + 1 < n$ elements. Thus, there is some $j \in H$ with $S \cup jS = H$. Since U is H-incomplete, $\bigcup_{h \in H} hU$ is empty and U_S is disjoint from jU_S, as desired. ∎

Thus if maximal H-incomplete sets miss only a few points from those fibres that they meet at all, the H-coincomplete and the H-equatorial filters will be equal.

Remark: Properties intermediate between H-antipolarity and H-incompleteness may be useful if Theorem 4 is to be extended to a larger class of groups. Some candidates might be
a) $\bigcap_{h \in K} hU = \emptyset$ for some cyclic proper subgroup K;
b) $\bigcap_{h \in K} hU = \emptyset$ for some proper subgroup K;
c) U meets no more than half the points in any orbit.
This last may be the most workable.

8 Loci of Degeneration

While all universal G-bundles are equivalent up to homotopy, the Stone-Čech functor will be sensitive to topological differences between such bundles. We will obtain our most precise results by concentrating on one particular model of the universal G-bundle, that constructed by Milnor. Recall that Milnor's EG is the infinite join $G * G * G * \cdots$; the quotient by the natural left action of G gives his BG.

Milnor's universal G-bundle has two advantages for us. First, it is metrizable. Second, if H is a subgroup of G, Milnor's $EH = H * H * H * \cdots$ sits inside EG as a closed subspace whose quotient is canonically of copy of Milnor's BH sitting inside Milnor's BG as a closed subspace. Actually there will be $|G/H|$ copies of EH sitting over BH: For each left coset gH the infinite join $gH * gH * gH * \cdots$ will be one such.

The degeneration of a fibre of $\beta EG \to \beta BG$ is measured by the conjugacy class of the stabilizer of any point in that fibre. When the hypotheses of Theorem 6 below are satisfied, as they are for p-groups G, we may associate to each conjugacy class of subgroups of G a non-empty locus in BG. We emphasize that the action of G on EG is not part of the data available to the Stone-Čech functor; rather the compactification process directly detects the symmetry of the bundle.

Theorem 6 *Let $EG \to BG$ be Milnor's model of the universal G-bundle. Suppose that for each subgroup H of G, there are fibres of $\beta p_H : \beta EH \to \beta BH$ consisting of a single point, at least whenever BH is normal. Then for each such H, there are points in βEG whose stabilizer is exactly H.*

We shall need a preliminary lemma. Given a family $\mathcal{F}$ of zero-sets in a space X, let $\mathcal{F}'$ denote the collection of zero-sets that have a non-empty intersection with every member of $\mathcal{F}$. If $\mathcal{F}$ is a filter, then $\mathcal{F}'$ is simply the union of the z-ultrafilters extending $\mathcal{F}$. Note that if $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{G}' \subseteq \mathcal{F}'$, and that $(\bigcup_i \mathcal{F}_i)' = \bigcap_i \mathcal{F}_i'$.

Recall that a space is *perfectly normal* if every closed set is a zero set. Metric spaces (e.g., Milnor's EG) are perfectly normal.

Lemma 7 *Let $\mathcal{F}$ be a filter on a perfectly normal space X. Then a closed set C belongs to $\mathcal{F}''$ if and only if every closed neighborhood of C belongs to $\mathcal{F}$.*

Proof Suppose $C \in \mathcal{F}''$. Let N be a closed neighborhood of C. Let D denote the closure of $X \setminus N$. As $C \cap D = \emptyset$, D itself must be disjoint from a set E in $\mathcal{F}$. Since $E \subseteq N$ and $\mathcal{F}$ is a filter, $N \in \mathcal{F}$.

Conversely, suppose that $C \notin \mathcal{F}''$. Then C is disjoint from some zero-set D that meets every element of $\mathcal{F}$. Being zero-sets, C and D are completely separated, so C has a closed neighborhood N disjoint from D. In particular N cannot be in $\mathcal{F}$. (Note that the converse makes no use of the hypothesis on X). ∎

Proof of Theorem 6: Suppose the theorem were false. Then there would be a subgroup H which was not the stabilizer of any point of βEG. Let $K_1, \ldots, K_n$ be an enumeration of the subgroups strictly larger than H. The H-equatorial filter is non-trivial, so for every H-stable point u there is a group K_i such that u is K_i-stable. That is, any ultrafilter extending the H-equatorial filter $\mathcal{H}$ extends at least one of the K_i-equatorial filters $\mathcal{K}_i$ – in other words, $\mathcal{H}' \subseteq \bigcup_i \mathcal{K}_i'$. Since $\mathcal{K}_i \subset \mathcal{K}_i''$ we will obtain a contradiction by showing show that

$$\bigcap_i \mathcal{K}_i \nsubseteq \mathcal{H}'' \ .$$

As noted above, over the canonical copy of BH in BG sits a canonical copy of EH together with its translates $g_k EH$ by elements of G (we assume g_1 is the identity). By normality, the $g_k EH$ have disjoint open neighborhoods U_k. If necessary, $U = U_1$ may be replaced by $\bigcap g_k^{-1} U_k$ so that U is disjoint from gU for any $g \notin H$. Since U is K_i-antipolar, its complement belongs to $\mathcal{K}_i$ for each i.

We claim that the complement of U is not an element of $\mathcal{H}''$. By Lemma 7, any closed neighborhood of an element of $\mathcal{H}''$ must be an element of $\mathcal{H}$. We will obtain our contradiction by producing an open set W, whose closure is contained in U, such that W cannot be covered by finitely many H-antipolar sets. Let W be the open neighborhood of EH obtained by using normality to separate EH from the complement of U. By hypothesis, there are fibres of any universal H-bundle consisting of a single point. Therefore the H-equatorial filter is not trivial, as it would have to be if W were covered by finitely many H-antipolar sets. ∎

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