Efficiency measurement in fuzzy additive data envelopment analysis

Adel Hatami-Marbini

Louvain School of Management,
Center of Operations Research and Econometrics (CORE),
Universite Catholique de Louvain,
34 Voie du Roman Pays, B-1348 Louvain-la-Neuve, Belgium
Fax: +32 10 47 4301
E-mail: adel.hatamimarbini@uclouvain.be
E-mail: adel_hatami@yahoo.com

Madjid Tavana*

Management Information Systems,
Lindback Distinguished Chair of Information Systems,
La Salle University,
Philadelphia, PA 19141, USA
Fax: +1 267 295 2854
E-mail: tavana@lasalle.edu
*Corresponding author

Ali Emrouznejad

Aston Business School,
Aston University, Birmingham, UK
E-mail: a.emrouznejad@aston.ac.uk

Saber Saati

Department of Mathematics,
Tehran-North Branch, Islamic Azad University,
P.O. Box 19585-936, Tehran, Iran
E-mail: s_saatim@iau-tnb.ac.ir

Abstract: Performance evaluation in conventional data envelopment analysis (DEA) requires crisp numerical values. However, the observed values of the input and output data in real-world problems are often imprecise or vague. These imprecise and vague data can be represented by linguistic terms characterised by fuzzy numbers in DEA to reflect the decision-makers' intuition and subjective judgements. This paper extends the conventional DEA models to a fuzzy framework by proposing a new fuzzy additive DEA model for evaluating the efficiency of a set of decision-making units (DMUs) with fuzzy inputs and outputs. The contribution of this paper is threefold: (1) we consider ambiguous, uncertain and imprecise input and output data in DEA, (2) we propose a new fuzzy additive DEA model derived from the α-level approach and (3) we demonstrate the practical aspects of our model with two numerical examples and show its comparability with five different fuzzy DEA methods in the literature.

Keywords: DEA; data envelopment analysis; fuzzy sets theory; DMUs; decision-making units; fuzzy additive model.

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Biographical notes: Adel Hatami-Marbini is a Doctoral student and a Research Assistant in the Louvain School of Management and the Center for Operations Research and Econometrics (CORE) at the Université catholique de Louvain (UCL) in Belgium. He received his MSc and BSc in Industrial Engineering from Islamic Azad University in Iran. His research interests are in operations research, data envelopment analysis, multicriteria decision-making and fuzzy sets theory. He has published in journals, such as *Omega, Computers and Industrial Engineering, Applied Soft Computing, Journal of Applied Sciences, Applied Mathematical Sciences, Int. J. Mathematics in Operational Research, Int. J. Operations Research and Information Systems, Int. J. Productivity and Quality Management, Int. J. Information and Decision Sciences and Industrial Engineering International*, among others.

Madjid Tavana is a Professor of Management Information Systems and Decision Sciences and the Lindback Distinguished Chair of Information Systems at La Salle University, where he served as the Chairman of the Management Department and Director of the Center for Technology and Management. He has been a distinguished research fellow at NASA's Kennedy Space Center, NASA's Johnson Space Center, Naval Research Laboratory – Stennis Space Center and Air Force Research Laboratory. He was awarded the prestigious Space Act Award by NASA in 2005. He holds an MBA, a PMIS and a PhD in Management Information Systems. He received his post-doctoral diploma in strategic information systems from the Wharton School of the University of Pennsylvania. He is the Editor-in-Chief for the *Int. J. Strategic Decision Sciences, the Int. J. Enterprise Information Systems* and the *Int. J. Applied Decision Sciences*. He has published in journals, such as *Decision Sciences, Interfaces, Information Systems, Annals of Operations Research, Information and Management*, *Journal of the Operational Research Society, Computers and Operations Research and Advances in Engineering Software*, among others.

Ali Emrouznejad is a Senior Lecturer in Operations and Information Management Group at the Aston Business School in Birmingham, UK. His area of research interests include performance measurement and management, efficiency and productivity analysis and data mining. He serves on the Editorial Board of several scientific journals. He is the Associate Editor of the *IMA Journal of Management Mathematics* and the Guest Editor to several special issues of journals, including *Journal of Operational Research Society, Annals of Operations Research, Journal of Medical Systems* and *Int. J. Energy Management Sector*. He is the co-founder of Performance Improvement Management Software (PIM-DEA).

Saber Saati is an Assistant Professor of Mathematics and Chairman of the Department of Mathematics at the Islamic Azad University, Tehran North Branch in Iran. He received his PhD in Operations Research at Science and

Research Branch of IAU in 2002. His research interests include data envelopment analysis, fuzzy programming and fuzzy MADM. He has published in journals, such as *Ricerca Operativa, Fuzzy Optimisation and Decision Making, Journal of Interdisciplinary Mathematics, Far East Journal of Applied Mathematics, Applied Mathematics and Computation, Iranian Journal of Fuzzy Systems, Advance in Fuzzy Sets and Systems, Int. J. Mathematical Analysis, Applied Mathematical Science, Journal of Industrial Engineering International, Australian Journal of Basic and Applied Sciences* and *Journal of Applied Sciences, Mathematical Sciences and Applied Soft Computing*.

1 Introduction

Data envelopment analysis (DEA) is a widely used mathematical programming approach for comparing the inputs and outputs of a set of homogenous decision-making units (DMUs) by evaluating their relative efficiency. A DMU is considered efficient when no other DMUs can produce more outputs, using an equal or lesser amount of inputs. The DEA generalises the usual efficiency measurement from a single-input single-output ratio to a multiple-input multiple-output ratio by using a ratio of the weighted sum of outputs to the weighted sum of inputs. The conventional DEA methods like CCR (Charnes et al., 1978) and BCC (Banker et al., 1984) require accurate measurement of both the inputs and outputs.

One of the main challenges associated with the application of DEA is the difficulty in quantifying some of the input and output data in real-world problems, where the observed values are often imprecise or vague. Imprecise or vague data may be the result of unquantifiable, incomplete and unobtainable information. In recent years, many researchers have formulated DEA models to deal with the uncertain input and output data. One way to manipulate uncertain data in DEA is via probability distributions. Nevertheless, probability distributions require either *a priori* predictable regularity or *a posteriori* frequency determinations which are difficult to construct. An alternative approach is to represent the imprecise or vague values by membership functions of the fuzzy sets theory. In this study, we propose a new fuzzy additive DEA model for evaluating the efficiency of a set of DMUs with fuzzy inputs and outputs. The proposed model is derived from the α-level approach introduced by Saati et al. (2002) and considers ambiguous, uncertain and imprecise input and output data in DEA. We also show the validity and efficacy of the proposed fuzzy DEA model with two numerical examples.

This paper is organised into seven sections. Section 2, presents a brief review of the existing fuzzy DEA literature. In Section 3, we review some preliminaries and definitions and in Section 4, we provide an overview of the basic DEA models. Section 5 illustrates the details of the proposed framework. In Section 6, we present a numerical comparison of our method with four commonly used methods in the literature to demonstrate the applicability of the proposed framework and exhibit the efficacy of the procedures and algorithms. Section 7 presents our conclusions and future research directions.

2 Fuzzy DEA literature review

The fuzzy DEA methods in the literature are categorised into four general groups by Lertworasirikul et al. (2003a):

1. the fuzzy ranking approach
2. the possibility approach
3. the tolerance approach
4. the α-level-based approach.

Sengupta (1992) proposed a fuzzy mathematical programming approach by incorporating fuzzy input and output data into a DEA model, and defining tolerance levels for the objective function and constraint violations. Guo and Tanaka (2001, 2008) and León et al. (2003) proposed three similar fuzzy DEA models by considering the uncertainties in fuzzy objectives and fuzzy constraints. Lertworasirikul et al. (2003b) proposed a fuzzy DEA model, using the credibility approach where fuzzy variables were replaced by expected credits according to the credibility measures.

Kao and Liu (2000) transformed fuzzy input and output data into intervals by using α-level sets. The α-level set approach was extended by Saati et al. (2002), who defined the fuzzy DEA model as a possibilistic programming problem and transformed it into an interval programming problem. Entani et al. (2002) extended the α-level set research by changing fuzzy input and output data into intervals. Dia (2004) proposed a fuzzy DEA model, where a fuzzy aspiration level and a safety α-level were used to transform the fuzzy DEA model into a crisp DEA. Soleimani-Damaneh et al. (2006) addressed some of the limitations of the fuzzy DEA models proposed by Kao and Liu (2000), León et al. (2003) and Lertworasirikul et al. (2003a) and suggested a fuzzy DEA model to produce crisp efficiencies.

Liu (2008) and Liu and Chuang (2009) extended the α-level set approach by proposing the assurance region approach in the fuzzy DEA model. Hatami-Marbini et al. (2009) proposed a fuzzy DEA model to assess efficiency scores with fuzzy data based on the ranking fuzzy numbers method. Hatami-Marbini and Saati (2009) extended a fuzzy BCC model, which considered fuzziness in the input and output data as well as the u_0 variable. Hatami-Marbini et al. (2010) further developed a four-phase fuzzy DEA framework based on the theory of displaced ideal. They constructed two hypothetical DMUs called the *ideal* and *nadir* DMUs and used them as reference points to evaluate a set of DMUs based on their Euclidean distance from these reference points. They considered the best relative efficiency of the fuzzy ideal DMU and the worst relative efficiency of the fuzzy nadir DMU for ranking the DMUs. Hatami-Marbini et al. (2010) proposed an interactive evaluation process for measuring the relative efficiencies of a set of DMUs in fuzzy DEA with consideration of the DMs' preferences. They constructed a linear programming model with fuzzy parameters and calculated the fuzzy efficiency of the DMUs for different α-levels.

Wang et al. (2009) developed two fuzzy DEA models, using fuzzy arithmetic to handle fuzziness in input and output data in DEA. Soleimani-Damaneh (2008) used the fuzzy signed distance and the fuzzy upper bound concepts to formulate a fuzzy additive model in DEA with fuzzy input–output data. Soleimani-Damaneh (2009) put forward a theorem on the fuzzy DEA model, which was proposed by Soleimani-Damaneh (2008) in order to show the existence of a distance-based upper bound for the objective function of the model. Khodabakhshi et al. (2010) formulated two alternative fuzzy and stochastic additive models to determine returns to scale (RTS) in DEA. They developed the fuzzy and stochastic DEA models based on the possibility approach and the concept of chance constraint programming, respectively.

3 Preliminaries and definitions

In this section, we introduce some basic definitions for fuzzy sets (Dubois and Prade, 1978, 1980; Kaufmann and Gupta, 1991; Klir and Yuan, 1995; Lee, 2005; Zadeh, 1965; Zimmerman, 1996).

Definition 1: *Let U be a universe set. A fuzzy set* $\tilde{A}$ *of U is defined by a membership function* $\mu_{\tilde{A}}(x) \to [0,1]$, *where* $\mu_{\tilde{A}}(x)$, $\forall x \in U$ *indicates the degree of membership of* $\tilde{A}$ *to U.*

Definition 2: *A fuzzy subset* $\tilde{A}$ *of real number R is convex iff*

$$\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \left(\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{A}}(y)\right), \forall x, y \in R, \forall \lambda \in [0,1]$$

where '$\wedge$' denotes the minimum operator.

Definition 3: *The* α*-level of fuzzy set* $\tilde{A}$ *denoted by* $\tilde{A}_\alpha$ *is the crisp set* $\tilde{A}_\alpha = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha\}$. *The support of* $\tilde{A}$ *is the crisp set* $Sup(\tilde{A}) = \{x \mid \mu_{\tilde{A}}(x) > 0\}$. $\tilde{A}$ *is normal iff* $Sup_{x \in U} \mu_{\tilde{A}}(x) = 1$, *where U is the universal set.*

Definition 4: $\tilde{A}$ *is a fuzzy number iff* $\tilde{A}$ *is a normal and convex fuzzy subset of R.*

Definition 5: *A fuzzy number* $\tilde{A} = (a^{m1}, a^{m2}, a^{1}, a^{u})$ *is called a generalised trapezoidal fuzzy number with membership function* $\mu_{\tilde{A}}$ *and has the following properties:*

1 $\mu_{\tilde{A}}$ *is a continuous mapping from R to the closed interval [0, 1]*

2 $\mu_{\tilde{A}}(x) = 0$ *for all* $x \in (-\infty, a^{l}]$

3 μ_{A} *is strictly increasing on* $[a^{l}, a^{m1}]$

4 $\mu_{\tilde{A}}(x) = 1$ *for all* $x \in [a^{m1}, a^{m2}]$

5 $\mu_{\tilde{A}}$ *is strictly decreasing on* $[a^{m2}, a^{u}]$ *and*

6 $\mu_{\tilde{A}}(x) = 0$ *for all* $x \in [a^{u}, +\infty)$.

The membership function $\mu_{\tilde{A}}$ of $\tilde{A}$ can be defined as follows:

$$\mu_{\tilde{A}}(x)=\begin{cases} f_a(x), & a^l \le x \le a^{m1} \\ 1, & a^{m1} \le x \le a^{m2} \\ g_a(x), & a^{m2} \le x \le a^u \\ 0, & \text{Otherwise} \end{cases} \tag{1}$$

where $f_a:[a^l,a^{m1}]\to[0,1]$ and $g_a:[a^{m2},a^u]\to[0,1]$.

The inverse functions of f_a and g_a denoted as f_a^{-1} and g_a^{-1} exist, since $f_a:[a^l,a^{m1}]\to[0,1]$ is continuous and strictly increasing, $f_a^{-1}:[0,1]\to[a^l,a^{m1}]$ is also continuous and strictly increasing. Similarly, since $g_a:[a^{m2},a^u]\to[0,1]$ is continuous and strictly decreasing, $g_a^{-1}:[0,1]\to[a^{m2},a^u]$ is also continuous and strictly increasing. That is, both $\int_0^1 f_a^{-1}\mathrm{d}y$ and $\int_0^1 g_a^{-1}\mathrm{d}y$ exist (Liou and Wang, 1992).

Particularly, we are working with a special type of trapezoidal fuzzy number with a membership function $\mu_{\tilde{A}}$ expressed as follows:

$$\mu_{\tilde{A}}(x)=\begin{cases} \dfrac{x-a^l}{a^{m1}-a^l}, & a^l \le x \le a^{m1} \\ 1, & a^{m1} \le x \le a^{m2} \\ \dfrac{a^u-x}{a^u-a^{m2}}, & a^{m2} \le x \le a^u \\ 0, & \text{Otherwise} \end{cases} \tag{2}$$

The trapezoidal fuzzy number $\tilde{A}=(a^{m1},a^{m2},a^l,a^u)$ is reduced to a real number A, if $a^l=a^{m1}=a^{m2}=a^u$. Conversely, a real number A can be written as a trapezoidal fuzzy number $\tilde{A}=(a,a,a,a)$. If $a^m=a^{m1}=a^{m2}$, then $\tilde{A}=(a^m,a^l,a^u)$ is called a triangular fuzzy number. A triangular fuzzy number has the following membership function:

$$\mu_{\tilde{A}}(x)=\begin{cases} \dfrac{x-a^l}{a^m-a^l}, & a^l \le x \le a^m \\ \dfrac{a^u-x}{a^u-a^m}, & a^m \le x \le a^u \\ 0, & \text{Otherwise} \end{cases} \tag{3}$$

For the sake of simplicity, and without loss of generality, we assume that all fuzzy numbers used throughout the paper are trapezoidal fuzzy numbers.

Definition 6: *Linguistic variables are those variables, whose values are not numbers but phrases or sentences expressed in a natural or artificial language. For example, 'very low', 'low', 'medium', 'high' or 'very high' are linguistic variables, because their values are linguistic rather than numerical. The concept of a linguistic variable is useful in dealing with situations which are too complex or too ill-defined to be reasonably* *described with quantitative values. Linguistic values can also be represented by fuzzy numbers.*

Definition 7: *The minimum t-norm is usually applied in fuzzy linear programming to assess a linear combination of fuzzy quantities. Therefore, for a given set of trapezoidal fuzzy numbers* $\tilde{A}_j = (a_j^{m1}, a_j^{m2}, a_j^{l}, a_j^{u})$, *j = 1,2,…,n and* $\lambda_j \geq 0$, $\sum_{j=1}^{n} \lambda_j \tilde{A}_j$ *is defined as follows:*

$$\sum_{j=1}^{n} \lambda_j \tilde{A}_j = \left(\sum_{j=1}^{n} \lambda_j a_j^{m1}, \sum_{j=1}^{n} \lambda_j a_j^{m2}, \sum_{j=1}^{n} \lambda_j a_j^{l}, \sum_{j=1}^{n} \lambda_j a_j^{u} \right) \tag{4}$$

where $\sum_{j=1}^{n} \lambda_j \tilde{A}_j$ *denotes the combination* $\lambda_1 \tilde{A}_1 \oplus \lambda_2 \tilde{A}_2 \oplus \cdots \oplus \lambda_n \tilde{A}_n$.

4 Basic DEA models

Assume that there are *n* DMUs to be evaluated, where every DMU$_j$, $j = 1,2,\ldots,n$, produces the same *s* outputs in different amounts, $y_{rj} (r = 1,2,\ldots,s)$, using the same *m* inputs, $x_{ij} (i = 1,2,\ldots,m)$, also in different amounts. The CCR model is proposed by Charnes et al. (1978) under the constant returns to scale (CRS) to evaluate the efficiency of a specific DMU$_p$. The primal and its dual DEA models are given in (5).

$$\begin{aligned}
&\textit{Primal CCR model (input oriented)}\\
&\min \quad Z_p = \theta\\
&\text{s.t.}\\
&0 \leq \theta x_{ip} - \sum_{j=1}^{n} \lambda_j x_{ij}, \quad \forall i\\
&y_{rp} \leq \sum_{j=1}^{n} \lambda_j y_{rj}, \quad \forall r\\
&0 \leq \lambda_j, \quad \forall j\\
&\textit{Dual CCR model (input oriented)}\\
&\max \quad \theta_p = \sum_{r=1}^{s} u_r y_{rp}\\
&\text{s.t.}\\
&1 = \sum_{i=1}^{m} v_i x_{ip}\\
&0 \geq \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij}, \quad \forall j\\
&0 \leq u_r, v_i, \quad \forall r, i
\end{aligned} \tag{5}$$

Banker et al. (1984) extended the CRS model of Charnes et al. (1978) to variable returns to scale (VRS) by proposing the BCC model. The BCC and CCR models differ only in that the former includes an additional convexity constraint $\sum_{j=1}^{n} \lambda_j = 1$ in the primal BCC model, and an additional variable u_0 in the dual BCC model as shown in (6).

Primal BCC model (input oriented)

$$
\begin{aligned}
&\min \quad w_p = \theta \\
&\text{s.t.} \\
&0 \le \theta x_{ip} - \sum_{j=1}^{n} \lambda_j x_{ij}, \quad \forall i \\
&y_{rp} \le \sum_{j=1}^{n} \lambda_j y_{rj}, \quad \forall r \\
&1 = \sum_{j=1}^{n} \lambda_j \\
&0 \le \lambda_j, \quad \forall j
\end{aligned}
\tag{6}
$$

Dual BCC model (input oriented)

$$
\begin{aligned}
&\max \quad w_p = \sum_{r=1}^{s} u_r y_{rp} + u_0 \\
&\text{s.t.} \\
&1 = \sum_{i=1}^{m} v_i x_{ip}, \\
&0 \ge \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_0, \quad \forall j \\
&0 \le u_r, v_i, \quad \forall r, i
\end{aligned}
$$

u_r and v_i in the dual models (5) and (6) are the weights assigned to the *r*th output and the *i*th input, respectively. The primal and dual models are referred to as the envelopment and the multiplier, respectively. Note that the optimal value of u_0 can be used to characterise the situation for RTSwhen a DMU$_p$ is efficient in the dual BCC model. The optimum value of the objective function in the CCR and BCC models is unity. Nevertheless, the DMU$_p$ can be inefficient even if the optimum value of the objective function is equal to unity. Consequently, Charnes et al. (1985) proposed the 'additive model' or 'Pareto–Koopmans model' to evaluate *n* DMUs with the following mathematical form:

$$
\begin{aligned}
&\max \quad \sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^- \\
&\text{s.t.} \\
&y_{rp} = \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+, \quad \forall r \\
&x_{ip} = \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^-, \quad \forall i \\
&1 = \sum_{j=1}^{n} \lambda_j \\
&0 \le \lambda_j, s_r^+, s_i^-; \forall j, r, i
\end{aligned}
\tag{7}
$$

where the optimal solutions are $(\lambda_j^*, s_r^{+*}, s_i^{-*})$. The CCR and BCC models in the input-oriented cases are: radial projection constructs and the inputs are relatively reduced, while the outputs remain fixed. There is no need to distinguish between an 'output' and an 'input' orientation in model (7), because the objective function in (7) simultaneously maximises outputs and minimises inputs in the sense of vector optimisations**.** The following theorems are referred to in Cooper et al. (2000):

Theorem 1: *DMU_p is efficient iff all the slacks are zero in an optimum solution in (7).*

Theorem 2: *Suppose that we define two variables (y'_{rp}, x'_{ip}) as*

$$
\begin{aligned}
&y_{rp} \le y_{rp} + s_r^{+*} = y'_{rp}, \quad \forall r \\
&x_{ip} \ge x_{ip} - s_i^{-*} = x'_{ip}, \quad \forall i
\end{aligned}
$$

then (y'_{rp}, x'_{ip}) is additive efficient.

Note that the slacks are all independent of each other. Hence an optimum is not reached until it is not possible to increase an output y'_{rp} or reduce an input x'_{ip} without decreasing some other output or increasing some other input.

In recent years, DEA has been applied to DMUs in many different settings, such as efficiency and effectiveness in supply chain management (Wong et al., 2008), operating entities (Parkan, 2006), banking (Azadeh et al., 2010a,b; Cooper et al., 2008), economic cooperation in developing countries (Emrouznejad and Thanassoulis, 2010), farming (Mulwa et al., 2009), hotel industry (Cheng et al., 2010), financial statement analysis (Ho, 2007) and healthcare (Dharmapala, 2009).

5 Fuzzy additive DEA

In the conventional DEA, all the data assume the form of specific numerical values. However, the observed value of the input and output data in real-world situations are sometimes inexact, incomplete, vague, ambiguous or imprecise. Some researchers have proposed various methods for dealing with the imprecise and ambiguous data in DEA. Cooper et al. (1999) have studied this problem in the context of interval data. However, many real-world problems use linguistic data, such as good, fair or poor and cannot be used as interval data. Fuzzy sets can represent imprecise or ambiguous data in DEA by formalising inaccuracy in decision-making (Collan et al., 2009). Fuzzy sets algebra developed by Zadeh (1965) is the formal body of theory that allows the treatment of imprecise estimates in uncertain environments. In most fuzzy DEA models, the input and output data are represented by fuzzy numbers (e.g. Hatami-Marbini et al., 2009b; Kao and Liu, 2000, 2003; Lertworasirikul et al., 2003a; Saati et al., 2002).

In this section, we propose a fuzzy additive model for evaluating the efficiency of a set of DMUs with fuzzy inputs and outputs derived from the α-level approach proposed by Saati et al. (2002). Consider n DMUs, each of which uses m different fuzzy inputs to secure s different fuzzy outputs. The additive model (6) can be expressed by the following fuzzy linear programming model:

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^- \\
\text{s.t.} \quad & \\
& \tilde{y}_{rp} = \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - s_r^+, \quad \forall r \\
& \tilde{x}_{ip} = \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} + s_i^-, \quad \forall i \\
& 1 = \sum_{j=1}^{n} \lambda_j \\
& 0 \le \lambda_j, s_r^+, s_i^-; \forall j, r, i
\end{aligned}
\tag{8}
$$

where $\tilde{y}_{rj} = (y_{rj}^{m1}, y_{rj}^{m2}, y_{rj}^{l}, y_{rj}^{u})$ and $\tilde{x}_{ij} = (x_{ij}^{m1}, x_{ij}^{m2}, x_{ij}^{l}, x_{ij}^{u})$, the fuzzy output and fuzzy input values of the jth DMU, respectively are expressed as trapezoidal fuzzy numbers. Let us also portray the crisp slacks s_r^+ and s_i^- in the above model as trapezoidal fuzzy numbers denoted as $\tilde{s}_r^+ = (s_r^+, s_r^+, s_r^+, s_r^+)$ and $\tilde{s}_i^- = (s_i^-, s_i^-, s_i^-, s_i^-)$, respectively. Therefore, (8) can be rewritten as follows:

$$
\begin{aligned}
&\max \quad \sum_{r=1}^{s}\left(s_r^+, s_r^+, s_r^+, s_r^+\right) + \sum_{i=1}^{m}\left(s_i^-, s_i^-, s_i^-, s_i^-\right) \\
&\text{s.t.} \\
&\left(y_{rp}^{m1}, y_{rp}^{m2}, y_{rp}^{l}, y_{rp}^{u}\right) = \sum_{j=1}^{n}\lambda_j\left(y_{rj}^{m1}, y_{rj}^{m2}, y_{rj}^{l}, y_{rj}^{u}\right) - \left(s_r^+, s_r^+, s_r^+, s_r^+\right), \quad \forall r \\
&\left(x_{ip}^{m1}, x_{ip}^{m2}, x_{ip}^{l}, x_{ip}^{u}\right) = \sum_{j=1}^{n}\lambda_j\left(x_{ij}^{m1}, x_{ij}^{m2}, x_{ij}^{l}, x_{ij}^{u}\right) + \left(s_i^-, s_i^-, s_i^-, s_i^-\right), \quad \forall i \\
&1 = \sum_{j=1}^{n}\lambda_j \\
&0 \le \lambda_j, s_r^+, s_i^-; \forall j, r, i
\end{aligned}
\tag{9}
$$

Model (9) can be solved by using the following four approaches proposed in the fuzzy DEA literature: the fuzzy ranking approach, the possibility approach, the tolerance approach and the α-level-based approach. We use the α-level-based approach in this paper to consider all the fuzzy information in performance assessment because the other three approaches result in some loss of information when they produce crisp results. Using the α-level-based approach, the inputs and outputs can be represented by different levels of confidence intervals and the fuzzy additive model is then transformed as follows:

$$
\begin{aligned}
&\max \quad \sum_{r=1}^{s}(s_r^+, s_r^+, s_r^+, s_r^+) + \sum_{i=1}^{m}(s_i^-, s_i^-, s_i^-, s_i^-) \\
&s.t. \\
&(\alpha y_{rp}^{m1} + (1-\alpha)y_{rp}^{l}, \alpha y_{rp}^{m2} + (1-\alpha)y_{rp}^{u}) = \sum_{j=1}^{n}\lambda_j(\alpha y_{rj}^{m1} + (1-\alpha)y_{rj}^{l}, \alpha y_{rj}^{m2} \\
&\qquad + (1-\alpha)y_{rj}^{u}) - (\alpha s_r^+ + (1-\alpha)s_r^+, \alpha s_r^+ + (1-\alpha)s_r^+), \quad \forall r \\
&(\alpha x_{ip}^{m1} + (1-\alpha)x_{ip}^{l}, \alpha x_{ip}^{m2} + (1-\alpha)x_{ip}^{u}) = \sum_{j=1}^{n}\lambda_j(\alpha x_{ij}^{m1} + (1-\alpha)x_{ij}^{l}, \alpha x_{ij}^{m2} \\
&\qquad + (1-\alpha)x_{ij}^{u}) + (\alpha s_i^- + (1-\alpha)s_i^-, \alpha s_i^- + (1-\alpha)s_i^-), \quad \forall i \\
&1 = \sum_{j=1}^{n}\lambda_j, \\
&0 \le \lambda_j, s_r^+, s_i^-; \forall j, r, i.
\end{aligned}
\tag{10}
$$

Model (10) is an interval linear programming model that cannot be solved by standard optimisation methods without further variation. Hence we convert the interval model (10) into a non-linear programming model through the following interval alteration variables:

$$\left(\alpha x_{ij}^{m1}+(1-\alpha)x_{ij}^{l},\alpha x_{ij}^{m2}+(1-\alpha)x_{ij}^{u}\right)=\hat{x}_{ij} \quad \forall j,i$$
$$\left(\alpha y_{rj}^{m1}+(1-\alpha)y_{rj}^{l},\alpha y_{rj}^{m2}+(1-\alpha)y_{rj}^{u}\right)=\hat{y}_{rj} \quad \forall r,j$$

The above substitutions in model (10) will result in the following non-linear programming model:

$$\begin{aligned}
&\max \quad \sum_{r=1}^{s} s_r^{+}+\sum_{i=1}^{m} s_i^{-}\\
&\text{s.t.}\\
&\hat{y}_{rp}=\sum_{j=1}^{n}\lambda_j\hat{y}_{rj}-s_r^{+}, \quad \forall r \qquad \text{(I)}\\
&\hat{x}_{ip}=\sum_{j=1}^{n}\lambda_j\hat{x}_{ij}+s_i^{-}, \quad \forall i \qquad \text{(II)}\\
&1=\sum_{j=1}^{n}\lambda_j\\
&\alpha x_{ij}^{m1}+(1-\alpha)x_{ij}^{l}\le\hat{x}_{ij}\le\alpha x_{ij}^{m2}+(1-\alpha)x_{ij}^{u}, \quad \forall i,j\\
&\alpha y_{rj}^{m1}+(1-\alpha)y_{rj}^{l}\le\hat{y}_{rj}\le\alpha y_{rj}^{m2}+(1-\alpha)y_{rj}^{u}, \quad \forall r,j\\
&0\le\lambda_j,s_r^{+},s_i^{-};\forall j,r,i
\end{aligned} \tag{11}$$

Note that model (11) is a non-linear programming. Therefore, (11) can be simplified as the following model by using simple variations:

$$\begin{aligned}
&\max \quad \sum_{r=1}^{s} s_r^{+}+\sum_{i=1}^{m} s_i^{-}\\
&\text{s.t.}\\
&0=\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j\hat{y}_{rj}-\hat{y}_{rp}\left(1-\lambda_p\right)-\hat{s}_r^{+}, \quad \forall r\\
&0=\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j\hat{x}_{ij}-\hat{x}_{ip}\left(1-\lambda_p\right)+\hat{s}_i^{-}, \quad \forall i\\
&1=\sum_{j=1}^{n}\lambda_j\\
&\alpha x_{ij}^{m1}+(1-\alpha)x_{ij}^{l}\le\hat{x}_{ij}\le\alpha x_{ij}^{m2}+(1-\alpha)x_{ij}^{u}, \quad \forall i,j\\
&\alpha y_{rj}^{m1}+(1-\alpha)y_{rj}^{l}\le\hat{y}_{rj}\le\alpha y_{rj}^{m2}+(1-\alpha)y_{rj}^{u}, \quad \forall r,j\\
&0\le\lambda_j,s_r^{+},s_i^{-};\forall j,r,i
\end{aligned} \tag{12}$$

In order to obtain a linear programming model, the following alternation variables are introduced in model (12):

$$\lambda_j \hat{x}_{ij} = \overline{x}_{ij} \Rightarrow \hat{x}_{ij} = \frac{\overline{x}_{ij}}{\lambda_j}, \quad \forall i, j, j \neq p$$

$$\lambda_j \hat{y}_{rj} = \overline{y}_{rj} \Rightarrow \hat{y}_{rj} = \frac{\overline{y}_{rj}}{\lambda_j}, \quad \forall r, j, j \neq p$$

$$\hat{x}_{ip}\left(1-\lambda_p\right) = \overline{x}^p_{ip} \Rightarrow \hat{x}_{ip} = \frac{\overline{x}^p_{ip}}{1-\lambda_p}, \quad j = p$$

$$\hat{y}_{rp}\left(1-\lambda_p\right) = \overline{y}^p_{rp} \Rightarrow \hat{y}_{rp} = \frac{\overline{y}^p_{rp}}{1-\lambda_p}, \quad j = p$$

Finally, model (12) is changed into the following linear programming model by applying the above transformations:

$$\begin{aligned}
&\max \quad \sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^- \\
&\text{s.t.} \\
&0 = \sum_{\substack{j=1 \\ j\neq p}}^{n} \overline{y}_{rj} - \overline{y}^p_{rp} - s_r^+, \quad \forall r \qquad \text{(a)} \\
&0 = \sum_{\substack{j=1 \\ j\neq p}}^{n} \overline{x}_{ij} - \overline{x}^p_{ip} + s_i^-, \quad \forall i \qquad \text{(b)} \\
&1 = \sum_{j=1}^{n} \lambda_j \\
&\lambda_j\left(\alpha x_{ij}^{m1} + (1-\alpha)x_{ij}^{l}\right) \le \overline{x}_{ij} \le \lambda_j\left(\alpha x_{ij}^{m2} + (1-\alpha)x_{ij}^{u}\right), \quad \forall i, j,\ j \neq p \qquad \text{(c)} \\
&\left(1-\lambda_p\right)\left(\alpha x_{ip}^{m1} + (1-\alpha)x_{ip}^{l}\right) \le \overline{x}^p_{ip} \le \left(1-\lambda_p\right)\left(\alpha x_{ip}^{m2} + (1-\alpha)x_{ip}^{u}\right), \quad \forall i \quad \text{(d)} \\
&\lambda_j\left(\alpha y_{rj}^{m1} + (1-\alpha)y_{rj}^{l}\right) \le \overline{y}_{rj} \le \lambda_j\left(\alpha y_{rj}^{m2} + (1-\alpha)y_{rj}^{u}\right), \quad \forall r, j,\ j \neq p \quad \text{(e)} \\
&\left(1-\lambda_p\right)\left(\alpha y_{rp}^{m1} + (1-\alpha)y_{rp}^{l}\right) \le \overline{y}^p_{rp} \le \left(1-\lambda_p\right)\left(\alpha y_{rp}^{m2} + (1-\alpha)y_{rp}^{u}\right), \quad \forall r \quad \text{(f)} \\
&0 \le \lambda_j, s_r^+, s_i^-; \forall j, r, i
\end{aligned} \tag{13}$$

On the one hand, the above model is a parametric programming model in which $\alpha \in [0,1]$ is a parameter. On the other hand, it is clear that the objective function of the fuzzy additive model (13) behaves like the objective function of (7). Hence according to Theorem 3, if the optimal value of the objective function (13) is equivalent to null, then DMU_p is efficient.

Theorem 3: *The optimal value of the objective function (13) is equivalent to null, if DMU$_p$ is efficient.*

Proof: Assume that DMU$_p$ is efficient and, therefore, $\lambda_p = 1$ and $\lambda_j = 0\,(j \neq p)$. The following results can be obtained with regards to the constraints (c)–(f) in (13):

$$\bar{x}_{ij} = 0, \quad \forall i, j,\, j \neq p, \quad \bar{x}_{ip}^{p} = 0, \quad \forall i$$

$$\bar{y}_{rj} = 0, \quad \forall r, j,\, j \neq p, \quad \bar{y}_{rp}^{p} = 0, \quad \forall r$$

If we substitute the above results into the constraints (a) and (b), then

$$s_r^{+} = 0, \quad \forall r, \quad s_i^{-} = 0, \ \forall i$$

This completes the proof. □

6 Numerical examples

To illustrate the fuzzy additive DEA approach, we consider two separate problems to compare our results with five different fuzzy DEA approaches in the literature. The first example is taken from Guo and Tanaka (2001). Lertworasirikul et al. (2003a,b) and Saati et al. (2002) have also applied this example to illustrate their approach. The second example was initially considered by León et al. (2003).

Example 1: Consider the performance-measurement problem of Guo and Tanaka (2001) with five DMUs and two fuzzy inputs and two fuzzy outputs presented in Table 1.

Guo and Tanaka (2001) proposed a fuzzy CCR model in which fuzzy constraints (including fuzzy equalities and fuzzy inequalities) were converted into crisp constraints by predefining a possibility level and using the comparison rule for fuzzy numbers. According to Guo and Tanaka (2001), a DMU is α-possibilistic efficient if the maximum value of the fuzzy efficiency at that α-level is greater than or equal to 1. The set of all possibilistic efficient DMUs is called the α-possibilistic non-dominated set. In the fuzzy CCR model, Lertworasirikul et al. (2003a) applied the possibility approach to the numerical example of Guo and Tanaka (2001). In their approach, a DMU became α-possibilistic efficient if its objective value was greater than or equal to 1 at the specified α-level. Lertworasirikul et al. (2003b) proposed a possibility approach for solving fuzzy BCC models. Saati et al. (2002) suggested a fuzzy CCR model as a possibilistic programming problem and transformed it into an interval programming problem, using an α-cut-based approach. They employed Guo and Tanaka's (2001) numerical example and demonstrated their method. They claimed that their model provided the best situation for each DMU by means of the comparison of the intervals for all fuzzy data. The results of Guo and Tanaka (2001), Lertworasirikul et al. (2003a,b) and Saati et al. (2002) and the method proposed in this study are shown in Table 2 for three different α-values.

As shown in this table, the efficiency is higher under the fuzzy VRS as compared to the fuzzy CRS. This can easily be seen by comparing the results of Lertworasirikul et al. (2003a,b). In terms of the methodology, the closest method to our proposed additive model, since both are VRS is from Lertworasirikul et al. (2003b). However, the results reported for Lertworasirikul et al. (2003b) are the efficiency scores, while the result reported using model 13 are the sum of the slacks. Obviously, higher slacks mean lower efficiency. For example, for the case $\alpha = 1$, we see that units *B*, *D* and *E* are efficient in both models (efficiency score of 1 in Lertworasirikul et al. (2003b) and zero sum of slacks in our proposed additive model (13)). According to Lertworasirikul et al. (2003b), unit *A* is less efficient (score = 0.889) than unit *B* (score = 0.935) and this is consistent with our results, since the sum of slacks for unit *A* (=0.982) is higher than sum of the slacks for unit *B* (=0.875). There are some disparities between the two models; however, we believe that our model captures the efficiency scores under fuzzy conditions more accurately.

Consider Table 3, where we reported the optimal solutions for slacks s_i^{-*} and s_r^{+*} in (13) with regard to several different possibility levels. One immediate result is that as α increases the slack decreases. Unit *B* and *E* are constantly efficient under all α-levels, and unit *D* is efficient under the case of α-level of 0.4 and above.

Example 2: *Consider example 2 in Table 4 investigated by León et al. (2003) with eight DMUs, one fuzzy input and one fuzzy output.*

León et al. (2003) proposed a fuzzy BCC model by using the fuzzy numbers of L-R. They changed the fuzzy constraints into a crisp condition by applying the Ramík and Rimanek (1985) principle. The efficiency values of our proposed models versus León et al. (2003) and Saati et al. (2002) are presented in Table 5.

On the one hand, as shown in Table 5, considering α = 0.75, units *A* and *C* are efficient according to (13) and if we rank the units according to their sum of slacks, we have *A* = *C>B>E>D>G>H>F*. On the other hand, for the same α = 0.75, the models of León et al. (2003) and Saati et al. (2002) give slightly different rankings, respectively, *A=C=G>B>D>E>H>F* and *A=C>E>B>D>G>F>H*. However, disparity between these three models is expected since the models of León et al. (2003) and Saati et al. (2002) are radial, and thus only consider slacks for output variables. The proposed additive model in this paper is non-radial in the sense that it maximises the sum of slacks on both input and output variables.

Table 1 The numerical example of Guo and Tanaka (2001)

	Inputs		*Outputs*	
DMU	*1*	*2*	*1*	*2*
A	(3.5, 4.0, 4.5)	(1.9, 2.1, 2.3)	(2.4, 2.6, 2.8)	(3.8, 4.1, 4.4)
B	(2.9, 2.9, 2.9)	(1.4, 1.5, 1.6)	(2.2, 2.2, 2.2)	(3.3, 3.5, 3.7)
C	(4.4, 4.9, 5.4)	(2.2, 2.6, 3.0)	(2.7, 3.2, 3.7)	(4.3, 5.1, 5.9)
D	(3.4, 4.1, 4.8)	(2.2, 2.3, 2.4)	(2.5, 2.9, 3.3)	(5.5, 5.7, 5.9)
E	(5.9, 6.5, 7.1)	(3.6, 4.1, 4.6)	(4.4, 5.1, 5.8)	(6.5, 7.4, 8.3)

Table 2 The comparison between our results and the results from four popular fuzzy DEA methods in the literature

Method	*α-level*	*A*	*B*	*C*	*D*	*E*
Guo and Tanaka (2001)	0	(0.66, 0.81, 0.99)	(0.88, 0.98, 1.09)	(0.60, 0.82, 1.12)	(0.71, 0.93, 1.25)	(0.61, 0.79, 1.02)
	0.5	(0.75, 0.83, 0.92)	(0.94, 0.97, 1.00)	(0.12, 0.83, 0.14)	(0.85, 0.97, 1.12)	(0.72, 0.82, 0.93)
	0.75	(0.77, 0.81, 0.99)	(0.80, 0.98, 1.09)	(0.22, 0.82, 0.30)	(0.71, 0.93, 1.25)	(0.61, 0.79, 1.02)
	1	(0.85, 0.85, 0.94)	(1.00, 1.00, 1.00)	(0.86, 0.86, 0.86)	(1.00, 1.00, 1.00)	(1.00, 1.00, 1.00)
Lertworasirikul et al. (2003a)	0	1.107	1.238	1.276	1.52	1.296
	0.5	0.963	1.112	1.035	1.258	1.159
	0.75	0.904	1.055	0.932	1.131	1.095
	1	0.855	1.000	0.861	1.000	1.000
Saati et al. (2002)	0.0	1	1	1	1	1
	0.5	0.954	1	1	1	1
	0.75	0.901	1	0.929	1	1
	1.0	0.855	1	0.862	1	1
Lertworasirikul et al. (2003b)	0	1.299	1.247	1.699	1.692	∞
	0.5	1.062	1.119	1.243	1.300	∞
	0.75	0.969	1.059	1.074	1.142	∞
	1	0.889	1.000	0.935	1.000	1.000
The proposed fuzzy additive model (13) → sum of the slacks	0.0	3.9	0	5.4	2.629	0
	0.5	2.434	0	3.058	0	0
	0.75	1.721	0	2.006	0	0
	1	0.982	0	0.875	0	0

Table 3 The optimal solutions of slacks $(s_1^{-*}, s_2^{-*}, s_1^{+*}, s_2^{+*})$ of model (13) for different α-level

α-level	*A*	*B*	*C*	*D*	*E*
0	(0.9214,0,0.7071,2.2714)	(0,0,0,0)	(0.5714,0,1.8571,2.9714)	(0.1857,0,1.7,0.7429)	(0,0,0,0)
0.1	(0.8590,0,0.5795,2.1633)	(0,0,0,0)	(0.5831,0,1.6048,2.7135)	(0.1194,0,1.5513,0.5543)	(0,0,0,0)
0.2	(0.7930,0,0.4568,2.0611)	(0,0,0,0)	(0.5870,0,1.3632,2.4689)	(0.0530,0,1.4051,0.3676)	(0,0,0,0)
0.3	(0.7237,0,0.3385,1.9644)	(0,0,0,0)	(0.5837,0,1.1316,2.2365)	(0.2206,0,1.2344,0)	(0,0,0,0)
0.4	(0.6395,0,0.2465,1.8450)	(0,0,0,0)	(0.5738,0,0.9092,2.0154)	(0,0,0,0)	(0,0,0,0)
0.5	(0.5531,0,0.1688,1.7125)	(0,0,0,0)	(0.5578,0,0.6953,1.8047)	(0,0,0,0)	(0,0,0,0)
0.6	(0.4720,0,0.0940,1.5800)	(0,0,0,0)	(0.5361,0,0.4893,1.6037)	(0,0,0,0)	(0,0,0,0)
0.7	(0.3961,0,0.0223,1.4475)	(0,0,0,0)	(0.5091,0,0.2905,1.4116)	(0,0,0,0)	(0,0,0,0)
0.8	(0.3219,0,0,1.2421)	(0,0,0,0)	(0.4772,0,0.0986,1.2279)	(0,0,0,0)	(0,0,0,0)
0.9	(0.2647,0,0,0.9994)	(0,0,0,0)	(0.4443,0,0,0.9101)	(0,0,0,0)	(0,0,0,0)
1	(0.2273,0,0,0.7545)	(0,0,0,0)	(0.4416,0,0,0.4331)	(0,0,0,0)	(0,0,0,0)

Table 4 The numerical example of León et al. (2003)

DMU	*Input*	*Output*
A	(1, 3, 4)	(2, 3, 4)
B	(3.5, 4, 4.5)	(1.5, 2.5, 3.5)
C	(3, 4.5, 6)	(5, 6, 7)
D	(6, 6.5 ,7)	(2.75, 4, 5.25)
E	(5, 7, 9)	(4.5, 5, 5.5)
F	(7.5, 8, 8.5)	(3, 3.5, 4)
G	(9, 10, 11)	(5.5, 6, 6.5)
H	(5.5, 6, 6.5)	(0.5, 2, 3.5)

Table 5 The comparison between our results and the results from León et al. (2003) and Saati et al. (2002)

Method	*α-level*	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
León et al. (2003)	0	1	1	1	0.7500	0.6429	0.6050	1	0.6923
	0.5	1	0.9412	1	0.6623	0.6172	0.5227	1	0.6400
	0.75	1	0.8675	1	0.6144	0.6010	0.4776	1	0.5854
	1	1	0.7500	1	0.5385	0.5714	0.4062	0.45	0.5000
Saati et al. (2002)	0	1	1	1	1	1	0.640	0.867	0.764
	0.5	1	0.764	1	0.706	0.835	0.462	0.628	0.457
	0.75	1	0.602	1	0.574	0.668	0.390	0.533	0.343
	1	0.750	0.469	1	0.462	0.536	0.328	0.450	0.250
The proposed fuzzy additive model (13) → sum of the slacks	0	6	7	2.333	8.25	8.5	9.5	9.5	10
	0.5	0	5	0	6.125	6	7.75	7.5	7.75
	0.75	0	4	0	5.063	4.75	6.875	6.5	6.625
	1	0	2.5	0	4	3.5	6	5.5	5.5

7 Conclusions and future research directions

In most of the real-world performance evaluation problems, some data are not known accurately. The conventional DEA lacks the flexibility to deal with imprecise or vague data. These imprecise and vague data can be represented by linguistic terms characterised by fuzzy numbers in DEA to reflect the decision-makers' intuition and subjective judgements. Furthermore, it is very expensive for organisations to collect precise data for efficiency analysis. Consequently, there is a strong impetus for developing cost-effective methodologies to capture imprecision in productivity and efficiency analysis.

This paper extends the conventional DEA model to a fuzzy framework by proposing a new additive model for evaluating the efficiency of a set of DMUs with fuzzy inputs and outputs. In this paper:

1 we consider ambiguous, uncertain and imprecise input and output data in DEA

2 we propose a new fuzzy additive DEA model derived from the α-level approach

3 we demonstrate the practical aspects of our model with two numerical examples.

From a future research point of view, it is suggested that researchers develop similar models for other non-radial DEA models like slack-based measure or directional measure of DEA.

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