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Weak approximation of killed diffusion using Euler schemes

Emmanuel Gobet

Laboratoire de Probabilités et Modèles Aléatoires. Case 7012, Université Paris VII. 2, place Jussieu, 75251 Paris Cedex 05, France

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Abstract

We study the weak approximation of a multidimensional diffusion $(X_t)_{0\leqslant t\leqslant T}$ killed as it leaves an open set D, when the diffusion is approximated by its continuous Euler scheme $(\tilde{X}_t)_{0\leqslant t\leqslant T}$ or by its discrete one $(\tilde{X}_{t_i})_{0\leqslant i\leqslant N}$, with discretization step T/N. If we set $\tau := \inf\{t>0\colon X_t \notin D\}$ and $\tilde{\tau}_c := \inf\{t>0\colon \tilde{X}_t \notin D\}$, we prove that the discretization error $\mathbb{E}_x[\mathbb{1}_{T<\tilde{\tau}_c} f(\tilde{X}_T)] - \mathbb{E}_x[\mathbb{1}_{T<\tau} f(X_T)]$ can be expanded to the first order in N^{-1}, provided support or regularity conditions on f. For the discrete scheme, if we set $\tilde{\tau}_d := \inf\{t_i>0\colon \tilde{X}_{t_i} \notin D\}$, the error $\mathbb{E}_x[\mathbb{1}_{T<\tilde{\tau}_d} f(\tilde{X}_T)] - \mathbb{E}_x[\mathbb{1}_{T<\tau} f(X_T)]$ is of order $N^{-1/2}$, under analogous assumptions on f. This rate of convergence is actually exact and intrinsic to the problem of discrete killing time.

Keywords: Weak approximation; Killed diffusion; Euler scheme; Error's expansion; Malliavin calculus; Itô's formula; Orthogonal projection; Local time on the boundary

1. Introduction

Let $(X_t)_{t\geqslant 0}$ be the diffusion taking its values in $\mathbb{R}^d$ defined by

$$X_t = x + \int_0^t B(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s, \tag{1}$$

where $(W_t)_{t\geqslant 0}$ is a Brownian motion in $\mathbb{R}^{d'}$. Let $\tau := \inf\{t>0\colon X_t\notin D\}$ be its first exit time from the open set $D\subset\mathbb{R}^d$. We are interested in computing $\mathbb{E}_x[\mathbb{1}_{T<\tau} f(X_T)]$, where T is a fixed time and f a measurable function, using a Monte-Carlo method. In other words, we focus on the law at time T of the diffusion killed when it leaves D. The results presented in this paper were announced without proofs in Gobet (1998b, 1999).

It is of interest to know how to evaluate such expectations, e.g. in financial mathematics. Indeed, let us consider a continuous monitored barrier option on the d-dimensional assets X_t, with characteristics f, T and D: it is a contract which gives to its owner the cashflow $f(X_T)$ at time T if the prices have stayed in D between 0 and T (the

E-mail address: gobet@ccr.jussieu.fr (E. Gobet)

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option remains active) and 0 otherwise. When the market is complete, the price of this option is unique and is given by the expectation under the neutral-risk probability of the discounted cashflow at time T: it leads to the computation of $\mathbb{E}_x[\mathbb{1}_{T<\tau} f(X_T)]$ (see Musiela and Rutkowski, 1998). The results we prove in this paper also enable us to approximate a continuous monitored barrier option by a discrete monitored one, and conversely (see Broadie et al., 1996).

Our approach is to evaluate $\mathbb{E}_x[\mathbb{1}_{T<\tau} f(X_T)]$ with a Monte-Carlo algorithm. Whenever this expectation can be viewed as a solution of a parabolic partial differential equation, we might prefer a Monte-Carlo method to a deterministic algorithm issued from numerical analysis if the dimension d is large ($d \geqslant 4$), if the operator is degenerate or if we need to compute $\mathbb{E}_x[\mathbb{1}_{T<\tau} f(X_T)]$ only for a few x and T (see the discussion in Lapeyre et al., 1998).

To evaluate the expectation of the functional of the diffusion, the simplest way to approximate the process is to use its discrete Euler scheme $(\tilde{X}_{t_i})_{0 \leqslant i \leqslant N}$ with discretization step T/N, defined if $t_i = i\,T/N$ is the ith discretization time by

$$\begin{aligned} \tilde{X}_0 &= x, \\ \tilde{X}_{t_{i+1}} &= \tilde{X}_{t_i} + B(\tilde{X}_{t_i})T/N + \sigma(\tilde{X}_{t_i})(W_{t_{i+1}} - W_{t_i}). \end{aligned} \tag{2}$$

Let $\tilde{\tau}_d := \inf\{t_i\colon \tilde{X}_{t_i} \notin D\}$ be its first exit time from D. We study in this paper the discretization error obtained by replacing $\mathbb{E}_x[\mathbb{1}_{T<\tau} f(X_T)]$ by $\mathbb{E}_x[\mathbb{1}_{T<\tilde{\tau}_d} f(\tilde{X}_T)]$.

A more sophisticated procedure consists in interpolating the previous discrete time process (2) into a continuous Euler scheme $(\tilde{X}_t)_{0 \leqslant t \leqslant T}$ by setting

$$\text{for } t \in [t_i, t_{i+1}) \quad \tilde{X}_t = \tilde{X}_{t_i} + B(\tilde{X}_{t_i})(t - t_i) + \sigma(\tilde{X}_{t_i})(W_t - W_{t_i}). \tag{3}$$

Note that the continuous Euler scheme is an Itô process verifying

$$\tilde{X}_t = x + \int_0^t B(\tilde{X}_{\varphi(s)})\,\mathrm{d}s + \int_0^t \sigma(\tilde{X}_{\varphi(s)})\,\mathrm{d}W_s, \tag{4}$$

where $\varphi(t) := \sup\{t_i\colon t_i \leqslant t\}$. Let $\tilde{\tau}_c := \inf\{t\colon \tilde{X}_t \notin D\}$ be its first exit time from D. We are also interested in studying the approximation of $\mathbb{E}_x[\mathbb{1}_{T<\tau} f(X_T)]$ by $\mathbb{E}_x[\mathbb{1}_{T<\tilde{\tau}_c} f(\tilde{X}_T)]$.

1.1. Monte-Carlo simulations

From the simulation point of view, the evaluation of $\mathbb{E}_x[\mathbb{1}_{T<\tilde{\tau}_d} f(\tilde{X}_T)]$ by a Monte-Carlo method works as follows: if $(Y^d_m)_{m \geqslant 1}$ is a sequence of independent copies of the random variable $Y^d := \mathbb{1}_{T<\tilde{\tau}_d} f(\tilde{X}_T)$, we approximate $\mathbb{E}_x[\mathbb{1}_{T<\tilde{\tau}_d} f(\tilde{X}_T)]$ by $(1/M)\sum_{m=1}^M Y^d_m$, for M large enough. The simulation of Y^d is straightforward whatever the dimension d is, because we only need realizations of $(\tilde{X}_{t_i})_{0 \leqslant i \leqslant N}$, which can be easily obtained using the simulation of N independent Gaussian variables for the increments $(W_{t_{i+1}} - W_{t_i})_{0 \leqslant i \leqslant N-1}$. For the continuous Euler scheme, the simulation of $Y^c := \mathbb{1}_{T<\tilde{\tau}_c} f(\tilde{X}_T)$ requires an additional step, because the process $(\tilde{X}_t)_{t_i \leqslant t \leqslant t_{i+1}}$ may have left D even if $\tilde{X}_{t_i} \in D$ and $\tilde{X}_{t_{i+1}} \in D$. We first obtain realizations of $(\tilde{X}_{t_i})_{0 \leqslant i \leqslant N}$ as before. Then, conditionally on the values $(\tilde{X}_{t_i})_{1 \leqslant i \leqslant N}$, $(\tilde{X}_t)_{t_j \leqslant t \leqslant t_{j+1}}$ has the law of some Brownian bridge. Using N extra independent Bernoulli variables, this enables us to simulate if $(\tilde{X}_t)_{0\leqslant t\leqslant T}$ has left D between two discretization times or not. Each parameter involved for the simulation of the Bernoulli variables is related to the quantity

$$\mathbb{P}(\forall t \in [t_i, t_{i+1}]\tilde{X}_t \in D / \tilde{X}_{t_i} = z_1, \tilde{X}_{t_{i+1}} = z_2) := p(z_1, z_2, T/N).$$

In the one-dimensional case, $p(z_1, z_2, \Delta)$ is the cumulative of the infimum and supremum of a linear Brownian bridge and has a simple expression (see Revuz and Yor, 1991, p.105):

1. if $D = (-\infty, b)$, we have
$$p(z_1, z_2, \Delta) = \mathbb{1}_{b>z_1}\mathbb{1}_{b>z_2}\left(1 - \exp\left(-2\,\frac{(b-z_1)(b-z_2)}{\sigma^2(z_1)\Delta}\right)\right);$$
2. if $D = (a, +\infty)$, we have
$$p(z_1, z_2, \Delta) = \mathbb{1}_{z_1>a}\mathbb{1}_{z_2>a}\left(1 - \exp\left(-2\,\frac{(a-z_1)(a-z_2)}{\sigma^2(z_1)\Delta}\right)\right);$$
3. if $D = (a, b)$, we have
$$\begin{aligned} p(z_1, z_2, \Delta) &= \mathbb{1}_{b>z_1>a}\mathbb{1}_{b>z_2>a}\left(1 - \sum_{k=-\infty}^{+\infty}\left[\exp\left(-2\frac{k(b-a)(k(b-a)+z_2-z_1)}{\sigma^2(z_1)\Delta}\right)\right.\right. \\ &\quad \left.\left. - \exp\left(-2\,\frac{(k(b-a)+z_1-b)(k(b-a)+z_2-b)}{\sigma^2(z_1)\Delta}\right)\right]\right). \end{aligned}$$

For higher dimension, in the case of a half-space, $p(z_1, z_2, \Delta)$ has also a simple expression (see Lépingle, 1993). But for more general domains, as far as we know, there are no tractable expressions for $p(z_1, z_2, \Delta)$. Nevertheless, the probability $p(z_1, z_2, \Delta)$ can be accurately approximated using an asymptotic expansion in Δ (see Baldi, 1995): this may be an appropriate way to evaluate $p(z_1, z_2, \Delta)$. So, in short, the discrete Euler scheme is very easy to implement for any dimension $d \geqslant 1$, whereas for the continuous Euler scheme, the simulation is simple in the one-dimensional case and more delicate in higher dimension.

1.2. Convergence results

Now, our main objective is to analyze the two errors

$$\mathscr{E}_{\mathrm{c}}(f, T, x, N) := \mathbb{E}_x[\mathbb{1}_{T<\tilde{\tau}_c} f(\tilde{X}_T)] - \mathbb{E}_x[\mathbb{1}_{T<\tau} f(X_T)] \tag{5}$$

and

$$\mathscr{E}_{\mathrm{d}}(f, T, x, N) := \mathbb{E}_x[\mathbb{1}_{T<\tilde{\tau}_{\mathrm{d}}} f(\tilde{X}_T)] - \mathbb{E}_x[\mathbb{1}_{T<\tau} f(X_T)] \tag{6}$$

as a function of N, the number of discretization steps. We first state an easy result, which shows that both errors tend to 0 when N goes to infinity under mild assumptions.

Proposition 1.1. *Assume that B and σ are globally Lipschitz functions, that D is defined by $D = \{x \in \mathbb{R}^d\colon F(x) > 0\}$, $\partial D = \{x \in \mathbb{R}^d\colon F(x) = 0\}$ for some globally Lipschitz function F. Provided that the condition* (C) *below is satisfied*

(C) : $\mathbb{P}_x(\exists t \in [0,T]\ X_t \notin D\,;\ \forall t \in [0,T]\ X_t \in \bar{D}) = 0,$

for all function $f \in C_b^0(\bar{D},\mathbb{R})$, we have

$$\lim_{N\to+\infty} \mathscr{E}_c(f,T,x,N) = \lim_{N\to+\infty} \mathscr{E}_d(f,T,x,N) = 0.$$

Remark 1.1. Condition (C) rules out the pathological situation where the paths may reach ∂D without leaving $\bar{D}$. A simple example of non-convergence in this situation is the following: take $d=1$, $D=(-\infty,\exp(1))$, $B(y)=y$, $\sigma(y)\equiv 0$, $X_0=1$, $T=1$ and $f\equiv 1$. In this deterministic situation, on the one hand, we have $\tau = 1$ (condition (C) is not fulfilled) and on the other, $\tilde{X}_t$ is an increasing function with $\tilde{X}_1 = (1 + N^{-1})^N < \exp(1)$, so that $\tilde{\tau}_c > 1$ and $\tilde{\tau}_d > 1$: thus, $\mathscr{E}_c(f,T,x,N)=\mathscr{E}_d(f,T,x,N)=1$ for all $N\geqslant 1$.

Remark 1.2. If D is of class C^2 with a compact boundary, the existence of such a function F holds. Indeed, on a neighbourhood of ∂D, let $F(x)$ be the algebraic distance between x and ∂D: this is a locally C^2 function, which we can extend to the whole space with the required properties (see Property 3.1 in Section 3).

Remark 1.3. Assume moreover that D is of class C^3 with a compact boundary. Then, we note that an uniform ellipticity condition on the diffusion implies condition (C). Indeed, from the strong Markov property, we have $\mathbb{P}_x(\exists t \in [0,T]\ X_t \notin D\,;\ \forall t \in [0,T]\ X_t \in \bar{D}) = \mathbb{P}_x(\tau=T) + \mathbb{E}_x(\mathbb{1}_{\tau<T}\mathbb{P}_{X_\tau}[\forall t \in [0,T-\tau]\ X_t \in \bar{D}])$. The first term on the r.h.s. equals 0 because τ has a density w.r.t. the Lebesgue measure. The second term on the r.h.s. also equals 0 using the 0–1 law to show that $\mathbb{P}_z(\forall t \in [0,s]\ X_t \in \bar{D}) = 0$ for $z \in \partial D$ and $s > 0$ (see Friedman, 1976).

Remark 1.4. For $d=1$ and $D=(-\infty,b)$, condition (C) becomes $\mathbb{P}_x(\sup_{t\in[0,T]} X_t=b)=0$. Thus, condition (C′) below implies condition (C):

(C′): $\exists y \in (x,b)$ such that $\sigma(y) \neq 0$.

This can be justified using the Nualart–Vives criterion for absolute continuity w.r.t. the Lebesgue measure of the law of the supremum of some process (see Nualart, 1995, Proposition 2.1.4); we omit the details. Condition (C’) shows that, in some sense, condition (C) is weak (much weaker than an uniform ellipticity condition e.g.).

Proof of Proposition 1.1. For the continuous Euler scheme, it is well known that $\lim_{n\to+\infty}\tilde{X} \overset{\mathbb{P}}{=} X$ uniformly on $[0,T]$. It easily implies $\lim_{n\to+\infty}\mathbb{1}_{T<\tilde{\tau}_c} f(\tilde{X}_T) \overset{\mathbb{P}}{=} \mathbb{1}_{T<\tau} f(X_T)$ since one has $\{T < \tilde{\tau}_c\} = \{\inf_{t\in[0,T]} F(\tilde{X}_t) > 0\}$ and the condition (C) is equivalent to $\mathbb{P}(\inf_{t\in[0,T]} F(X_t) = 0) = 0$. The result for $\mathscr{E}_c(f,T,x,N)$ follows. For $\mathscr{E}_d(f,T,x,N)$, analogous arguments apply. □

We now focus on the rate of convergence of the errors under stronger assumptions. Our main results state that under regularity assumptions on B, σ, D and an uniform ellipticity condition, one has

- for the continuous Euler scheme,

  $$\mathscr{E}_c(f,T,x,N) = C N^{-1} + \mathrm{o}(N^{-1}),$$

  provided that f is a measurable function with support strictly included in D (Theorem 2.1). The support condition can be weakened if f is smooth enough (Theorem 2.2).
- for the discrete Euler scheme,

  $$\mathscr{E}_d(f,T,x,N) = \mathrm{O}(N^{-1/2}),$$

  for functions f satisfying analogous hypotheses as before (Theorem 2.3). The rate $N^{-1/2}$ is optimal and intrinsic to the choice of a discrete killing time (Theorem 2.4).

1.3. Background results

Known results about the Euler scheme (4) concern the approximation of $\mathbb{E}_x[f(X_T)]$: the error can be expanded in terms of powers of N^{-1} (see Talay and Tubaro (1990) if f is smooth and Bally and Talay (1996a) if f is only measurable with hypoellipticity conditions). A different point of view is to study the convergence in law of the renormalized error $(\sqrt{N}(\tilde{X}_t^N - X_t))_{t\geqslant 0}$ (see Kurtz and Protter (1991)).

When we consider the weak approximation of killed diffusion, we know from Siegmund and Yuh (1982) that the error $\mathscr{E}_d(f,T,x,N)$ can be expanded to the first order in $N^{-1/2}$ in the case of a Brownian motion in dimension 1, for f equal to a characteristic function of an interval strictly included in D (this implies in particular that f vanishes on a neighbourhood of the boundary ∂D):

$$\mathscr{E}_d(f,T,x,N) = C N^{-1/2} + \mathrm{o}(N^{-1/2}).$$

Their proof uses random walk techniques and cannot be adapted to others situations. For a more general multidimensional diffusion, Costantini et al. (1998) prove that, for all $\eta > 0$,

$$|\mathscr{E}_d(f,T,x,N)| \leqslant C_\eta N^{-1/2+\eta},$$

provided that the domain is bounded, smooth and convex and that the function $f \in C^{3,\beta}(\bar{D},\mathbb{R})$ (i.e. f is $C^3(\bar{D},\mathbb{R})$ with third derivatives satisfying β-Hölder conditions with $\beta \in (0,1)$) with some conditions of vanishing on ∂D. Our results improve theirs since we show that

1. the convergence rate of $\mathscr{E}_d(f,T,x,N)$ to 0 is in fact of order $N^{-1/2}$;
2. the domain needs not be convex;
3. provided a support condition, the function f needs only be measurable.

1.4. Outline of the paper

To derive the estimates of the errors, following the approach of Bally and Talay (1996a), we transform both approximation errors ($\mathscr{E}_c(f,T,x,N)$ and $\mathscr{E}_d(f,T,x,N)$)

using the parabolic PDE satisfied by the function $(T,x) \mapsto \mathbb{E}_x[\mathbb{1}_{t<\tau} f(X_T)]$, so that the global errors will be decomposed into a sum of local errors: in Section 2, we first recall some standard regularity results concerning the associate PDE and then, we state the main results of the paper.

Their proofs are given in Section 3: for the continuous Euler scheme, the analysis of local errors involves standard stochastic calculus. But to handle the case of measurable functions f, we need some crucial controls on the law of killed processes, which are given in Lemma 3.1. Their proofs use Malliavin calculus techniques and require some particular and careful treatment due to the exit time: they are postponed to Section 4. For the discrete Euler scheme, additional techniques are needed: in particular, we project orthogonally on $\bar{D}$ the Euler scheme, to obtain a non-standard Itô's formula. It involves a local time on the boundary which we accurately estimate using a exterior cone condition on ∂D. These boundary estimates are exposed in Lemmas 3.6 and 3.7, but their proofs are given in Section 5. Section 6 deals with some extensions.

1.5. General notation

We consider a domain $D \subset \mathbb{R}^d$, i.e. an open connected set, with a non-empty boundary ∂D. We assume that $X_0 = x \in D$. For $s \in \partial D$, $n(s)$ denotes the unit inner normal at s, when it is well defined.

For $(t,x) \in [0,T] \times \mathbb{R}^d$, we set

$$v(t,x) := \mathbb{E}_x[\mathbb{1}_{T-t<\tau} f(X_{T-t})], \tag{7}$$

where $\tau := \inf\{t > 0:\ X_t \notin D\}$ (with the convention $\tau = +\infty$ if $\forall t > 0\, X_t \in D$).

For sets Λ and Λ' in $\mathbb{R}^d$, for $z \in R^d$, $d(z,\Lambda)$ stands for the distance between z and Λ, $d(\Lambda,\Lambda')$ for the distance between Λ and Λ'.

For $r \geqslant 0$, set $V_{\partial D}(r) := \{z \in \mathbb{R}^d:\ d(z,\partial D) \leqslant r\}$ and $D(r) := \{z \in \mathbb{R}^d:\ d(z,D) < r\}$. We also introduce the stopping time $\tilde{\tau}(r)$ which will permit to localize $(\tilde{X}_t)_{0\leqslant t\leqslant T}$ near D:

$$\tilde{\tau}(r) := \inf\{t > 0:\ \tilde{X}_t \notin D(r)\}. \tag{8}$$

We will keep the same notation $K(T)$ for all finite, non-negative and non-decreasing functions, independent of x, N or f, which will appear in proofs (i.e. they depend on D, the coefficients $B(.)$, $\sigma(.)$ of (1) and so on).

For smooth functions $g(t,x)$, we denote by $\partial_x^\alpha g(t,x)$ the derivative of g w.r.t. x according to the multi-index α, and by $|\alpha|$ the length of α.

If $(V_t)_{t\geqslant 0}$ is a process taking its values in $\mathbb{R}^d$, $(V_{i,t})_{t\geqslant 0}$ will denote its d coordinates.

2. Hypotheses and results

From now on, we assume that the following three assumptions are satisfied. The first one concerns the regularity of the coefficients of the diffusion process:

(H1) $B(.)$ is a $C_b^\infty(\mathbb{R}^d,\mathbb{R}^d)$ function and $\sigma(.)$ is a $C_b^\infty(\mathbb{R}^d,\mathbb{R}^d \otimes \mathbb{R}^{d'})$ function.

The next assumption is an uniform ellipticity condition on the diffusion:

(H2) there is $\sigma_0 > 0$ such that $\forall x \in \mathbb{R}^d$ $\sigma(x)\sigma^*(x) \geqslant \sigma_0^2 I_{\mathbb{R}^d\otimes\mathbb{R}^d}$.

Moreover, we require that D is smooth enough. Let us recall

Definition 2.1 (Gilbarg and Trudinger, 1977, pp. 88–89). For $d \geqslant 2$, the domain D is of class C^k $(k \geqslant 1)$ if for each point $s \in \partial D$, there is a ball $O = O(s)$ and a one-to-one mapping ψ of O onto $O' \subset \mathbb{R}^d$ such that

$$\begin{aligned}&\psi(O \cap D) \subset \mathbb{R}^d_+ = \{y \in \mathbb{R}^d\colon\ y_1 > 0\},\\ &\psi(O \cap \partial D) \subset \partial\mathbb{R}^d_+ = \{y \in \mathbb{R}^d\colon\ y_1 = 0\},\\ &\psi \in C^k(O) \quad \text{and} \quad \psi^{-1} \in C^k(O').\end{aligned}$$

For $d \geqslant 2$, we assume that

(H3) The domain D is of class C^∞ and ∂D is compact.

For some of the next results, hypotheses (H1)–(H3) may be weakened (see Section 6).

Under assumptions (H1)–(H3), we know that the function $v(t,x)$ (defined in (7)) is related to the transition density at time $T-t$ of the killed diffusion, denoted by $q_{T-t}(x,y)$, by the relation

$$v(t,x) = \int_D q_{T-t}(x,y)\, f(y)\, \mathrm{d}y \tag{9}$$

for a bounded measurable function f. Moreover, if we fix y, $q_s(x,y)$ is a $C^\infty((0,T]\times \bar{D}, \mathbb{R})$ function in (s,x), vanishing for $x \in \partial D$. It satisfies Kolmogorov's backward equation. Furthermore, for all multi-index α, there are a constant $c > 0$ and a function $K(T)$, such that

$$\forall (s,x,y) \in (0,T]\times\bar{D}\times\bar{D}\ |\partial_x^\alpha q_s(x,y)| \leqslant \frac{K(T)}{s^{(|\alpha|+d)/2}} \exp\left(-c\,\frac{\|y-x\|^2}{s}\right). \tag{10}$$

These classical results can be found in Theorem 16.3 of Ladyzenskaja et al. (1968, p. 413), and Chapter 3 of Friedman (1964) (see also Cattiaux (1991) for hypoellipticity conditions). Thus, $v(t,x)$ is of class $C^\infty([0,T)\times\bar{D}, \mathbb{R})$ and satisfies a parabolic partial differential equation of second order with Cauchy and Dirichlet conditions, i.e.

$$\begin{aligned}&\partial_t v + L v = 0 && \text{for } (t,x) \in [0,T)\times\bar{D},\\ &v(t,x) = 0 && \text{for } (t,x) \in [0,T]\times D^c,\\ &v(T,x) = f(x) && \text{for } x \in D,\end{aligned} \tag{11}$$

where L is the infinitesimal generator of the diffusion

$$Lu(x) = \sum_{i=1}^d B_i(x)\,\frac{\partial u}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d (\sigma(x)\sigma^*(x))_{i,j}\,\frac{\partial^2 u}{\partial x_i \partial x_j}(x).$$

Remark 2.1. Note that $v \in C^\infty([0,T)\times\bar{D}, \mathbb{R}) \cap C^0([0,T)\times\mathbb{R}^d, \mathbb{R})$: spatial derivatives of v have jumps at the boundary. This simple fact will lead, for the analysis of the discrete Euler scheme, to technical difficulties for writing some Itô's formula: to solve this, we will orthogonally project the process on $\bar{D}$.

We analyze the discretization errors $\mathscr{E}_c(f,T,x,N)$ and $\mathscr{E}_d(f,T,x,N)$ for two classes of function. This corresponds to the following assumptions.

(H4) f is a bounded measurable function, satisfying $d(\mathrm{Supp}(f),\partial D)\geqslant 2\varepsilon>0$.
(H5-k) ($k\in\mathbb{N}$) f is a $C^{m,\beta}(\bar{D},\mathbb{R})$ function with $m\geqslant 2k$, $\beta\in(0,1)$, satisfying the following condition of vanishing on ∂D: $\forall z\in\partial D$ $f(z)=Lf(z)=\cdots=L^{(k)}f(z)=0$.

We recall (see Ladyzenskaja et al., 1968, pp. 7,8) that for $(m,\beta)\in\mathbb{N}\times(0,1)$, $C^{m,\beta}(\bar{D},\mathbb{R})$ is the Banach space whose elements are continuous functions $u(x)$ in D having in $\bar{D}$ continuous derivatives up to order m and a finite value for the quantity

$$\|u\|_D^{(m,\beta)}=\sum_{j=0}^{m}\sum_{|j'|=j}\sup_{x\in D}|\partial_x^{j'}u(x)|+\sum_{|m'|=m}\sup_{x,x'\in D}\frac{|\partial_x^{m'}u(x)-\partial_x^{m'}u(x')|}{\|x-x'\|^{\beta}},$$

i.e. the norm on $C^{m,\beta}(\bar{D},\mathbb{R})$ (the summation $\sum_{|j'|=j}$ is taken over all multi-index j' of length j).

Denote by $\mathscr{L}_z$, the operator on C^2 functions defined by

$$\mathscr{L}_z u(x)=\sum_{i=1}^{d}B_i(z)\frac{\partial u}{\partial x_i}(x)+\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}(\sigma(z)\sigma^*(z))_{i,j}\frac{\partial^2 u}{\partial x_i\partial x_j}(x) \tag{12}$$

(for $t\in[t_i,t_{i+1})$, $\mathscr{L}_{\tilde{X}_{t_i}}$ is the infinitesimal generator of $\tilde{X}_t$). For $(t,x)\in[0,T)\times\bar{D}$, set

$$\Theta(t,x)=\tfrac{1}{2}\left(L^2v(t,x)-2\mathscr{L}_zLv(t,x)+\mathscr{L}_z^2v(t,x)\right)\Big|_{z=x}. \tag{13}$$

2.1. Analysis of the continuous Euler scheme

We first state the expansion result for bounded measurable functions f, with support strictly included in D.

Theorem 2.1. *Assume that assumptions* (H1)–(H3) *are fulfilled and f satisfies* (H4). *Then, there is a function $K(T)$ such that*

$$|\mathscr{E}_c(f,T,x,N)|\leqslant K(T)\frac{\|f\|_\infty}{1\wedge\varepsilon^4}N^{-1}. \tag{14}$$

Moreover, we have

$$\mathscr{E}_c(f,T,x,N)=T\int_0^T \mathrm{d}t\,\mathbb{E}_x[\mathbb{1}_{t<\tau}\,\Theta(t,X_t)]\,N^{-1}+\mathrm{o}(N^{-1}), \tag{15}$$

with $|T\int_0^T \mathrm{d}t\,\mathbb{E}_x[\mathbb{1}_{t<\tau}\Theta(t,X_t)]|\leqslant K(T)\|f\|_\infty/(1\wedge\varepsilon^4)$.

The support condition on f can be weakened to vanishing conditions on ∂D if f is smooth enough. This is the statement of the following theorem.

Theorem 2.2. *Assume that assumptions* (H1)–(H3) *are fulfilled and f satisfies* (H5-2). *Then, there is a function $K(T)$ such that*

$$|\mathscr{E}_c(f,T,x,N)|\leqslant K(T)\|f\|_D^{(m,\beta)}N^{-1}. \tag{16}$$

Moreover, we have

$$\mathscr{E}_c(f,T,x,N) = T\int_0^T \mathrm{d}t\, \mathbb{E}_x[\mathbb{1}_{t<\tau}\Theta(t,X_t)]N^{-1} + \mathrm{o}(N^{-1}), \tag{17}$$

with $|T\int_0^T \mathrm{d}t\, \mathbb{E}_x[\mathbb{1}_{t<\tau}\Theta(t,X_t)]| \leqslant K(T)\|f\|_D^{(m,\beta)}$.

The existence of the expansion of the error enables to reach a higher rate of convergence using linear combinations of results obtained with different step-sizes (Romberg extrapolation technique: see Talay and Tubaro, 1990).

2.2. Analysis of the discrete Euler scheme

Theorem 2.3. *Assume that assumptions* (H1)–(H3) *are fulfilled. If f satisfies* (H4), *there is a function $K(T)$ such that*

$$|\mathscr{E}_d(f,T,x,N)| \leqslant K(T)\frac{\|f\|_\infty}{1\wedge\varepsilon^4}\frac{1}{\sqrt{N}}. \tag{18}$$

If f satisfies (H5-1), *there is a function $K(T)$ such that*

$$|\mathscr{E}_d(f,T,x,N)| \leqslant K(T)\|f\|_D^{(m,\beta)}\frac{1}{\sqrt{N}}. \tag{19}$$

The rate of convergence $N^{-1/2}$ is the best we can obtain in a general situation because we know this rate is achieved in the special case of a linear Brownian motion (see Siegmund and Yuh, 1982).

Moreover, if we set

$$\tau_d := \inf\{t_i\colon X_{t_i} \notin D\},$$

we have

Theorem 2.4. *Assume that assumptions* (H1)–(H3) *are fulfilled. If f satisfies* (H4), *there is a function $K(T)$ such that*

$$|\mathbb{E}_x[\mathbb{1}_{T<\tau}f(X_T)] - \mathbb{E}_x[\mathbb{1}_{T<\tau_d}f(X_T)]| \leqslant K(T)\frac{\|f\|_\infty}{1\wedge\varepsilon^4}\frac{1}{\sqrt{N}}. \tag{20}$$

If f satisfies (H5-1), *there is a function $K(T)$ such that*

$$|\mathbb{E}_x[\mathbb{1}_{T<\tau}f(X_T)] - \mathbb{E}_x[\mathbb{1}_{T<\tau_d}f(X_T)]| \leqslant K(T)\|f\|_D^{(m,\beta)}\frac{1}{\sqrt{N}}. \tag{21}$$

Theorem 2.4 shows that the rate $N^{-1/2}$ is intrinsic to the problem of discrete killing time: even if there is no approximation of the values of the process at discretization times, the error is still of order $N^{-1/2}$. This fact will appear more clearly in the proof of these results (see Remark 3.5): we will see that the global error $\mathscr{E}_d(f,T,x,N)$ can be separated into two contributions, the first one coming from the approximation due to Euler scheme of the infinitesimal generator L, and the second coming from the approximation of the "continuous" exit time by the discrete one.

Note that Theorems 2.3 and 2.4 deal with the case of functions f vanishing on the boundary (as for Siegmund and Yuh, 1982; Costantini et al., 1998). Nevertheless, presumably, the approach we develop in this sequel may be the appropriate one to prove that the rate of convergence $N^{-1/2}$ remains true for a function smooth near ∂D (without conditions of vanishing on ∂D).

3. Proof of Theorems 2.1–2.4

To begin, we state a technical Lemma, involving controls on the law of some killed processes: this result is crucial to handle the case of measurable functions f for the analysis of the errors. Its proof based on Malliavin calculus techniques is given in Section 4.

Let $\psi \in C_b^\infty(\mathbb{R}^d,\mathbb{R})$ be a cutting function near ∂D verifying $\mathbb{1}_{V_{\partial D}(\varepsilon/2)} \leqslant 1-\psi \leqslant \mathbb{1}_{V_{\partial D}(\varepsilon)}$ and $\|\partial_x^\alpha \psi\|_\infty \leqslant C_{|\alpha|}/(1\wedge\varepsilon^{|\alpha|})$ for all multi-index α ($\varepsilon>0$ is defined by (H4)).

Lemma 3.1. *Assume that* (H1)–(H4) *are satisfied. Then, for all multi-index* α, *there is a function* $K(T)$, *such that*

$$\forall (s,x) \in [0,T)\times V_{\partial D}(\varepsilon) \quad |\partial_x^\alpha v(s,x)| \leqslant K(T)\frac{\|f\|_\infty}{1\wedge\varepsilon^{|\alpha|}}. \tag{22}$$

Moreover, for $R\geqslant 0$, *for all multi-indices* α *and* α', *for all* $g \in C_b^{|\alpha|}(\mathbb{R}^d,\mathbb{R})$, *there is a function* $K(T)$ (*depending on* $\|g\|_{C_b^{|\alpha|}}$), *such that for* $0\leqslant t\leqslant T$ *and* $0\leqslant s<T$, *we have*

$$|\mathbb{E}_x[\mathbb{1}_{s<\tilde{\tau}_c} g(\tilde{X}_t)\partial_x^\alpha v(s,\tilde{X}_s)]| \leqslant \frac{\|f\|_\infty}{1\wedge\varepsilon^{|\alpha|}}\frac{K(T)}{T^{|\alpha|/2}}, \tag{23}$$

$$|\mathbb{E}_x[\mathbb{1}_{s<\tau} g(X_t)\partial_x^\alpha v(s,X_s)]| \leqslant \frac{\|f\|_\infty}{1\wedge\varepsilon^{|\alpha|}}\frac{K(T)}{T^{|\alpha|/2}}, \tag{24}$$

$$|\mathbb{E}_x[\mathbb{1}_{s<\tilde{\tau}_d\wedge\tilde{\tau}(R)} g(\tilde{X}_t)\partial_x^\alpha [v\partial_x^{\alpha'}\psi](s,\tilde{X}_s)]| \leqslant \frac{\|f\|_\infty}{1\wedge\varepsilon^{|\alpha|+|\alpha'|}}\frac{K(T)}{T^{|\alpha|/2}}. \tag{25}$$

The constant R introduced in the lemma above will be defined later in the proof of Theorems (see Property 3.1).

We first address the analysis of the continuous Euler scheme, which is more easy than the discrete one.

3.1. Continuous Euler scheme

3.1.1. Proof of Theorem 2.1

We have

$$\begin{aligned}\mathscr{E}_c(f,T,x,N) &= \mathbb{E}_x[\mathbb{1}_{T<\tilde{\tau}_c} f(\tilde{X}_T)] - \mathbb{E}_x[v((T-T/N)\wedge\tilde{\tau}_c, \tilde{X}_{(T-T/N)\wedge\tilde{\tau}_c})]\\ &\quad + \mathbb{E}_x[v((T-T/N)\wedge\tilde{\tau}_c, \tilde{X}_{(T-T/N)\wedge\tilde{\tau}_c})] - \mathbb{E}_x[v(0,\tilde{X}_0)]\\ &:= C_1(N)+C_2(N).\end{aligned} \tag{26}$$

It results from Lemma 3.2 below (whose proof will be given at the end) that $C_1(N)$ yields a negligible contribution:

$$|C_1(N)| \leqslant K(T)\frac{\|f\|_\infty}{1\wedge\varepsilon^4}N^{-3/2}. \tag{27}$$

Lemma 3.2. *Assume that* (H1)–(H4) *are satisfied. Then, there is a function* $K(T)$, *such that*

$$|\mathbb{E}_x[\mathbb{1}_{T<\tilde{\tau}_c} f(\tilde{X}_T)] - \mathbb{E}_x[v((T-T/N)\wedge\tilde{\tau}_c, \tilde{X}_{(T-T/N)\wedge\tilde{\tau}_c})]| \leqslant K(T)\frac{\|f\|_\infty}{1\wedge\varepsilon^4}N^{-3/2}. \quad (28)$$

To transform $C_2(N)$, apply Itô's formula to $v\in C^{1,2}([0,T)\times\bar{D},\mathbb{R})$, between times 0 and $(T-T/N)\wedge\tilde{\tau}_c$. Using the notation (4) and (12), it readily follows that

$$\begin{aligned} C_2(N) &= \mathbb{E}_x\left[\int_0^{t_{N-1}\wedge\tilde{\tau}_c} \mathrm{d}s(\partial_t v + \mathscr{L}_z v)|_{z=\tilde{X}_{\varphi(s)}}(s,\tilde{X}_s)\right] \\ &= \int_0^{t_{N-1}} \mathrm{d}s\, \mathbb{E}_x[\mathbb{1}_{s<\tilde{\tau}_c}(\mathscr{L}_z v - Lv)|_{z=\tilde{X}_{\varphi(s)}}(s,\tilde{X}_s)], \end{aligned}$$

where we used $\partial_t v = -Lv$ in $[0,T)\times\bar{D}$. Since spatial derivatives of v have jumps on ∂D (see Remark 2.1), we may stop the paths at time $\tilde{\tau}_c$ to avoid some discontinuity problems. Using $\mathbb{1}_{s<\tilde{\tau}_c} = \mathbb{1}_{\varphi(s)<\tilde{\tau}_c} - \mathbb{1}_{\varphi(s)<\tilde{\tau}_c\leqslant s}$, we obtain

$$\begin{aligned} C_2(N) \;&=\; \int_0^{t_{N-1}} \mathrm{d}s\, \mathbb{E}_x[\mathbb{1}_{\varphi(s)<\tilde{\tau}_c}(\mathscr{L}_z v - Lv)|_{z=\tilde{X}_{\varphi(s)}}(s\wedge\tilde{\tau}_c, \tilde{X}_{s\wedge\tilde{\tau}_c})] \\ &\quad - \int_0^{t_{N-1}} \mathrm{d}s\, \mathbb{E}_x[\mathbb{1}_{\varphi(s)<\tilde{\tau}_c\leqslant s}(\mathscr{L}_z v - Lv)|_{z=\tilde{X}_{\varphi(s)}}(s\wedge\tilde{\tau}_c, \tilde{X}_{s\wedge\tilde{\tau}_c})] \\ &:= C_3(N) - C_4(N). \end{aligned} \quad (29)$$

When we explicit $\mathscr{L}_z - L$, we can assert that for $g_\alpha = B_i$ or $(\sigma\sigma^*)_{i,j}$, we have

$$\begin{aligned} (\mathscr{L}_z v - Lv)|_{z=\tilde{X}_{\varphi(s)}}(s\wedge\tilde{\tau}_c, \tilde{X}_{s\wedge\tilde{\tau}_c}) = \sum_{1\leqslant|\alpha|\leqslant 2} & c_\alpha \partial_x^\alpha v(s\wedge\tilde{\tau}_c, \tilde{X}_{s\wedge\tilde{\tau}_c}) \\ & \times[g_\alpha(\tilde{X}_{s\wedge\tilde{\tau}_c}) - g_\alpha(\tilde{X}_{\varphi(s)})]. \end{aligned}$$

On the event $\{\varphi(s)<\tilde{\tau}_c\leqslant s\}$, the involved derivatives of v are computed on ∂D: so, using the estimates (22), they are uniformly bounded. Using Lemma 3.3 below (proved later), it readily follows that

$$\begin{aligned} |C_4(N)| &\leqslant \frac{K(T)\|f\|_\infty}{1\wedge\varepsilon^2}\sum_{i=0}^{N-2}\int_{t_i}^{t_{i+1}} \mathrm{d}s\, \mathbb{E}_x\left[\mathbb{1}_{t_i<\tilde{\tau}_c\leqslant s} \max_{0\leqslant i\leqslant N-1}\sup_{u\in[t_i,\,t_{i+1}]}\|\tilde{X}_u - \tilde{X}_{t_i}\|_{\mathbb{R}^d}\right] \\ &\leqslant \frac{K(T)\|f\|_\infty}{1\wedge\varepsilon^2}\mathbb{E}_x\left[\left(\max_{0\leqslant i\leqslant N-1}\sup_{u\in[t_i,\,t_{i+1}]}\|\tilde{X}_u - \tilde{X}_{t_i}\|_{\mathbb{R}^d}\right)\right. \\ &\quad \left.\times\sum_{i=0}^{N-2}\int_{t_i}^{t_{i+1}} \mathrm{d}s\,\mathbb{1}_{t_i<\tilde{\tau}_c\leqslant t_{i+1}}\right] \\ &\leqslant \frac{K(T)\|f\|_\infty}{1\wedge\varepsilon^2}\frac{\log(N+1)}{N^{3/2}}, \end{aligned} \quad (30)$$

where we used $\sum_{i=0}^{N-2}\mathbb{1}_{t_i<\tilde{\tau}_c\leqslant t_{i+1}}\leqslant 1$.

Lemma 3.3. *Assume that* (H1) *is satisfied. Then, for* $p \geqslant 1$, *there is a function* $K(T)$, *such that*

$$\left[\mathbb{E}_x\left(\max_{0\leqslant i\leqslant N-1}\sup_{s\in[t_i,t_{i+1}]}\|\tilde{X}_s-\tilde{X}_{t_i}\|_{\mathbb{R}^d}\right)^p\right]^{1/p}\leqslant K(T)N^{-1/2}\sqrt{\log(N+1)}. \qquad (31)$$

For the term $C_3(N)$, apply once again Itô's formula on the event $\{\varphi(s)<\tilde{\tau}_c\}$, between times $\varphi(s)$ and $s\wedge\tilde{\tau}_c$, to obtain

$$C_3(N)=\int_0^{t_{N-1}}\mathrm{d}s\int_{\varphi(s)}^{s}\mathrm{d}t\,\mathbb{E}_x[\mathbb{1}_{t<\tilde{\tau}_c}(L^2v-2\mathscr{L}_zLv+\mathscr{L}_z^2v)|_{z=\tilde{X}_{\varphi(t)}}(t,\tilde{X}_t)],$$

where we used the PDE satisfied by v (the term from Itô's formula corresponding to $\varphi(s)$ equals to 0 because $\mathscr{L}_zv(s,z)=Lv(s,z)$). Applying the estimate (23), we immediately obtain

$$|C_3(N)|\leqslant\frac{K(T)}{N}\frac{\|f\|_\infty}{1\wedge\varepsilon^4}.$$

We complete the proof of (14) by combining this last estimate with (26), (27), (29) and (30). Now, note that to obtain (15), it is enough to prove that

$$C_3(N)=\frac{T}{N}\int_0^T\mathrm{d}s\,\mathbb{E}_x[\mathbb{1}_{s<\tau}\Theta(s,X_s)]+\mathrm{o}(N^{-1}).$$

We proceed as follows:

$$\begin{aligned}
C_3(N)&=\int_0^{t_{N-1}}\mathrm{d}s\int_{\varphi(s)}^{s}\mathrm{d}t\,\mathbb{E}_x[\mathbb{1}_{t<\tilde{\tau}_c}(L^2v-2\mathscr{L}_zLv+\mathscr{L}_z^2v)|_{z=\tilde{X}_{\varphi(t)}}(t,\tilde{X}_t)]\quad\text{(Step 1)}\\
&\approx\int_0^{t_{N-1}}\mathrm{d}s\int_{\varphi(s)}^{s}\mathrm{d}t\,2\mathbb{E}_x[\mathbb{1}_{\varphi(t)<\tilde{\tau}_c}\Theta(\varphi(t),\tilde{X}_{\varphi(t)})]\\
&:=\frac{T}{N}\sum_{i=0}^{N-2}\int_{t_i}^{t_{i+1}}\mathrm{d}s\,\mathbb{E}_x[\mathbb{1}_{t_i<\tilde{\tau}_c}\Theta(t_i,\tilde{X}_{t_i})]\quad\text{(Step 2)}\\
&\approx\frac{T}{N}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\mathrm{d}s\,\mathbb{E}_x[\mathbb{1}_{s<\tilde{\tau}_c}\Theta(s,\tilde{X}_s)]:=\frac{T}{N}\int_0^T\mathrm{d}s\,\mathbb{E}_x[\mathbb{1}_{s<\tilde{\tau}_c}\Theta(s,\tilde{X}_s)]\ \text{(Step 3)}\\
&\approx\frac{T}{N}\int_0^T\mathrm{d}s\,\mathbb{E}_x[\mathbb{1}_{s<\tau}\Theta(s,X_s)]\quad\text{(Step 4)},
\end{aligned}$$

where the symbol $\approx$ means that the remainder term is an $\mathrm{o}(N^{-1})$ (recall that the function $\Theta(t,x)$ is defined by (13)).

To obtain Step 2 from Step 1, proceed as for the analysis of $C_2(N)$ by applying Itô's formula between $\varphi(t)$ and $t\wedge\tilde{\tau}_c$, using estimates (22) and (23). Very similar arguments apply to the passage from Step 2 to Step 3. The last step consists in proving that

$$\int_0^T\mathrm{d}s\,\mathbb{E}_x[\mathbb{1}_{s<\tilde{\tau}_c}\Theta(s,\tilde{X}_s)]-\int_0^T\mathrm{d}s\,\mathbb{E}_x[\mathbb{1}_{s<\tau}\Theta(s,X_s)]=\mathrm{o}(1).$$

This directly follows from the Lebesgue-dominated convergence Theorem by noting that on the one hand, each integrand is bounded, using the estimates (23) and (24) from Lemma 3.1. On the other, for $s\in[0,T)$, we have $\lim_{N\to+\infty}\mathbb{E}_x[\mathbb{1}_{s<\tilde{\tau}_c}\Theta(s,\tilde{X}_s)]=$

$\mathbb{E}_x[\mathbb{1}_{s<\tau}\Theta(s,X_s)]$ applying Proposition 1.1 with $f(z)=\Theta(s,z)$. This completes the proof of Theorem 2.1. □

It remains to prove Lemmas 3.2 and 3.3.

Proof of Lemma 3.2. Using the properties (11), note that

$$\begin{aligned}&\mathbb{E}_x[\mathbb{1}_{T<\tilde{\tau}_c}f(\tilde{X}_T)] - \mathbb{E}_x[v((T-T/N)\wedge\tilde{\tau}_c,\tilde{X}_{(T-T/N)\wedge\tilde{\tau}_c})]\\ &\quad = \mathbb{E}_x[v(T\wedge\tilde{\tau}_c,\tilde{X}_{T\wedge\tilde{\tau}_c})] - \mathbb{E}_x[v((T-T/N)\wedge\tilde{\tau}_c,\tilde{X}_{(T-T/N)\wedge\tilde{\tau}_c})].\end{aligned}$$

The reason why we treat this term apart from $C_2(N)$ in (26) is only technical. Since $v\in C^{1,2}([0,T)\times\mathbb{R}^d,\mathbb{R})$, we may apply Itô's formula between $(T-T/N)\wedge\tilde{\tau}_c$ and $(T-\delta)\wedge\tilde{\tau}_c$ and take the limit when $\delta\to 0$: this last step is difficult to prove because $v(t,x)$ may not be continuous in $t=T$ if f is only a measurable function. To solve this difficulty, we approximate f by some continuous functions, using a density argument.

1. *Assume that the function f is continuous and satisfies* (H4). Then, we have $\lim_{t\to T,y\to x} v(t,y)=v(T,x)=f(x)$ for $x\in D$. Fix $\delta\in(0,T/N)$. The term to estimate can be rewritten as

$$\begin{aligned}&\mathbb{E}_x[\mathbb{1}_{T<\tilde{\tau}_c}f(\tilde{X}_T)] - \mathbb{E}_x[v((T-T/N)\wedge\tilde{\tau}_c,\tilde{X}_{(T-T/N)\wedge\tilde{\tau}_c})]\\ &\quad = \mathbb{E}_x[v(T\wedge\tilde{\tau}_c,\tilde{X}_{T\wedge\tilde{\tau}_c})] - \mathbb{E}_x[v((T-\delta)\wedge\tilde{\tau}_c,\tilde{X}_{(T-\delta)\wedge\tilde{\tau}_c})]\\ &\qquad + \mathbb{E}_x[v((T-\delta)\wedge\tilde{\tau}_c,\tilde{X}_{(T-\delta)\wedge\tilde{\tau}_c})] - \mathbb{E}_x[v((T-T/N)\wedge\tilde{\tau}_c,\tilde{X}_{(T-T/N)\wedge\tilde{\tau}_c})].\\ &\quad := E_1(\delta,N)+E_2(\delta,N). \end{aligned}\tag{32}$$

A very similar computation as the one made to estimate $C_2(N)$ from (26) leads to

$$|E_2(\delta,N)|\leqslant K(T)\frac{\|f\|_\infty}{1\wedge\varepsilon^4}N^{-3/2},\tag{33}$$

uniformly in δ. Using the continuity of $v(t,x)$ and $\tilde{X}_{t\wedge\tilde{\tau}_c}$, we conclude by the Lebesgue-dominated convergence Theorem that $\lim_{\delta\to0}E_1(\delta,N)=0$. Combining this fact with (32) and (33), the required estimate (28) is proved for bounded continuous functions f with $d(\mathrm{Supp}(f),\partial D)\geqslant 2\varepsilon$.

2. *Assume now that f is only measurable and satisfies* (H4). Denote by $\tilde{\mu}_1$ and $\tilde{\mu}_2$ the two measures defined by $\mathbb{E}_x[\mathbb{1}_{T<\tilde{\tau}_c}f(\tilde{X}_T)] := \int f\,\mathrm{d}\tilde{\mu}_1$ and

$$\begin{aligned}\mathbb{E}_x\left[v\left(\left(T-\frac{T}{N}\right)\wedge\tilde{\tau}_c,\tilde{X}_{(T-T/N)\wedge\tilde{\tau}_c}\right)\right] &= \mathbb{E}_x[\mathbb{1}_{T-T/N<\tilde{\tau}_c}\mathbb{E}_{\tilde{X}_{T-T/N}}[\mathbb{1}_{T/N<\tau}f(X_{T/N})]]\\ &:= \int f\,\mathrm{d}\tilde{\mu}_2.\end{aligned}$$

By a density argument, f can be approximated in $L^1(\tilde{\mu}_1+\tilde{\mu}_2)$ by a sequence of continuous functions denoted by $(f_p)_{p\geqslant0}$: moreover, in that loss of generality assume that each function f_p satisfies $\|f_p\|_\infty\leqslant\|f\|_\infty$ and $d(\mathrm{Supp}(f_p,\partial D))\geqslant 2\varepsilon$, so that the result for continuous functions applies, uniformly in p. This completes the proof. □

Proof of Lemma 3.3. In fact, we prove that Lemma 3.3 holds for any Itô process $(Y_t)_{t\geqslant0}$, defined by $\mathrm{d}Y_t=b_t\,\mathrm{d}t+\sigma_t\,\mathrm{d}W_t$, with adapted and uniformly bounded coefficients.

Without loss of generality, we can assume that $Y_t\in\mathbb{R}$, $(W_t)_{t\geqslant0}$ is a linear Brownian motion and $b_t\equiv0$. Since $\sup_{s\in[t_i,t_{i+1}]}|Y_s-Y_{t_i}|\leqslant\sup_{s\in[t_i,t_{i+1}]}\int_{t_i}^s\sigma_u\,\mathrm{d}W_u-\inf_{s\in[t_i,t_{i+1}]}$

$\int_{t_i}^{s} \sigma_u \,\mathrm{d}W_u$, it is sufficient to prove the estimate for $\sup_{s\in[t_i,t_{i+1}]} \int_{t_i}^{s} \sigma_u \,\mathrm{d}W_u$ (the other one will follow by replacing σ by $-\sigma$). The Bernstein exponential inequality for martingales yields

$$\mathbb{P}\left[\sup_{s\in[t_{N-1},t_N]} \int_{t_{N-1}}^{s} \sigma_u \,\mathrm{d}W_u > z/\mathscr{F}_{t_{N-1}}\right] \leqslant \exp\left(-N\frac{z^2}{2\|\sigma^2\|_\infty T}\right).$$

Hence,

$$\begin{aligned}
&\mathbb{P}_x\left(\max_{0\leqslant i\leqslant N-1}\ \sup_{s\in[t_i,t_{i+1}]} \int_{t_i}^{s} \sigma_u \,\mathrm{d}W_u \leqslant z\right)\\
&\quad\geqslant \mathbb{P}_x\left(\max_{0\leqslant i\leqslant N-2}\ \sup_{s\in[t_i,t_{i+1}]} \int_{t_i}^{s} \sigma_u \,\mathrm{d}W_u \leqslant z\right)\left(1-\exp\left(-N\frac{z^2}{2\|\sigma^2\|_\infty T}\right)\right)\\
&\quad\geqslant \left(1-\exp\left(-N\frac{z^2}{2\|\sigma^2\|_\infty T}\right)\right)^N,
\end{aligned}$$

where we have iterated the conditioning. It follows that

$$\begin{aligned}
&\mathbb{E}_x\left(\max_{0\leqslant i\leqslant N-1}\ \sup_{s\in[t_i,t_{i+1}]} \int_{t_i}^{s} \sigma_u \,\mathrm{d}W_u\right)^p\\
&\quad\leqslant \int_{\mathbb{R}^+} \mathrm{d}z\, p z^{p-1}\left(1-\left(1-\exp\left(-N\frac{z^2}{2\|\sigma^2\|_\infty T}\right)\right)^N\right).
\end{aligned}$$

Cut the integral at the point $\xi(N) = 2\|\sigma\|_\infty\sqrt{T}N^{-1/2}\sqrt{\log(N+1)}$. The first term corresponding to the integral between 0 and $\xi(N)$ is obviously bounded by $\xi^p(N)$. Using that $1-(1-u)^N \leqslant Nu$ for $u\in[0,1]$, the second one can be easily bounded by $K(T)N^{-p/2}\exp(-N\xi^2(N)/(4\|\sigma^2\|_\infty T)) = K(T)N^{-p/2}(N+1)^{-1}$. This leads to the required estimate and completes the proof. □

3.1.2. Proof of Theorem 2.2

We mimic the arguments of Theorem 2.1: in that case, the fourth spatial derivatives of v are uniformly bounded with Hölder conditions for the fourth ones (see Lemma 3.4 below). This is enough to obtain the expected results: we omit the details. The estimates (22)–(24) used under assumption (H4) have to be replaced by those given by

Lemma 3.4. *Under* (H1)–(H3) *and* (H5-2), *v is at least a $C^{2,4}([0,T]\times\bar{D},\mathbb{R})$ function, and there is a function $K(T)$, such that for all multi-index α of length $|\alpha|\leqslant 4$, we have*

$$\forall(t,x)\in[0,T]\times\bar{D}\quad |\partial_x^\alpha v(t,x)| \leqslant K(T)\|f\|_D^{(m,\beta)}. \tag{34}$$

Moreover, for all multi-index α of length $|\alpha| = 4$, we have

$$\sup_{(t,x),(t',x')\in[0,T]\times\bar{D}} \left(\frac{|\partial_x^\alpha v(t,x) - \partial_x^\alpha v(t,x')|}{\|x-x'\|^\beta} + \frac{|\partial_x^\alpha v(t,x) - \partial_x^\alpha v(t',x)|}{|t-t'|^{\beta/2}} \right)$$
$$\leqslant K(T)\|f\|_D^{(m,\beta)}. \qquad (35)$$

This is immediately derived from classical results for linear equations of parabolic type: we refer to Ladyzenskaja et al. (1968, Theorem 5.2, p. 320) for fuller statement.

3.2. Discrete Euler scheme

Because the case $d = 1$ masks some problems, we focus on the case $d \geqslant 2$ in the following.

First, for the convenience of stochastic calculus for continuous-time processes, we consider, in the sequel, the continuous Euler scheme (4) killed at the discrete time $\tilde{\tau}_d := \inf\{t_i\colon \tilde{X}_{t_i} \notin D\}$: this new point of view does not change of course $\mathscr{E}_d(f,T,x,N)$.

The first stage for the proof of Theorems 2.3 and 2.4 (and analogously to the analysis of $\mathscr{E}_c(f,T,x,N)$) is to note that, from properties (11), we have

$$\mathscr{E}_d(f,T,x,N) = \mathbb{E}_x[v(T\wedge\tilde{\tau}_d, \tilde{X}_{T\wedge\tilde{\tau}_d}) - v(0,\tilde{X}_0)]. \qquad (36)$$

Then, we would like to apply Itô's formula to explicit $v(T\wedge\tilde{\tau}_d, \tilde{X}_{T\wedge\tilde{\tau}_d}) - v(0,\tilde{X}_0)$. Unfortunately, the situation is not classical at all because spatial derivatives of v are discontinuous at the boundary (see Remark 2.1) and the process $(\tilde{X}_{t\wedge\tilde{\tau}_d})_{t\geqslant 0}$ probably crosses ∂D. Intuitively, if such a decomposition exists, it should involve a local time on the boundary (note that for the analysis of $\mathscr{E}_c(f,T,x,N)$, $(\tilde{X}_{t\wedge\tilde{\tau}_c})_{t\geqslant 0}$ has been stopped just before crossing ∂D, so that we only need classical Ito's formula).

To solve this problem, our approach is the following. Consider $Z_t := \mathrm{Proj}_{\bar{D}}(\tilde{X}_t)$, the orthogonal projection on $\bar{D}$ of $\tilde{X}_t$: this process takes its values in $\bar{D}$. As v vanishes outside D, we have $v(t,\tilde{X}_t) = v(t,Z_t)$. If $(Z_t)_{t\geqslant 0}$ remains a continuous semimartingale, then classical Itô's formula can be applied to $v(t,Z_t)$ because $v \in C^{1,2}([0,T)\times\bar{D},\mathbb{R})$. So, the main task is to show that Z_t is still a continuous semimartingale and to obtain a tractable decomposition for it. In the case of a half-space $D = \{z \in \mathbb{R}^d\colon z_1 > 0\}$, this fact is clear because $Z_t = ((\tilde{X}_{1,t})^+, \tilde{X}_{2,t}, \ldots, \tilde{X}_{d,t})^*$ and we conclude using Tanaka's formula. For a general domain, by an appropriate mapping, we can transform D locally near ∂D in a half-space and thus, apply the arguments of the first case. Actually, the orthogonal projection is not uniquely defined on the whole space (except if D is convex), but only near D: so, we will use localization arguments, owing to the stopping time $\tilde{\tau}(r)$ (introduced in (8)).

We now bring together in Property 3.2 below few basic facts from differential geometry about the functions "distance to the boundary" and "orthogonal projection on $\bar{D}$" (for the proofs, see e.g. Appendix, Gilbarg and Trudinger, 1977, pp. 381–384).

Property 3.1. *For a domain D of class C^3 with compact boundary ∂D, there is a constant $R > 0$ such that the following three properties hold.*

1. Local diffeomorphism (*which locally maps the boundary into a half-space*). *For all $s \in \partial D$, there are two open bounded sets U^s and V^s, a C^2-diffeomorphism F^s ($G^s = (F^s)^{-1}$) from U^s ($s \in U^s$) into $(-2R, 2R) \times V^s$, such that*

$$F^s : \begin{cases} U^s \subset \mathbb{R}^d \to (-2R, 2R) \times V^s \subset \mathbb{R} \times \mathbb{R}^{d-1}, \\ x \mapsto (z_1, z) := (z_1, z_2, \ldots, z_d) \quad \text{such that } x = g^s(z) + z_1 n(g^s(z)), \end{cases}$$

where g^s is a mapping of ∂D in a neighbourhood of s.
2. Distance to ∂D. *Let $s \in \partial D$. On U^s, the function $F_1^s(.)$ is the algebraic distance to ∂D (thus it does not depend on s and we denote it by F_1), i.e. $|F_1(x)| = d(x, \partial D)$ and $F_1(x) > 0$ (resp. $F_1(x) < 0$) if $x \in D \cap U^s$ (resp. $x \in \bar{D}^c \cap U^s$). It is a C^3 function on $\bigcup_{s \in \partial D} U^s = V_{\partial D}(2R)$, which we extend into a $C_b^3(\mathbb{R}^d, \mathbb{R})$ function, with the conditions $F_1(.) > 0$ on D and $F_1(.) < 0$ on $\bar{D}^c$. Note that $\partial D = \{x \in \mathbb{R}^d \colon F_1(x) = 0\}$.*
3. Orthogonal projection on $\bar{D}$. *Let $s \in \partial D$. For $x \in U^s$, the orthogonal projection on $\bar{D}$ of x is uniquely defined by*

$$\mathrm{Proj}_{\bar{D}}(x) = G^s([F_1(x)]^+, F_2^s(x), \ldots, F_d^s(x)). \tag{37}$$

Since ∂D is compact, there exists a finite number of points $(s_i)_{1 \leqslant i \leqslant k}$ in ∂D (we associate to them G^i, F^i, U^i and V^i respectively) such that $V_{\partial D}(3R/2) \subset \bigcup_{1 \leqslant i \leqslant k} U^i$. Consider an open set U^0 with $d(\partial D, \overline{U^0}) > 0$, such that $D(3R/2) \subset \bigcup_{0 \leqslant i \leqslant k} U^i$. Now, we construct a partition of unity, subordinate to the cover $(U^i)_{0 \leqslant i \leqslant k}$, i.e. non-negative C_b^∞ functions $(\phi^i)_{0 \leqslant i \leqslant k}$ verifying $\mathrm{Supp}(\phi^i) \subset U^i$ and $\sum_{i=0}^k \phi^i \equiv 1$ on $\overline{D_R}$. So far, functions F^i (resp. G^i) have been well defined only on U^i (resp. $(-2R, 2R) \times V^i$): we extend them in smooth functions on $\mathbb{R}^d$.

We now can state

Proposition 3.1. *Consider a domain D of class C^3 with compact boundary (with the constant $R > 0$ defined in Property 3.1). Let $(Y_t)_{t \geqslant 0}$ be a continuous semimartingale, taking its values in $\overline{D(R)}$ (with $Y_0 \in D$). Then, the orthogonal projection of Y_t on $\bar{D}$ denoted by $\mathrm{Proj}_{\bar{D}}(Y_t)$ defines a continuous semimartingale, whose decomposition is*

$$d(\mathrm{Proj}_{\bar{D}}(Y_t)) = \mathbb{1}_{Y_t \in D} dY_t + \mathbb{1}_{Y_t \notin D} dY_t^{\partial D} + \tfrac{1}{2} n(Y_t) dL_t^0(F_1(Y)),$$

where

- *$Y_t^{\partial D}$ is a continuous semimartingale with $Y_0^{\partial D} = 0$, with decomposition*

$$dY_t^{\partial D} := \sum_{i=1}^k \phi^i(Y_t) d(G^i(0, F_2^i(Y_t), \ldots, F_d^i(Y_t))).$$

- *$L_t^0(F_1(Y))$ is the one-dimensional local time of the continuous semimartingale $F_1(Y)$ at time t and level 0.*

Proof. According to Property 3.1 and using the partition of unity above, we have for $x \in \overline{D(R)}$: $\mathrm{Proj}_{\bar{D}}(x) = \phi^0(x)x + \sum_{i=1}^k \phi^i(x) G^i([F_1(x)]^+, F_2^i(x), \ldots, F_d^i(x))$. Since F^i are C^2 functions, $(F_j^i(Y_t))_{t \geqslant 0}$ $(2 \leqslant j \leqslant d)$ and $([F_1(Y_t)]^+)_{t \geqslant 0}$ remain continuous semimartingales owing to Itô's formula for the first ones and Tanaka's formula for the latter:

because G^i are C^2 functions, we deduce that $\mathrm{Proj}_{\bar{D}}(Y_t)$ is a continuous semimartingale. To obtain its decomposition, we study coordinatewise. Denote $Z_t := \mathrm{Proj}_{\bar{D}}(Y_t)$ and fix $j \in \{1,\dots,d\}$. First, we have

$$
\begin{aligned}
dZ_{j,t} &= \phi^0(Y_t)dY_{j,t} + \sum_{i=1}^{k} \phi^i(Y_t)d(G^i_j([F_1(Y_t)]^+, F^i_2(Y_t),\dots,F^i_d(Y_t)))\\
&\quad + \sum_{i=0}^{k} Z_{j,t}d(\phi^i(Y_t)) + \sum_{i=0}^{k} d\langle\phi^i(Y_.), Z_{j,.}\rangle_t\\
&= \phi^0(Y_t)dY_{j,t} + \sum_{i=1}^{k} \phi^i(Y_t)d(G^i_j([F_1(Y_t)]^+, F^i_2(Y_t),\dots,F^i_d(Y_t))),
\end{aligned}
$$

since $\sum_{i=0}^{k} \phi^i \equiv 1$ on $\overline{D_R}$. Using $\mathrm{d}([F_1(Y_t)]^+) = \mathbb{1}_{F_1(Y_t)>0}\, d(F_1(Y_t)) + \frac{1}{2}dL^0_t(F_1(Y))$ (see Revuz and Yor, 1991), a straightforward computation with Itô's formula for C^2 functions leads to

$$
\begin{aligned}
dZ_{j,t} &= \sum_{i=1}^{k} \phi^i(Y_t)\frac{\partial G^i_j}{\partial z_1}(0, F^i_2(Y_t),\dots,F^i_d(Y_t))\frac{1}{2}dL^0_t(F_1(Y))\\
&\quad + \mathbb{1}_{Y_t\in D}\phi^0(Y_t)dY_{j,t} + \mathbb{1}_{Y_t\in D}\sum_{i=1}^{k}\phi^i(Y_t)\left(\sum_{l=1}^{d}\frac{\partial G^i_j}{\partial z_l}(F^i(Y_t))d(F^i_l(Y_t))\right.\\
&\quad \left. + \frac{1}{2}\sum_{l=1}^{d}\sum_{m=1}^{d}\frac{\partial^2 G^i_j}{\partial z_l \partial z_m}(F^i(Y_t))d\langle F^i_l(Y_.), F^i_m(Y_.)\rangle_t\right)\\
&\quad + \mathbb{1}_{Y_t\notin D}\sum_{i=1}^{k}\phi^i(Y_t)\left(\sum_{l=2}^{d}\frac{\partial G^i_j}{\partial z_l}(0, F^i_2(Y_t),\dots,F^i_d(Y_t))d(F^i_l(Y_t))\right.\\
&\quad \left. + \frac{1}{2}\sum_{l=2}^{d}\sum_{m=2}^{d}\frac{\partial^2 G^i_j}{\partial z_l \partial z_m}(0, F^i_2(Y_t),\dots,F^i_d(Y_t))d\langle F^i_l(Y_.), F^i_m(Y_.)\rangle_t\right),
\end{aligned}
$$

where we used that $\{F_1(Y_t)\leqslant 0\} = \{Y_t \notin D\}$ and $\phi^0(Y_t) = \phi^0(Y_t)\mathbb{1}_{Y_t\in D}$.

The terms involving $dL^0_t(F_1(Y))$ can be identified with $\frac{1}{2}n_j(Y_t)$ using Property 3.1. For terms corresponding to $\mathbb{1}_{Y_t\in D}$, we obtain $dY_{j,t}$, combining the simple fact that $Y_{j,t} = \sum_{i=0}^{k} \phi^i(Y_t)Y_{j,t} = \phi^0(Y_t)Y_{j,t} + \sum_{i=1}^{k} \phi^i(Y_t)G^i_j(F_1(Y_t), F^i_2(Y_t),\dots,F^i_d(Y_t))$ and a computation as before. Terms with $\mathbb{1}_{Y_t\notin D}$ can be rewritten vectorially as $\sum_{i=1}^{k} \phi^i(Y_t)\, d(G^i(0, F^i_2(Y_t),\dots,F^i_d(Y_t))) := dY^{\partial D}_t$. This completes the proof. □

Remark 3.1. The restriction to some set $\overline{D(R)}$ is necessary to have $\mathrm{Proj}_{\bar{D}}(.)$ well defined. If D is convex, $R = +\infty$.

Remark 3.2. Another way to proceed is to show that the function $z \to \mathrm{Proj}_{\bar{D}}(z)$ is locally the difference of convex functions: it is well known that these functions preserve continuous semimartingales (see e.g. Bouleau, 1984). Rather than its semimartingale character, what is of interest for our objective is to obtain a nice decomposition of

$\mathrm{Proj}_{\bar{D}}(Y_t)$. From this point of view, the approach we had is simpler and has the advantage that we are able to interpret and control quite easily the terms of the decomposition.

We now are able to deduce how to explicit (36).

Corollary 3.1. *Consider a domain D of class C^3 with compact boundary (with the constant $R>0$ defined in Property 3.1). Let $(Y_t)_{t\geqslant 0}$ be a continuous semimartingale, taking its values in $\overline{D(R)}$ (with $Y_0 \in D$). Let $u(t,x) \in C^0(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}) \cap C^{1,2}(\mathbb{R}^+ \times \bar{D}, \mathbb{R})$. We assume that for $t\geqslant 0$, the support of $u(t,.)$ is included in $\bar{D}$.*

Then, $(u(t,Y_t))_{t\geqslant 0}$ is a continuous semimartingale, with decomposition

$$
\begin{aligned}
d(u(t,Y_t)) = {} & \mathbb{1}_{Y_t\in D}\frac{\partial u}{\partial t}(t,Y_t)\,\mathrm{d}t + \frac{1}{2}\sum_{l=1}^{d}\frac{\partial u}{\partial x_l}(t,Y_t)n_l(Y_t)\,\mathrm{d}L_t^0(F_1(Y)) \\
& + \mathbb{1}_{Y_t\in D}\left(\sum_{l=1}^{d}\frac{\partial u}{\partial x_l}(t,Y_t)\,\mathrm{d}Y_{l,t} + \frac{1}{2}\sum_{l=1}^{d}\sum_{m=1}^{d}\frac{\partial^2 u}{\partial x_l \partial x_m}(t,Y_t)\,\mathrm{d}\langle Y_{l,.}, Y_{m,.}\rangle_t\right) \\
& + \mathbb{1}_{Y_t\notin D}\left(\sum_{l=1}^{d}\frac{\partial u}{\partial x_l}(t,\mathrm{Proj}_{\bar{D}}(Y_t))\,\mathrm{d}Y_{l,t}^{\partial D}\right. \\
& \left. + \frac{1}{2}\sum_{l=1}^{d}\sum_{m=1}^{d}\frac{\partial^2 u}{\partial x_l \partial x_m}(t,\mathrm{Proj}_{\bar{D}}(Y_t))\mathrm{d}\langle Y_{l,.}^{\partial D}, Y_{m,.}^{\partial D}\rangle_t\right).
\end{aligned}
$$

Proof. Since $u(t,.)$ vanishes outside D for all $t\geqslant 0$, we have $u(t,Y_t)-u(0,Y_0) = u(t,\mathrm{Proj}_{\bar{D}}(Y_t)) - u(0,\mathrm{Proj}_{\bar{D}}(Y_0))$. Now, combine Proposition 3.1 and classical Itô's formula to complete the proof. □

Remark 3.3. If $d=1$, Corollary 3.1 reduces to Itô–Tanaka's formula (see Revuz and Yor, 1991).

Before proving Theorems 2.3 and 2.4, we state three technical Lemmas, the two last will be proved in Section 5.

Lemma 3.5. *Under* (H1)–(H3) *and* (H5-1), *v is at least a $C^{1,2}([0,T]\times\bar{D},\mathbb{R})$ function, and there is a function $K(T)$, such that for all multi-index α of length $|\alpha|\leqslant 2$, we have*

$$
\forall (s,x)\in[0,T]\times\bar{D} \quad |\partial_x^\alpha v(s,x)| \leqslant K(T)\|f\|_D^{(m,\beta)}. \tag{38}
$$

Proof. Apply the same arguments as for Lemma 3.4.

Lemma 3.6. *Under* (H1)–(H3), *there is a function $K(T)$, such that, for $s\in(0,T]$ and $x\in D$, we have*

$$
\mathbb{P}_x(\tilde{X}_{\varphi(s)}\in D,\ \tilde{X}_s\notin D) \leqslant \frac{K(T)}{\sqrt{N}}\frac{1}{\sqrt{s}} \quad \textit{and} \quad \mathbb{P}_x(X_{\varphi(s)}\in D,\ X_s\notin D) \leqslant \frac{K(T)}{\sqrt{N}}\frac{1}{\sqrt{s}}.
$$

Lemma 3.7. *Under* (H1)–(H3), *there is a function* $K(T)$, *such that*, *for* $x \in D$, *we have*

$$\mathbb{E}_x[L^0_{T\wedge\tilde{\tau}_d}(F_1(\tilde{X}))] \leqslant \frac{K(T)}{\sqrt{N}} \quad \textit{and} \quad \mathbb{E}_x[L^0_{T\wedge\tau_d}(F_1(X))] \leqslant \frac{K(T)}{\sqrt{N}}.$$

We now are able to prove Theorem 2.3.

1. *Suppose first that f is continuous on D and satisfies* (H4). Fix $\delta > 0$. Considering (11), we obtain easily

$$\begin{aligned}\mathscr{E}_d(f,T,x,h) &= \mathbb{E}_x[\mathbb{1}_{\tilde{\tau}(R)<T<\tilde{\tau}_d} f(\tilde{X}_T)] \\ &\quad + \mathbb{E}_x[v(T\wedge\tilde{\tau}_d\wedge\tilde{\tau}(R),\tilde{X}_{T\wedge\tilde{\tau}_d\wedge\tilde{\tau}(R)}) \\ &\quad - v((T-\delta)\wedge\tilde{\tau}_d\wedge\tilde{\tau}(R),\tilde{X}_{(T-\delta)\wedge\tilde{\tau}_d\wedge\tilde{\tau}(R)})] \\ &\quad + \mathbb{E}_x[v((T-\delta)\wedge\tilde{\tau}_d\wedge\tilde{\tau}(R),\tilde{X}_{(T-\delta)\wedge\tilde{\tau}_d\wedge\tilde{\tau}(R)}) - v(0,\tilde{X}_0)] \\ &:= E_1(N) + E_2(\delta,N) + E_3(\delta,N).\end{aligned}$$

Observe that $E_1(N)$ is exponentially small and is bounded by $K(T)N^{-1/2}\|f\|_\infty$. For this, use classical upper bound for large deviations probability (see Lemma 4.1 in Section 4)

$$\forall x \in \mathbb{R}^d \ \forall \eta \geqslant 0 \quad \mathbb{P}_x\left(\sup_{t\in[S,S']} \|\tilde{X}_t - \tilde{X}_S\| \geqslant \eta\right) \leqslant K(T)\exp\left(-c\frac{\eta^2}{\Delta}\right),$$

for S and S' two stopping times, bounded by T, such that $0 \leqslant S' - S \leqslant \Delta$.

From the continuity of v at $t = T$ (because f is continuous) and the continuity of $\tilde{X}_{.\wedge\tilde{\tau}_d\wedge\tilde{\tau}(R)}$, we prove that $\lim_{\delta\to 0} E_2(\delta,N) = 0$ using the Lebesgue-dominated convergence Theorem.

To complete the proof, it suffices now to show that

$$|E_3(\delta,N)| \leqslant K(T)\frac{\|f\|_\infty}{1\wedge\varepsilon^4}\frac{1}{\sqrt{N}}, \tag{39}$$

uniformly in δ. For this, we apply Itô's formula from Corollary 3.1 with $u = v$, $Y_t = \tilde{X}_{t\wedge\tilde{\tau}(R)}$ between times 0 and $(T-\delta)\wedge\tilde{\tau}_d\wedge\tilde{\tau}(R)$. If we introduce the operator $\mathscr{L}_z$ (see (12)) and if we take into account that $\partial_t v + Lv = 0$ in $[0,T)\times\bar{D}$, we easily obtain

$$\begin{aligned}E_3(\delta,N) &= \mathbb{E}_x\left[\int_0^{(T-\delta)\wedge\tilde{\tau}_d\wedge\tilde{\tau}(R)} \mathrm{d}t \mathbb{1}_{\tilde{X}_t\in D}(\mathscr{L}_z v - Lv)|_{z=\tilde{X}_{\varphi(t)}}(t,\tilde{X}_t)\right] \\ &\quad + \mathbb{E}_x\left[\int_0^{(T-\delta)\wedge\tilde{\tau}_d\wedge\tilde{\tau}(R)} \frac{1}{2}\sum_{l=1}^d \frac{\partial v}{\partial x_l}(t,\tilde{X}_t)n_l(\tilde{X}_t)\,\mathrm{d}L^0_t(F_1(\tilde{X}))\right] \\ &\quad + \mathbb{E}_x\left[\int_0^{(T-\delta)\wedge\tilde{\tau}_d\wedge\tilde{\tau}(R)} \sum_{l=1}^d \frac{\partial v}{\partial x_l}(t,\mathrm{Proj}_{\bar{D}}(\tilde{X}_t))\mathbb{1}_{\tilde{X}_t\notin D}\,\mathrm{d}\tilde{X}^{\partial D}_{l,t}\right. \\ &\quad \left. + \frac{1}{2}\sum_{l=1}^d\sum_{m=1}^d \frac{\partial^2 v}{\partial x_l \partial x_m}(t,\mathrm{Proj}_{\bar{D}}(\tilde{X}_t))\mathbb{1}_{\tilde{X}_t\notin D}\mathrm{d}\langle \tilde{X}^{\partial D}_{l,.},\tilde{X}^{\partial D}_{m,.}\rangle_t\right] \\ &:= E_4(\delta,N) + E_5(\delta,N) + E_6(\delta,N).\end{aligned} \tag{40}$$

Since $\mathrm{d}L_t^0(F_1(\tilde{X}))$ is a non-negative measure, using inequality (22) and Lemma 3.7, we deduce

$$|E_5(\delta,N)| \leqslant K(T)\frac{\|f\|_\infty}{1\wedge\varepsilon}\mathbb{E}_x[L^0_{T\wedge\tilde{\tau}_d}(F_1(\tilde{X}))] \leqslant \frac{K(T)}{\sqrt{N}}\frac{\|f\|_\infty}{1\wedge\varepsilon}, \tag{41}$$

uniformly in δ. Since $\tilde{X}$ is an Itô process with adapted and bounded coefficients, $\tilde{X}^{\partial D}$ has the same property (see Proposition 3.1). Hence, by using inequality (22) and Lemma 3.6, we have

$$\begin{aligned}|E_6(\delta,N)| &\leqslant K(T)\frac{\|f\|_\infty}{1\wedge\varepsilon^2}\mathbb{E}_x\left[\int_0^{(T-\delta)\wedge\tilde{\tau}_d\wedge\tilde{\tau}(R)} \mathbb{1}_{\tilde{X}_t\notin D}\,\mathrm{d}t\right]\\ &\leqslant K(T)\frac{\|f\|_\infty}{1\wedge\varepsilon^2}\mathbb{E}_x\left[\int_0^T \mathbb{1}_{\tilde{X}_{\varphi(t)}\in D}\mathbb{1}_{\tilde{X}_t\notin D}\,\mathrm{d}t\right] \leqslant \frac{K(T)}{\sqrt{N}}\frac{\|f\|_\infty}{1\wedge\varepsilon^2},\end{aligned} \tag{42}$$

uniformly in δ. It remains to control $E_4(\delta,N)$. If ψ is the cutting function introduced at the beginning of Section 3, we have

$$\begin{aligned}E_4(\delta,N) &= \mathbb{E}_x\left[\int_0^{(T-\delta)} \mathrm{d}t\mathbb{1}_{\tilde{X}_t\in D}\mathbb{1}_{t<\tilde{\tau}_d\wedge\tilde{\tau}(R)}(\mathscr{L}_z - L)|_{z=\tilde{X}_{\varphi(t)}}[(1-\psi)v](t,\tilde{X}_t)\right]\\ &\quad + \mathbb{E}_x\left[\int_0^{(T-\delta)} \mathrm{d}t\mathbb{1}_{\tilde{X}_t\in D}\mathbb{1}_{t<\tilde{\tau}_d\wedge\tilde{\tau}(R)}(\mathscr{L}_z - L)|_{z=\tilde{X}_{\varphi(t)}}[\psi v](t,\tilde{X}_t)\right]\\ &:= E_7(\delta,N) + E_8(\delta,N).\end{aligned} \tag{43}$$

When we explicit $\mathscr{L}_z - L$, we can assert that for $g_\alpha = B_i$ or $(\sigma\sigma^*)_{i,j}$, we have

$$\begin{aligned}&(\mathscr{L}_z - L)|_{z=\tilde{X}_{\varphi(t)}}[(1-\psi)v](t,\tilde{X}_t)\\ &\quad= \sum_{1\leqslant|\alpha|\leqslant 2} C_\alpha\partial_x^\alpha[(1-\psi)v](t,\tilde{X}_t)[g_\alpha(\tilde{X}_t) - g_\alpha(\tilde{X}_{\varphi(t)})].\end{aligned}$$

Since $(1-\psi)$ has support included in $V_{\partial D}(\varepsilon)$, derivatives of $[(1-\psi)v]$ are controlled by (22), whereas the increments of $\tilde{X}$ are estimated by $\|\|\tilde{X}_t - \tilde{X}_{\varphi(t)}\|_{R^d}\|_{L^p} \leqslant K(T)N^{-1/2}$. It readily follows that

$$|E_7(\delta,N)| \leqslant \frac{K(T)}{\sqrt{N}}\frac{\|f\|_\infty}{1\wedge\varepsilon^2}, \tag{44}$$

uniformly in δ.

To control $E_8(\delta,N)$, we need to transform its expression. Since $\mathrm{Supp}(\psi v)\subset\bar{D}$, we have

$$\begin{aligned}&\mathbb{1}_{\tilde{X}_t\in D}\mathbb{1}_{t<\tilde{\tau}_d\wedge\tilde{\tau}(R)}(\mathscr{L}_z - L)|_{z=\tilde{X}_{\varphi(t)}}[\psi v](t,\tilde{X}_t)\\ &\quad= \mathbb{1}_{\varphi(t)<\tilde{\tau}_d\wedge\tilde{\tau}(R)}(\mathscr{L}_z - L)|_{z=\tilde{X}_{\varphi(t)}}[\psi v](t\wedge\tilde{\tau}_d\wedge\tilde{\tau}(R), \tilde{X}_{t\wedge\tilde{\tau}_d\wedge\tilde{\tau}(R)}).\end{aligned}$$

Hence, $E_8(\delta,N)$ can be rewritten as

$$E_8(\delta,N)=\int_0^{T-\delta}\mathrm{d}t\,\mathbb{E}_x[\mathbb{1}_{\varphi(t)<\tilde\tau_d\wedge\tilde\tau(R)}(\mathscr{L}_z-L)|_{z=\tilde X_{\varphi(t)}}[\psi v](t\wedge\tilde\tau_d\wedge\tilde\tau(R),\tilde X_{t\wedge\tilde\tau_d\wedge\tilde\tau(R)})]$$

$$=\int_0^{T-\delta}\mathrm{d}t\,\mathbb{E}_x\left[\mathbb{1}_{\varphi(t)<\tilde\tau_d\wedge\tilde\tau(R)}\int_{\varphi(t)}^{t\wedge\tilde\tau_d\wedge\tilde\tau(R)}\mathrm{d}s[\partial_t+\mathscr{L}_z]\right.$$

$$\left.\times(\mathscr{L}_z[\psi v]-L[\psi v])|_{z=\tilde X_{\varphi(t)}}(s,\tilde X_s)\right],$$

by an application of Itô's formula to $\tilde X$, between times $\varphi(t)$ and $t\wedge\tilde\tau_d\wedge\tilde\tau(R)$, to the function $(\mathscr{L}_z[\psi v]-L[\psi v])(s,x)\in C^{1,2}([0,T)\times\mathbb{R}^d,\mathbb{R})$: note that the term corresponding to $\varphi(t)$ vanishes because $\mathscr{L}_z u(z)=Lu(z)$. Since the coefficients of L and $\mathscr{L}$ do not depend on t, we obtain

$$E_8(\delta,N)=\int_0^{T-\delta}\mathrm{d}t\int_{\varphi(t)}^{t}\mathrm{d}s$$
$$\mathbb{E}_x[\mathbb{1}_{s<\tilde\tau_d\wedge\tilde\tau(R)}(L[\psi Lv]-\mathscr{L}_z[L(\psi v)+\psi Lv]+\mathscr{L}_z^2[\psi v])|_{z=\tilde X_{\varphi(t)}}(s,\tilde X_s)],$$

exploiting again $\partial_t v+Lv=0$ in $[0,T)\times\bar D$. Verify that

$$(L[\psi Lv]-\mathscr{L}_z[L(\psi v)+\psi Lv]+\mathscr{L}_z^2[\psi v])(s,x)$$

$$=\sum_{\substack{|\alpha|\leqslant 4,|\alpha|+|\alpha'|\leqslant 4,\\ y=x,y=z}} g_{\alpha,\alpha'}(y)\partial_x^{\alpha}[v\partial_x^{\alpha'}\psi](s,x),$$

where the functions $g_{\alpha,\alpha'}(y)$ depend only on B_i, $(\sigma\sigma^*)_{i,j}$ and their derivatives up to $4-|\alpha|$. From inequality (25) of Lemma 3.1, we deduce that

$$\mathbb{E}_x[\mathbb{1}_{s<\tilde\tau_d\wedge\tilde\tau(R)}(-L[\psi Lv]+\mathscr{L}_z[L(\psi v)+\psi Lv]-\mathscr{L}_z^2[\psi v])|_{z=\tilde X_{\varphi(t)}}(s,\tilde X_s)]$$
$$\leqslant\frac{\|f\|_\infty}{1\wedge\varepsilon^4}\frac{K(T)}{T^2},$$

hence

$$|E_8(\delta,N)|\leqslant\frac{\|f\|_\infty}{1\wedge\varepsilon^4}\frac{K(T)}{N},\qquad(45)$$

uniformly in δ. Combining estimations (41), (42), (44), (45) with (40) and (43), we prove that estimate (39) holds: this completes the proof if f is continuous in D.

2. *Suppose now that f is only measurable and satisfies* (H4). Denote by μ and $\tilde\mu$ the two measures defined by $\mathbb{E}_x[\mathbb{1}_{T<\tau}f(X_T)]:=\int f\,\mathrm{d}\mu$ and $\mathbb{E}_x[\mathbb{1}_{T<\tilde\tau_d}f(\tilde X_T)]:=\int f\,\mathrm{d}\tilde\mu$. By a density argument, f can be approximated in $L^1(\mu+\tilde\mu)$ by a sequence of continuous functions denoted by $(f_p)_{p\geqslant 0}$: moreover, we can impose that each function f_p satisfies $\|f_p\|_\infty\leqslant\|f\|_\infty$ and $d(\mathrm{Supp}(f_p,\partial D))\geqslant 2\varepsilon$, so that the result for continuous function applies, uniformly in p. This finishes the proof.

3. *Suppose that f satisfies* (H5-1). In that case, the first and second spatial derivatives of v are uniformly bounded (see Lemma 3.5). This enables us to proceed analogously to the case 1 (in this simpler situation, $E_4(\delta,N)$ can be directly bounded with the same arguments as for $E_7(\delta,N)$). □

For the proof of Theorem 2.4, the same reasoning applies. Note that $E_4(\delta,N)=0$ because there is no approximation of the infinitesimal generator of the diffusion. □

Remark 3.4. Actually, a slight change in the proof shows that the error corresponding to the approximation of L is negligible w.r.t. $N^{-1/2}$: $E_4(\delta,N)=\mathrm{o}(N^{-1/2})$. For this, consider the cutting function ψ such that $\mathbb{1}_{V_{\partial D}((\varepsilon\wedge N^{-\gamma})/2)} \leqslant 1-\psi \leqslant \mathbb{1}_{V_{\partial D}(\varepsilon\wedge N^{-\gamma})}$. By writing $(\mathscr{L}_z v - Lv) = (1-\psi)(\mathscr{L}_z v - Lv) + \psi(\mathscr{L}_z v - Lv)$, show that the first contribution (corresponding to $E_7(\delta,N)$ in the proof) is of order $\mathrm{O}(N^{(1+\gamma)/2})$, whereas the second one (corresponding to $E_8(\delta,N)$) is of order $\mathrm{O}(N^{1-8\gamma})$. The choice of $\gamma\in(0,1/16)$ leads to the required estimate.

Remark 3.5. The analysis of the error presented in the proof of Theorem 2.3 makes two different type errors appear, which have interesting interpretations. On the one hand, we make an error by approximating the diffusion process by its Euler scheme (term $E_4(\delta,N)$), but this error is smaller than $N^{-1/2}$ (see the previous remark). On the other, we make an error by considering a discrete killing time instead of a continuous one (terms $E_5(\delta,N)$ and $E_6(\delta,N)$): these terms give the rate of convergence $N^{-1/2}$. In this sense, $N^{-1/2}$ is intrinsic to the problem of discrete killing time.

4. Proof of Lemma 3.1

From the equality (9) and the estimate (10), we deduce

$$\begin{aligned}|\partial_x^\alpha v(s,x)| &\leqslant \|f\|_\infty \frac{K(T)}{(T-s)^{|\alpha|/2}} \int_{\mathrm{Supp}(f)} \frac{\mathrm{d}y}{(T-s)^{d/2}} \exp\left(-c\frac{\|y-x\|^2}{2(T-s)}\right)\\ &\quad\times\exp\left(-c\frac{\|y-x\|^2}{2(T-s)}\right)\\ &\leqslant \|f\|_\infty \frac{K(T)}{(T-s)^{|\alpha|/2}} \exp\left(-c\frac{d^2(\mathrm{Supp}(f),x)}{2(T-s)}\right),\end{aligned} \tag{46}$$

where we used for one of the two exponential terms that for $y\in\mathrm{Supp}(f)$, we have $\|y-x\|\geqslant d(\mathrm{Supp}(f),x)$. Hence, inequality (22) easily follows using $d^2(\mathrm{Supp}(f),x)\geqslant\varepsilon^2$ for $x\in V_{\partial D}(\varepsilon)$ and $\sup_{\lambda>0}(\varepsilon^2/\lambda)^{|\alpha|/2}\exp(-(c\varepsilon^2/2\lambda)) < +\infty$ uniformly in ε. Obviously, from (46), we also deduce

$$\forall (s,x)\in[0,T)\times\bar{D} \quad |\partial_x^\alpha v(s,x)| \leqslant \|f\|_\infty \frac{K(T)}{(T-s)^{|\alpha|/2}}. \tag{47}$$

It now remains to prove the estimates (23)–(25): actually, for $s\leqslant T/2$, they are obvious using (47). The difficult case is for $T/2\leqslant s<T$. To handle this, following the approach of Bally and Talay (1996a) for the approximation of $\mathbb{E}_x[f(X_T)]$, we use Malliavin calculus: it needs particular treatment because of the characteristic function with the exit time. For this, we adapt some techniques from Cattiaux (1991, Theorem 3.3). Furthermore, we need some results concerning Malliavin calculus for elliptic Itô processes: they are proved by Kusuoka and Stroock (1984). First, we briefly introduce the required material for the sequel (for a detailed exposition, see Nualart, 1995).

4.1. Basic results on Malliavin calculus for elliptic Itô processes

Fix a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ and let $(W_t)_{t\geqslant 0}$ be a d'-dimensional Brownian motion. For $h(.) \in H = L^2([0,T], \mathbb{R}^{d'})$, $W(h)$ is the Wiener stochastic integral $\int_0^T h(t)\,\mathrm{d}W_t$.

Let $\mathscr{S}$ denote the class of random variables of the form $F = f(W(h_1),\dots,W(h_n))$ where $f \in C_p^\infty(\mathbb{R}^n)$, $(h_1,\dots,h_n)\in H^n$ and $n\geqslant 1$.

For $F\in\mathscr{S}$, we define its derivative $\mathscr{D}F$ as the H-valued random variable given by

$$\mathscr{D}F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1),\dots,W(h_n))h_i.$$

The operator $\mathscr{D}$ is closable as an operator from $L^p(\Omega)$ to $L^p(\Omega;H)$, for $p\geqslant 1$. Its domain is denoted by $\mathbb{D}^{1,p}$ w.r.t. the norm $\|F\|_{1,p} = [\mathbb{E}|F|^p + \mathbb{E}(\|\mathscr{D}F\|_H^p)]^{1/p}$.

We can define the iteration of the operator $\mathscr{D}$, in such a way that for a smooth random variable F, the derivative $\mathscr{D}^k F$ is a random variable with values on $H^{\otimes k}$. As in the case $k=1$, the operator $\mathscr{D}^k$ is closable from $S\subset L^p(\Omega)$ into $L^p(\Omega; H^{\otimes k})$, $p\geqslant 1$. If we define the norm

$$\|F\|_{k,p} = \left[\mathbb{E}|F|^p + \sum_{j=1}^k \mathbb{E}(\|\mathscr{D}^j F\|^p_{H^{\otimes j}})\right]^{1/p},$$

we denote its domain by $\mathbb{D}^{k,p}$. Set $\mathbb{D}^\infty = \bigcap_{p\geqslant 1}\bigcap_{k\geqslant 1}\mathbb{D}^{k,p}$.

We associate with $F=(F^1,\dots,F^m)\in(\mathbb{D}^\infty)^m$ its Malliavin covariance matrix, denoted by $\gamma_F = (\gamma_F^{i,j})_{1\leqslant i,j\leqslant m}$, which is defined by $\gamma_F^{i,j} = \langle \mathscr{D}F^i, \mathscr{D}F^j\rangle_H$. A crucial tool of the theory is the following integration by parts formula.

Proposition 4.1. *Let $F\in(\mathbb{D}^\infty)^m$ such that its Malliavin matrix satisfies $\gamma_F^{-1}\in\bigcap_{p\geqslant 1}L^p$. Let $f\in C_b^\infty(\mathbb{R}^m)$ and $G\in\mathbb{D}^\infty$. For any multi-index α, there is a random variable $H_\alpha(F,G)\in\bigcap_{p\geqslant 1}L^p$ such that*

$$\mathbb{E}[G\partial_x^\alpha f(F)] = \mathbb{E}[f(F)H_\alpha(F,G)]. \tag{48}$$

Now, we intend to apply such a result for $F=Y_t$, some Itô process (e.g. $F=X_t$ or $F=\tilde{X}_t$). We restrict our attention on a specific class of elliptic Itô processes defined in the following proposition.

Proposition 4.2. *Assume that assumptions* (H1) *and* (H2) *are satisfied. Consider v, a map from $\mathbb{R}^+$ into $\mathbb{R}^+$, satisfying the non-anticipative condition: $0\leqslant v(s)\leqslant s$ for any s. Let $(Y_t^v)_{t\geqslant 0}$ be the d-dimensional Itô process defined by*

$$Y_t^v = x + \int_0^t B(Y^v_{v(s)})\,\mathrm{d}s + \int_0^t \sigma(Y^v_{v(s)})\,\mathrm{d}W_s.$$

Then, for $t>0$, $Y_t^v\in\mathbb{D}^\infty$ and for $k\geqslant 1$, $p>1$, there is a function $K(T)$ (which does not depend on v), such that

$$\sup_{t\in[0,T]}\|Y_t^v(x)\|_{k,p}\leqslant K(T)(1+\|x\|).$$

The Malliavin covariance matrix of Y_t^v is invertible a.s. and its inverse, denoted by Γ_t^v belongs to $\bigcap_{p>1} L^p$. Moreover, we have

$$\|\Gamma_t^v(x)\|_{L^p} \leqslant \frac{K(T)}{t^d}, \tag{49}$$

uniformly in x and v.

Integration by parts formula: for all $p > 1$, for all multi-index α, for $s \in [0,T]$ and $t \in (0,T]$, for f and g any $C_b^{|\alpha|}(\mathbb{R}^d,\mathbb{R})$ functions, there are a random variable $H_\alpha(g(Y_t^v), Y_s^v) \in L^p$ and some function $K(T)$ (uniform in v,x,s,t,f and g) such that

$$\mathbb{E}_x[\partial_x^\alpha f(Y_t^v)g(Y_s^v)] = \mathbb{E}_x[f(Y_t^v)H_\alpha(g(Y_s^v), Y_t^v)], \tag{50}$$

with

$$[\mathbb{E}_x|H_\alpha(g(Y_s^v), Y_t^v)|^p]^{1/p} \leqslant \frac{K(T)}{t^{|\alpha|/2}}\|g\|_{C_b^{|\alpha|}}. \tag{51}$$

Proof. These results are derived from Kusuoka and Stroock (1984). The estimates of Sobolev norms $\| \ \|_{k,p}$ are given in Theorem 2.19 and the inequality (49) is stated in Theorem 3.5. To obtain (51), combine Theorem 1.20 and Corollary 3.7. □

Remark 4.1. Note that the choice $v(s)=\varphi(s)=\sup\{t_i\colon t_i\leqslant s\}$ corresponds to $Y_t^v=\tilde{X}_t$, whereas $v(s)=s$ corresponds to $Y_t^v=X_t$. Obviously, for the latter, to obtain (50), assumption (H2) can be considerably weakened to some hypoellipticity conditions.

We now come back to

4.2. Proof of the estimates (23)–(25) for $s\geqslant T/2$

We first prove the estimate (25). Let $s\geqslant T/2$. Set $\tilde{\tau}' = \tilde{\tau}_d \wedge \tilde{\tau}(R)$. Since $\mathbb{1}_{s<\tilde{\tau}'} = 1 - \mathbb{1}_{s\geqslant\tilde{\tau}'}\mathbb{1}_{t\geqslant\tilde{\tau}'} - \mathbb{1}_{s\geqslant\tilde{\tau}'>t}$, the term to estimate can be rewritten as

$$\begin{aligned}\mathbb{E}_x[\mathbb{1}_{s<\tilde{\tau}_d\wedge\tilde{\tau}(R)}g(\tilde{X}_t)\partial_x^\alpha[v\partial_x^{\alpha'}\psi](s,\tilde{X}_s)] &= \mathbb{E}_x[g(\tilde{X}_t)\partial_x^\alpha[v\partial_x^{\alpha'}\psi](s,\tilde{X}_s)]\\ &\quad - \mathbb{E}_x[\mathbb{1}_{s\geqslant\tilde{\tau}'}\mathbb{1}_{t\geqslant\tilde{\tau}'}g(\tilde{X}_t)\partial_x^\alpha[v\partial_x^{\alpha'}\psi](s,\tilde{X}_s)]\\ &\quad - \mathbb{E}_x[\mathbb{1}_{s\geqslant\tilde{\tau}'>t}g(\tilde{X}_t)\partial_x^\alpha[v\partial_x^{\alpha'}\psi](s,\tilde{X}_s)]\\ &:= B_1 - B_2 - B_3.\end{aligned} \tag{52}$$

Term B_1. Applying Proposition 4.2, we immediately obtain

$$|B_1| = |\mathbb{E}_x[v(s,\tilde{X}_s)\partial_x^{\alpha'}\psi(\tilde{X}_s)H_\alpha(g(\tilde{X}_t),\tilde{X}_s)]| \leqslant \frac{\|f\|_\infty}{1\wedge\varepsilon^{|\alpha'|}}\frac{K(T)}{s^{|\alpha|/2}} \leqslant \frac{\|f\|_\infty}{1\wedge\varepsilon^{|\alpha'|}}\frac{K(T)}{T^{|\alpha|/2}}, \tag{53}$$

using estimate (51) and taking into account that $s\geqslant T/2$.

Term B_2. Denote $s'=s-\tilde{\tau}'$; $t'=t-\tilde{\tau}'$; $t_j=\sup\{t_i\colon t_i\leqslant\tilde{\tau}'\}$; $\tilde{t}_0=0$, $\tilde{t}_1=t_{j+1}-\tilde{\tau}'>0$ and $\tilde{t}_{k+1}=\tilde{t}_k+T/N$ for $k\geqslant 1$. Now, apply strong Markov property on $(W_t)_{t\geqslant 0}$ at time $\tilde{\tau}'$ ($(W'_u=W_{u+\tilde{\tau}'}-W_{\tilde{\tau}'})_{u\geqslant 0}$ is a new Brownian motion, independent of $\mathscr{F}_{\tilde{\tau}'}$) to obtain

$$B_2 = \mathbb{E}_x[\mathbb{1}_{s\geqslant\tilde{\tau}'}\mathbb{1}_{t\geqslant\tilde{\tau}'}\mathbb{E}_{\tilde{X}_{\tilde{\tau}'},\tilde{X}_{t_j},\tilde{t}_1,T/N}\{g(Y_{t'})\partial_x^\alpha[v\partial_x^{\alpha'}\psi](t,Y_{s'})\}],$$

where the index $(\tilde{X}_{\tilde{\tau}'}, \tilde{X}_{t_j}, \tilde{t}_1, T/N)$ refers to the law of the process $(Y_u)_{u\geqslant 0}$ defined by

$$0\leqslant u < \tilde{t}_1 \quad Y_u = \tilde{X}_{\tilde{\tau}'} + \int_0^u B(\tilde{X}_{t_j})\,\mathrm{d}r + \int_0^u \sigma(\tilde{X}_{t_j})\,\mathrm{d}W'_r,$$

$$u\geqslant \tilde{t}_1 \quad Y_u = Y_{\tilde{t}_1} + \int_{\tilde{t}_1}^u B(Y_{v(r)})\,\mathrm{d}r + \int_{\tilde{t}_1}^u \sigma(Y_{v(r)})\,\mathrm{d}W'_r, \tag{54}$$

with $v(r) := \sup\{\tilde{t}_k : \tilde{t}_k \leqslant r\}$. In other words, $(Y_u)_{u\geqslant 0}$ is an Itô process as defined in Proposition 4.2 (except that on $[0, \tilde{t}_1)$, the coefficients depend on $\mathscr{F}_{\tilde{\tau}_c}$, but this slight modification does not change the results). Hence

$$\begin{aligned}&\mathbb{E}_{\tilde{X}_{\tilde{\tau}'}, \tilde{X}_{t_j}, \tilde{t}_1, T/N}\{g(Y_{t'})\partial_x^{\alpha}[v\partial_x^{\alpha'}\psi](t, Y_{s'})\}\\ &\quad = \mathbb{E}_{\tilde{X}_{\tilde{\tau}'}, \tilde{X}_{t_j}, \tilde{t}_1, T/N}\{[v\partial_x^{\alpha'}\psi](t, Y_{s'})H_\alpha(g(Y_{t'}), Y_{s'})\}.\end{aligned} \tag{55}$$

On the one hand, using (51), there is a function $K(T)$ (uniform in $\tilde{\tau}', \tilde{X}_{t_j}, \tilde{t}_1$ and T/N) such that

$$\sqrt{\mathbb{E}_{\tilde{X}_{\tilde{\tau}'}, \tilde{X}_{t_j}, \tilde{t}_1, T/N}\{H_\alpha^2(g(Y_{t'}), Y_{s'})\}} \leqslant \frac{K(T)}{s'^{|\alpha|/2}}.$$

On the other, since $v(t,.)\partial_x^{\alpha'}\psi(.)$ vanishes on $D^c \cup V_{\partial D}(\varepsilon/2)$ and $Y_0 = \tilde{X}_{\tilde{\tau}'} \in D^c$, we deduce

$$\begin{aligned}\sqrt{\mathbb{E}_{\tilde{X}_{\tilde{\tau}'}, \tilde{X}_{t_j}, \tilde{t}_1, T/N}\{[v\partial_x^{\alpha'}\psi]^2(t, Y_{s'})\}} &\leqslant \frac{\|f\|_\infty}{1\wedge \varepsilon^{|\alpha'|}} C_{|\alpha'|}\sqrt{\mathbb{P}_{\tilde{X}_{\tilde{\tau}'}, \tilde{X}_{t_j}, \tilde{t}_1, T/N}\{\|Y_{s'} - Y_0\| \geqslant \varepsilon/2\}}\\ &\leqslant \frac{\|f\|_\infty}{1\wedge \varepsilon^{|\alpha'|}} K(T)\exp\left(-c\frac{\varepsilon^2}{8s'}\right),\end{aligned}$$

using the large deviation estimate from Lemma 4.1 below. By applying the Schwarz inequality in (55), we conclude that

$$\begin{aligned}|B_2| &\leqslant \mathbb{E}_x\left[\frac{\|f\|_\infty}{1\wedge \varepsilon^{|\alpha'|}} K(T)\exp\left(-c\frac{\varepsilon^2}{8(s-\tilde{\tau}')}\right)\frac{1}{(s-\tilde{\tau}')^{|\alpha|/2}}\right]\\ &\leqslant \frac{\|f\|_\infty}{1\wedge \varepsilon^{|\alpha|+|\alpha'|}} K(T).\end{aligned} \tag{56}$$

Term B_3. Analogous arguments apply and enable us to show

$$|B_3| \leqslant \frac{\|f\|_\infty}{1\wedge \varepsilon^{|\alpha|+|\alpha'|}} K(T), \tag{57}$$

which details we omit. Substituting (53), (56) and (57) into (52), this completes the proof of (25) for $s\geqslant T/2$.

For the proof of estimate (23), we note that for $R=0$, one has $\tilde\tau(R)=\tilde\tau_c\leqslant\tilde\tau_d$, so that estimate (25) with $\alpha'=\emptyset$ can be rewritten as

$$|\mathbb{E}_x[\mathbb{1}_{s<\tilde\tau_c}g(\tilde X_t)\partial_x^\alpha[v\psi](s,\tilde X_s)]|\leqslant\frac{\|f\|_\infty}{1\wedge\varepsilon^{|\alpha|}}\frac{K(T)}{T^{|\alpha|/2}}. \tag{58}$$

On the other hand, using the definition of ψ and the estimates (22), we easily derive the following upper bound:

$$|\mathbb{E}_x[\mathbb{1}_{s<\tilde\tau_c}g(\tilde X_t)\partial_x^\alpha[v(1-\psi)](s,\tilde X_s)]|\leqslant\frac{\|f\|_\infty}{1\wedge\varepsilon^{|\alpha|}}K(T). \tag{59}$$

Now, estimate (23) obviously results from (58) and (59).

For (24), same arguments apply, and we omit the details. Lemma 3.1 is proved. □

Lemma 4.1. *Let $(Y_t)_{t\geqslant0}$ be an Itô process defined by $\mathrm{d}Y_t=b_t\,\mathrm{d}t+\sigma_t\,\mathrm{d}W_t$, with adapted and uniformly bounded coefficients. Let S and S' be two stopping times upper bounded by T, such that $0\leqslant S'-S\leqslant\Delta\leqslant T$. Then, there exists a constant $c>0$ and a function $K(T)$, such that*

$$\forall\eta>0\quad\mathbb{P}\left(\sup_{t\in[S,S']}\|Y_t-Y_S\|_{\mathbb{R}^d}\geqslant\eta\right)\leqslant K(T)\exp\left(-c\frac{\eta^2}{\Delta}\right).$$

Proof. This Lemma deals with some classical estimates for large deviation probabilities. There is no loss in considering that $Y_t\in\mathbb{R}$, up to dividing η by d.

Then, if $\eta\leqslant2\|b\|_\infty\Delta$, the estimation is obvious, since $\mathbb{P}_x(\sup_{t\in[S,S']}|Y_t-Y_S|\geqslant\eta)\leqslant\exp(c\eta^2/\Delta-c\eta^2/\Delta)\leqslant\exp(4c\|b\|_\infty^2T)\exp(-c\eta^2/\Delta)$.

If $\eta>2\|b\|_\infty\Delta$, then $\mathbb{P}(\sup_{t\in[S,S']}|Y_t-Y_S|\geqslant\eta)\leqslant\mathbb{P}(\sup_{t\in[S,S']}|\int_S^t\sigma_s\,\mathrm{d}W_s|\geqslant\eta/2)$. Apply the Bernstein exponential inequality for martingales (see Revuz and Yor, 1991, p. 145) to $M_t=\int_0^t\mathbb{1}_{S<u\leqslant S'}\sigma_u\,\mathrm{d}W_u$ satisfying $\langle M\rangle_T\leqslant\|\|\sigma\|_{\mathbb{R}^{d'}}^2\|_\infty\Delta$ to complete the proof. □

5. Proof of Lemmas 3.6 and 3.7

5.1. Proof of Lemma 3.6

Because the arguments we develop can also apply to X, we only sketch the proof for $\tilde X$, i.e.

$$\forall(s,x)\in(0,T]\times D\quad\mathbb{P}_x(\tilde X_{\varphi(s)}\in D,\ \tilde X_s\notin D)\leqslant\frac{K(T)}{\sqrt N}\frac{1}{\sqrt s}. \tag{60}$$

If $\varphi(s)=0$, estimate (60) is obvious because $\mathbb{P}_x(\tilde X_s\notin D)\leqslant1\leqslant\sqrt{T/Ns}$.

If $\varphi(s)>0$, we apply Markov property in $\varphi(s)$ and we directly obtain

$$\begin{aligned}\mathbb{P}_x(\tilde X_{\varphi(s)}\in D,\ \tilde X_s\notin D)&\leqslant\mathbb{E}_x[\mathbb{1}_{\tilde X_{\varphi(s)}\in D}\mathbb{P}_{\tilde X_{\varphi(s)}}(\|\tilde X'_{s-\varphi(s)}-\tilde X'_0\|\geqslant d(\tilde X'_0,\partial D))]\\&\leqslant K(T)\mathbb{E}_x\left[\mathbb{1}_{\tilde X_{\varphi(s)}\in D}\exp\left(-c\frac{d^2(\tilde X_{\varphi(s)},\partial D)}{s-\varphi(s)}\right)\right],\end{aligned} \tag{61}$$

using Lemma 4.1 with $\Delta = s - \varphi(s)$. To evaluate the last expectation, we exploit that the law of $\tilde{X}_t(x)$ has a density $\tilde{p}_t(x, y)$ w.r.t. the Lebesgue measure on $\mathbb{R}^d$ and that moreover, there exist a constant $c' > 0$ and a function $K(T)$, such that

$$\forall (t,x,y) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d \quad \tilde{p}_t(x,y) \leqslant \frac{K(T)}{t^{d/2}} \exp\left(-c' \frac{\|x-y\|^2}{t}\right) \tag{62}$$

(see Kusuoka and Stroock, 1984, or Bally and Talay, 1996b).

In the case of $D = \{y \in \mathbb{R}^d \colon y_1 > 0\}$, a straightforward computation involving Gaussian densities leads to

$$\begin{aligned}
&\mathbb{E}_x\left[\mathbb{1}_{\tilde{X}_{\varphi(s)} \in D} \exp\left(-c\frac{d^2(\tilde{X}_{\varphi(s)}, \partial D)}{s - \varphi(s)}\right)\right] \\
&\leqslant \int_{\mathbb{R}^d} \mathrm{d}y \frac{K(T)}{(\varphi(s))^{d/2}} \exp\left(-c'\frac{\|x-y\|^2}{\varphi(s)} - c\frac{y_1^2}{s-\varphi(s)}\right) \\
&\leqslant K(T)\sqrt{\frac{s-\varphi(s)}{s}} \exp\left(-(c \wedge c')\frac{x_1^2}{s}\right),
\end{aligned}$$

which completes the proof in this case, taking in account (61) and that $s - \varphi(s) \leqslant TN^{-1}$.

For the general case, we use Property 3.1 to map locally D as a half-space: in these maps, $d(y, \partial D)$ has a simple expression and this enables us to reduce to the first case. We omit the details (for a detailed proof, see Lemma 3.4.5 in Gobet, 1998a).

5.2. Proof of Lemma 3.7

We first prove the estimate for $\mathbb{E}_x[L^0_{T\wedge\tilde{\tau}_d}(F_1(\tilde{X}))]$. Tanaka's formula yields

$$\frac{1}{2}L^0_{T\wedge\tilde{\tau}_d}(F_1(\tilde{X})) = (F_1(\tilde{X}_{T\wedge\tilde{\tau}_d}))^- - (F_1(x))^- + \int_0^{T\wedge\tilde{\tau}_d} \mathbb{1}_{F_1(\tilde{X}_t)\leqslant 0}\, \mathrm{d}(F_1(\tilde{X}_t)).$$

Recall that $\{F_1(y) \leqslant 0\} = \{y \notin D\}$ (see Property 3.1). Since $F_1(\tilde{X}_t)$ is an Itô process with bounded coefficients, by using Lemma 3.6, we easily deduce

$$\mathbb{E}_x[L^0_{T\wedge\tilde{\tau}_d}(F_1(\tilde{X}))] \leqslant 2\mathbb{E}_x[(F_1(\tilde{X}_{T\wedge\tilde{\tau}_d}))^-] + \frac{K(T)}{\sqrt{N}}.$$

Now, it remains to bound $\mathbb{E}_x[(F_1(\tilde{X}_{T\wedge\tilde{\tau}_d}))^-]$. Obviously,

$$\mathbb{E}_x[(F_1(\tilde{X}_{T\wedge\tilde{\tau}_d}))^-] = \sum_{i=1}^{N} \mathbb{E}_x[\mathbb{1}_{t_i=\tilde{\tau}_d}(F_1(\tilde{X}_{t_i}))^-]. \tag{63}$$

Using $\{t_i = \tilde{\tau}_d\} = \{t_{i-1} < \tilde{\tau}_d\} \cap \{\tilde{X}_{t_i} \notin D\}$ and $F_1 > 0$ on D, we deduce

$$\mathbb{E}_x[\mathbb{1}_{t_i=\tilde{\tau}_d}(F_1(\tilde{X}_{t_i}))^-] = \mathbb{E}_x[\mathbb{1}_{t_{i-1}<\tilde{\tau}_d}\mathbb{E}_{\tilde{X}_{t_{i-1}}}[(F_1(\tilde{X}_{\frac{T}{N}}))^-]]. \tag{64}$$

If we set $\tilde{\tau}_c = \inf\{t > 0 \colon \tilde{X}_t \notin D\}$, simple computation yields

$$\begin{aligned}
\mathbb{E}_{\tilde{X}_{t_{i-1}}}[(F_1(\tilde{X}_{\frac{T}{N}}))^-] &= \mathbb{E}_{\tilde{X}_{t_{i-1}}}[\mathbb{1}_{\tilde{\tau}_c<\frac{T}{N}}(F_1(\tilde{X}_{\frac{T}{N}}))^-] \\
&= \mathbb{E}_{\tilde{X}_{t_{i-1}}}[\mathbb{1}_{\tilde{\tau}_c<\frac{T}{N}}\mathbb{E}[(F_1(\tilde{X}_{\frac{T}{N}}))^-/\mathscr{F}_{\tilde{\tau}_c}]]
\end{aligned}$$

$$= \mathbb{E}_{\tilde{X}_{t_{i-1}}}[\mathbb{1}_{\tilde{\tau}_c < \frac{T}{N}} \mathbb{E}[(F_1(\tilde{X}_{\frac{T}{N}}))^- - (F_1(\tilde{X}_{\tilde{\tau}_c}))^- / \mathscr{F}_{\tilde{\tau}_c}]]$$

$$\leqslant \mathbb{E}_{\tilde{X}_{t_{i-1}}}\left[\mathbb{1}_{\tilde{\tau}_c < \frac{T}{N}} C\sqrt{\frac{T}{N} - \tilde{\tau}_c}\right] \leqslant C\sqrt{\frac{T}{N}}\mathbb{P}_{\tilde{X}_{t_{i-1}}}\left[\tilde{\tau}_c < \frac{T}{N}\right], \tag{65}$$

using classical estimates on the increments of Itô process with bounded coefficients. At last, we state a technical (and interesting for itself) Lemma: we will prove it finally.

Lemma 5.1. *Under* (H1)–(H3), *there is a function* $K(T)$, *such that*

$$\forall (u,z) \in \left(0, \frac{T}{N}\right] \times \bar{D} \quad 1 \leqslant \frac{\mathbb{P}_z(\tilde{\tau}_c < u)}{\mathbb{P}_z(\tilde{X}_u \notin D)} \leqslant K(T). \tag{66}$$

Hence, using (64) and (65), we have $\mathbb{E}_x[\mathbb{1}_{t_i = \tilde{\tau}_d}(F_1(\tilde{X}_{t_i}))^-] \leqslant N^{-1/2}K(T)\mathbb{P}_x(t_i = \tilde{\tau}_d)$, and by substituting into (63), we conclude that $\mathbb{E}_x[(F_1(\tilde{X}_{T\wedge\tilde{\tau}_d}))^-] \leqslant K(T)N^{-1/2}$ and the proof is complete. □

Proof of Lemma 5.1. This lemma states an approximated reflection principle (indeed, if $\tilde{X}$ is a linear Brownian motion and $D = (-\infty, b)$, $\mathbb{P}_z(\tilde{\tau}_c < u) = 2\mathbb{P}_z(\tilde{X}_u \notin D)$ for $z \leqslant b$).

Note that the lower bound in (66) is obvious. Let $z \in \bar{D}$ and $0 < u \leqslant T/N$. First, we have

$$\mathbb{P}_z(\tilde{\tau}_c < u) = \mathbb{E}_z(\mathbb{1}_{\tilde{\tau}_c < u}\mathbb{P}[\tilde{X}_u \notin D/\mathscr{F}_{\tilde{\tau}_c}]) + \mathbb{E}_z(\mathbb{1}_{\tilde{\tau}_c < u}\mathbb{P}[\tilde{X}_u \in D/\mathscr{F}_{\tilde{\tau}_c}]). \tag{67}$$

Now, we observe that it is enough to prove that on $\{\tilde{\tau}_c < u\}$ we have

$$\mathbb{P}(\tilde{X}_u \notin D/\mathscr{F}_{\tilde{\tau}_c}) \geqslant \frac{1}{K(T)}, \tag{68}$$

for some function $K(T)$, which does not depend on $\mathscr{F}_{\tilde{\tau}_c}$ and u. Indeed,

$$\mathbb{P}(\tilde{X}_u \in D/\mathscr{F}_{\tilde{\tau}_c}) = \mathbb{P}(\tilde{X}_u \notin D/\mathscr{F}_{\tilde{\tau}_c})\frac{[1 - \mathbb{P}(\tilde{X}_u \notin D/\mathscr{F}_{\tilde{\tau}_c})]}{\mathbb{P}(\tilde{X}_u \notin D/\mathscr{F}_{\tilde{\tau}_c})} \leqslant K(T)\mathbb{P}(\tilde{X}_u \notin D/\mathscr{F}_{\tilde{\tau}_c}),$$

and by substituting in (67), this ends the proof of (66), noting that $\mathbb{P}_z(\tilde{\tau}_c < u, \tilde{X}_u \notin D) = \mathbb{P}_z(\tilde{X}_u \notin D)$.

To show that (68) holds, the basic idea is to bound from below $\mathbb{P}(\tilde{X}_u \notin D/\mathscr{F}_{\tilde{\tau}_c})$ by $\mathbb{P}(\tilde{X}_u \in \mathscr{K}/\mathscr{F}_{\tilde{\tau}_c})$, where $\mathscr{K}$ is a truncated cone included in D^c: this technical fact is used to prove Zaremba's cone condition for the Dirichlet problem (see e.g. Karatzas and Shreve, 1988, p. 250).

Let us define the cone $\mathscr{K}(s,w,\theta)$ with origin $s \in \mathbb{R}^d$, direction $w \in \mathbb{R}^d \setminus \{0\}$ and aperture $\theta \in (0,\pi)$ by $\mathscr{K}(s,w,\theta) := \{y \in \mathbb{R}^d\colon (y-s).w \geqslant \|y-s\|\|w\| \cos(\theta)\}$. Let $B(s,r)$ be the ball with centre $s \in \mathbb{R}^d$ and radius $r > 0$. Since D satifies an uniform exterior sphere condition, it satisfies also an uniform truncated exterior cone condition, i.e. for all $\theta \in (0,\pi/2)$, there is a radius $R(\theta) > 0$ such that

$$\forall s \in \partial D \quad \mathscr{K}(s, -n(s), \theta) \cap B(s, R(\theta)) \subset D^c$$

(recall that $n(s)$ is the unit inner normal at s). Set $\theta = \pi/3$ and denote $R' = R(\pi/3)$. Using an explicit lower bound for the Gaussian density of $\tilde{X}_u$ conditionally on $\mathscr{F}_{\tilde{\tau}_c}$, we easily obtain

$$\begin{aligned}\mathbb{P}(\tilde{X}_u \notin D/\mathcal{F}_{\tilde{\tau}_c}) &\geqslant \mathbb{P}(\tilde{X}_u \in \mathcal{K}(\tilde{X}_{\tilde{\tau}_c}, -n(\tilde{X}_{\tilde{\tau}_c}), \pi/3) \cap B(\tilde{X}_{\tilde{\tau}_c}, R)/\mathcal{F}_{\tilde{\tau}_c})\\ &\geqslant \frac{1}{K(T)} \int_{B(\tilde{X}_{\tilde{\tau}_c}, R')} \mathrm{d}y \frac{\mathbb{1}_{\{(y-\tilde{X}_{\tilde{\tau}_c}).(-n(\tilde{X}_{\tilde{\tau}_c}))\geqslant \|y-\tilde{X}_{\tilde{\tau}_c}\|/2\}}}{(u-\tilde{\tau}_c)^{d/2}}\\ &\quad\times \exp\left(-\frac{\|y-\tilde{X}_{\tilde{\tau}_c}\|^2}{c\,(u-\tilde{\tau}_c)}\right)\\ &\geqslant \frac{1}{K(T)} \int_{B(0,R'/\sqrt{u-\tilde{\tau}_c})} \mathrm{d}y' \mathbb{1}_{\{y'.(-n(\tilde{X}_{\tilde{\tau}_c}))\geqslant \|y'\|/2\}} \exp\left(-\frac{\|y'\|^2}{c}\right),\end{aligned}$$

using a simple change of variables. To conclude, use the spheric symmetry and $u - \tilde{\tau}_c \leqslant T$. □

To estimate $\mathbb{E}_x[L^0_{T\wedge\tau_d}(F_1(X))]$, we proceed in a very similar way. To prove analogous estimate to (68), we use the following lower bound for the density $p_u(s, y)$ of the law of $X_u(y)$ (see Aronson, 1967):

$$p_u(s, y) \geqslant \frac{1}{K(T)u^{d/2}} \exp\left(-\frac{\|s-y\|^2}{cu}\right). \qquad \square$$

Remark 5.1. If we think of local time in terms of the number of crossings, the statement of Lemma 3.7 $\mathbb{E}_x[L^0_{T\wedge\tilde{\tau}_d}(F_1(\tilde{X}))] \propto N^{-1/2}$ is not surprising. Indeed, for some process, we know that its number of crossings of level 0 renormalized by $CN^{-1/2}$ converges in probability to its local time at 0, under some conditions (see e.g. Azais, 1989). Here, the number of crossings of $F_1(\tilde{X})$ at time $T \wedge \tilde{\tau}_d$ equals 0 or 1: hence, heuristically $L^0_{T\wedge\tilde{\tau}_d}(F_1(\tilde{X}))$ is of order $N^{-1/2}$.

6. Extensions

Some interesting situations are not covered by the compactness assumption on ∂D in (H3). In fact, this is a technical hypothesis which we use to go from a local description to a global one. It can be relaxed if D is limited by one or two parallel hyperplans, for example.

The boundedness hypothesis on f in (H4) can be weakened to $|f(x)| \leqslant C\exp(c|x|)$: since the coefficients of (1) are bounded, classical exponential estimates enable us to replace $\|f\|_\infty$ by $C\exp(c|x|)$.

For each theorem of this paper, the C^∞ assumptions on B, σ and D can be weakened to C^k_b on B, σ and $C^{k'}$ on D, for suitable integers k and k'. Being a little careful, we can show that Theorem 2.1 holds with $k = k' = 7$, Theorem 2.2 with $k = 3$ and $k' = 5$, Theorems 2.3 and 2.4 for f satisfying (H4) with $k = k' = 5$, Theorems 2.3 and 2.4 for f satisfying (H5-1) with $k = 2$ and $k' = 3$.

In the one-dimensional case, to obtain Theorem 2.2, it is sufficient to assume that $f|_{\partial D} = 0$ and $f \in C^3_b(\bar{D}, \mathbb{R})$, instead of slightly stronger vanishing conditions for f at the boundary under (H5). Because $d = 1$, the estimates we can derive for the explosion of derivatives of v when $t \to T$ are more tractable than those derivable in higher-dimensional cases (see Gobet, 1998a, for details). These techniques enable us also to prove that $\mathscr{E}_c(f,T,x,N) = O(N^{-1/2})$ if $f \in C_b^1(\bar{D},\mathbb{R})$ without conditions on ∂D.

Instead of (H2), we can also consider hypoellipticity assumptions on L, with a non-characteristic boundary condition: in that case, we may extend theorems for measurable functions with support strictly included in D. Indeed, estimates like (10) remain valid (see Cattiaux, 1991). Actually, the main difficulty comes from the proof of Lemma 3.1, because the law of $\tilde{X}_t$ may be degenerate (i.e. we cannot directly apply the integration by parts formula): nevertheless, it is possible to handle this case, using perturbation and localization on the Malliavin covariance matrix of $\tilde{X}_t$ (see Bally and Talay, 1996a).

Following the approach of Bally and Talay (1996b), the choice of f as an approximation of the identity permits also the analysis of the approximation between the transition densities of the two killed processes. An adaptation of Lemma 3.1 (to obtain estimates involving $\|F\|_\infty$ instead of $\|f\|_\infty$, where F is the cumulative of f) proves that

$$\forall (x,y) \in D\times D \quad |q_T(x,y) - \tilde{q}_T(x,y)| \leqslant \frac{1}{N}\frac{K(T)}{T^{d/2}(1\wedge d_{\partial D}(y))^q}\exp\left(-c\frac{(x-y)^2}{T}\right),$$

for some positive constants c and q, where $\tilde{q}_T(x,y)$ is the transition density of the killed continuous Euler scheme. The term $(1 \wedge d_{\partial D}(y))^q$ is related to the condition $d(\mathrm{supp}(f),\partial D) \geqslant 2\varepsilon$ from Theorem 2.1.

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