Defining implication relation for classical logic

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Abstract
Classical logic is unfortunately defective with its problematic definition of implication relation, the “material implication”. This work presents a corrected definition of implication relation to replace material implication in classical logic. Common “paradoxes” of material implication are avoided while simplicity and usefulness of the system are reserved with this definition of implication relation.

Keywords. implication relation; material implication; classical logic; propositional logic; paradox

1 Introduction
In classical logic, material implication is defined by disjunction with negation. Specifically, “if \( p \) then \( q \)” or “\( p \) implies \( q \)” is defined as “not \( p \) or \( q \)”, i.e., \( p \rightarrow q \equiv \neg p \lor q \). It is well known that this definition is problematic in that it leads to counterintuitive tautologies or “paradoxes” such as

(1) \( \vdash \neg p \rightarrow (p \rightarrow q) \) (a false proposition implies any proposition),
(2) \( \vdash p \rightarrow (q \rightarrow p) \) (a true proposition is implied by any proposition), and
(3) \( \vdash (p \rightarrow q) \lor q \rightarrow p \) (for two propositions, at least one implies the other).

These tautologies can be verified by the rule of replacement for material implication, i.e., replacing \( p \rightarrow q \) with \( \neg p \lor q \). There are many such paradoxes of material implication, see, e.g., (Bronstein, 1936) and (Lojko, 2012) where lists sixteen “paradoxes” of material implication.

It should be noted that some paradoxes of material implication are even worse than just counterintuitive or “paradoxical” because they are actually wrong. Let us take \( (p \rightarrow q) \lor (q \rightarrow p) \) as an example. It means that, for any statements \( p \) and \( q \), either “If \( p \) then \( q \)” is true, or “If \( q \) then \( p \)” is true, or both are true. This is absurd since it is possible that it is not the case even if \( p \) and \( q \) are “relevant”. For instance, suppose \( n \) is an integer, let \( p \) be “\( n = 0 \)” and \( q \) be “\( n = 1 \)”, then it must be that “if \( n = 0 \) then \( n = 1 \)” is true or “if \( n = 1 \) then \( n = 0 \)” is true, or both are true, and this is an absurdity.
Many efforts have been made to resolve this problem such as: relevance logic which requires that antecedent and consequent be relevant, modal logic which uses concept of strict implication, intuitionistic logic which rejects the law of excluded middle, and inquisitive logic (see, e.g., (Ciardelli, 2011)) which considers inquisitive semantics of sentences rather than just their descriptive aspects. These developments of non-classical logic are important to modern logics and other disciplines.

However, classical logic exists on its own reason, and we are reluctant to discard it, see e.g. (Fulda, 1989). Indeed, classical logic, with its natural principles such as the law of excluded middle, is not only fundamental, but also simple and useful. Classical logic is an important part of logic education, so it has been an essential part of most logic textbooks. On the other hand, implication is a kernel concept in logic. So it is not satisfying that the both simple and useful classical logic has an unnatural and even wrong definition of (material) implication appeared in many textbooks.

This work is to present a correct definition of implication relation to replace material implication in classical logic. Moreover, we will do this by using only concepts that already exist in classical logic rather than using those specifically introduced in non-classical logics such as relevance logic, modal logic and intuitionistic logic. Another objective of this work is to prevent from developing “too narrow” a definition of implication, as Bronstein (1936) commented on E. J. Nelson’s “intensional logic”: “Although his system does avoid the ‘paradoxes’, it does so only by unduly narrowing his conception of implication.”

Although classical logics include at least propositional logic and first-order logic, this work concentrates on classical propositional logic, since concepts concerning implication relation are the same.

2 Informal analysis

2.1 Root of the defectiveness

The definition of material implication in classical logic is based on the “fact”: for propositions $p$ and $q$, “if $p$ then $q$” is logically equivalent to “it is not that $p$ and not $q$”, and this is again logically equivalent to “not $p$ or $q$”. It is this “logical equivalence” that leads to the definition $p \rightarrow q \overset{\text{def}}{=} \neg p \lor q$ for “material implication”.

This definition makes the material implication a truth-functional connective just like conjunction and disjunction, i.e., the truth-value of the compound proposition $p \rightarrow q$ is a function of the truth-values of its sub-propositions. It means that the truth-value of “if $p$ then $q$” is determined solely by the combination of truth-values of $p$ and $q$. This is unnatural as shown in the following.

(1) When we know that $p$ is true and $q$ is true, can we decide that “if $p$ then $q$” is true (or false)? No, not sure.
(2) When we know that \( p \) is true and \( q \) is false, can we decide that “if \( p \) then \( q \)” is true (or false)? Yes, we can decide that it must be false.

(3) When we know that \( p \) is false and \( q \) is true, can we decide that “if \( p \) then \( q \)” is true (or false)? No, not sure.

(4) When we know that \( p \) is false and \( q \) is false, can we decide that “if \( p \) then \( q \)” is true (or false)? No, not sure.

In only one case, namely the second one, the truth-value of the compound proposition “if \( p \) then \( q \)” can be determined by the truth-value combination of \( p \) and \( q \). This indicates that “if \( p \) then \( q \)” is not logically equivalent to “not \( p \) or \( q \)”, since the truth-value of the latter is indeed solely determined by the truth-value combination of \( p \) and \( q \). On the other hand, since “if \( p \) then \( q \)” is surely false when \( p \) is true and \( q \) is false, it suggests that when “if \( p \) then \( q \)” is true it must not be the case that \( p \) is true and \( q \) is false (which is equivalent to “not \( p \) or \( q \)”).

Thus, it should be that “if \( p \) then \( q \)” logically implies “not \( p \) or \( q \)” but not vice versa. Some researchers have already addressed this issue, although they might have different motivations or explained it in different ways, such as MacColl (1880), Bronstein (1936), Woods (1967), Dale (1974), and Lojko (2012).

Therefore, the definition of material implication in classical logic is not only unnatural but also, even more severely, not correct. This is why it leads to some unacceptable results as exemplified previously. The use of the (mistaken) equivalence between “if \( p \) then \( q \)” and “not \( p \) or \( q \)” is the source of paradoxes of material implication that makes the classical logic unfortunately defective.

### 2.2 Relations between propositions

In classical logic, “not \( p \)”, “\( p \) and \( q \)”, “\( p \) or \( q \)”, and “if \( p \) then \( q \)” are all viewed as the same kind of compound sentences formed with operations or functions, called “logical connectives” or “logical operators”. However, “if \( p \) then \( q \)” is actually different. Essentially, “implication” should not be viewed as an operation but a relation. In mathematics, “\( 1 + 2 \)” is an expression formed by an operation (+) while “\( 1 < 2 \)” is a sentence formed by a relation (<). We cannot say that a mathematical expression like “\( 1 + 2 \)” is “true” or “false”, while we can say that a mathematical sentence like “\( 1 < 2 \)” is “true”. So, in mathematics, a function expression has usually no truth-value while a relation expression has. In propositional logic, a function expression such as \( p \wedge q \) or \( p \vee q \), unlike their mathematical counterpart, does have a truth-value, but this is because the output of such a function happens to be a proposition that has a truth-value itself. In contrast, when we say that \( p \rightarrow q \) is “true” or “false”, we concern about the implication “\( \rightarrow \)” itself being “true” or “false”. So, a relation expression is a “higher level” sentence than a function expression if it happens to be a sentence.

Therefore, it is natural and correct to put implication in the level of relations between propositions, rather than the level of functions of propositions. Hence the implication relation is not truth-functional (Woods, 1967).
We now analyze possible relations between any two propositions $p$ and $q$. There are three distinct cases as shown in Figure 1. We use here ad hoc equals sign to denote “equivalence”.

Case 1: “Disjoint” that is characterized by the equation $p \land q = \bot$;

Case 2: “Joint” that is characterized by the equations $p \land q \neq \bot$ and $p \land q \neq p$ and $p \land q \neq q$; and

Case 3: “Inclusion” that is characterized by the equation $p \land q = p$.

The three cases are mutually exclusive except for trivial circumstances such as that $p$ or $q$ is $\bot$ or $\top$. Let us focus on case 3 characterized by $p \land q = p$. In this case, whenever $p$ is true $q$ must be true, so it is just the case that “if $p$ then $q$”. This indicates that we can use $p \land q = p$ to define the implication, noting that $R \overset{\text{def}}{=} \{(p, q) \mid p \land q = p\}$ is a binary relation on the set of propositions.

### 3 Formal definitions and properties

#### 3.1 The definitions

We start with a propositional language constructed from some proposition symbols representing some atomic statements.

**Definition 3.1.1** (Propositional language $\mathcal{L}(\sigma)$). Let $\sigma = \{p, q, ...\}$ be a finite set of propositional symbols (signature of the language), $\Omega = \{\top, \bot, \neg, \land, \lor, \rightarrow, \leftrightarrow\}$ be the set of logical symbols, and $\delta = \{(), \}\}$ be the set of delimiters (a pair of parentheses); where $\sigma, \Omega$ and $\delta$ are mutually disjoint. Let $\Sigma = \sigma \cup \Omega \cup \delta$ be the alphabet. The propositional language, denoted by $\mathcal{L}(\sigma)$, over the alphabet is the set of formulas formed only by the following rules:

1. $\top, \bot \in \mathcal{L}(\sigma)$.
2. If $\varphi \in \sigma$ then $\varphi \in \mathcal{L}(\sigma)$.
3. If $\varphi \in \mathcal{L}(\sigma)$ then $\neg \varphi \in \mathcal{L}(\sigma)$.
4. If $\varphi, \psi \in \mathcal{L}(\sigma)$, then $\varphi \land \psi, \varphi \lor \psi, \varphi \rightarrow \psi, \varphi \leftrightarrow \psi \in \mathcal{L}(\sigma)$.
5. $(\varphi) \in \mathcal{L}(\sigma)$ iff $\varphi \in \mathcal{L}(\sigma)$. 


For assigning semantics to $\mathcal{L}(\sigma)$, let us define formally truth functions and truth assignment functions first.

**Definition 3.1.2** (Truth functions). Let $\{T, F\}$ be the set of truth values. Generally, an $n$-ary truth function space $$[\{T, F\}^n \to \{T, F\}] = \{T, F\}^{[T, F]}_n$$ contains $2^{2^n}$ truth functions. The following three specific truth functions are needed for our purpose.

The unary function for negation 
$$\tau_\neg : \{T, F\} \to \{T, F\},$$
the binary function for conjunction 
$$\tau_\land : \{T, F\}^2 \to \{T, F\},$$
and the binary function for disjunction 
$$\tau_\lor : \{T, F\}^2 \to \{T, F\}.$$

The three functions are defined exactly the same as in standard classical logic, where truth tables are commonly used.

**Definition 3.1.3** (Truth assignment functions and interpretation space $V(\sigma)$). A formula in $\mathcal{L}(\sigma)$ is interpreted as a statement with a truth value being either $T$ or $F$. So a truth assignment function $\nu : \mathcal{L}(\sigma) \to \{T, F\}$ represents an interpretation of the language. Since all logical symbols (and a pair of delimiters) have constant semantics, the set of all possible interpretations of $\mathcal{L}(\sigma)$ is determined only by its signature $\sigma$. Specifically, the set of all possible truth assignment functions over the set $\sigma$—denoted by $U(\sigma) \subseteq \{\sigma \to \{T, F\}\} = \{T, F\}^\sigma$—determines the set of all possible interpretations of the language.

For each function $\mu \in U(\sigma)$, we can define a function $\nu$ by extending the domain from $\sigma$ to the whole set of formulas $\mathcal{L}(\sigma)$, i.e., $\nu$ is an extension of $\mu$. We denote the set of all the extended functions by $V(\sigma)$ ("valuation"). The set $V(\sigma)$ contains all possible truth assignment functions over $\mathcal{L}(\sigma)$, so it represents all possible interpretations of the language and we call $V(\sigma)$ the interpretation space for the language.

By definition $V(\sigma)$ and $U(\sigma)$ are related by a one-to-one correspondence, so the number of all possible interpretations is $|V(\sigma)| = |U(\sigma)| \leq |\{T, F\}^\sigma| = 2^\sigma$.

**Definition 3.1.4** (Semantics $M(\sigma)$ of $\mathcal{L}(\sigma)$, with implication relation defined by semantic equivalence). The semantics of $\mathcal{L}(\sigma)$ is given by defining truth assignment functions in the interpretation space $V(\sigma)$. The specific scheme for doing this, denoted by $M(\sigma)$ ("meaning, model"), is as follows. Let $\nu \in V(\sigma)$ be a truth assignment function or an interpretation.
(1) The interpretation of $\top$ and $\bot$:

$$\nu(\top) = T,$$
$$\nu(\bot) = F.$$  

(2) The interpretation of propositional symbols:

$$\forall p \in \sigma \ (\nu(p) = \mu_p \text{ is given}).$$

(3) The interpretation of general formulas. Assume that for $\phi, \psi \in \mathcal{L}(\sigma)$, $\forall \mu \in V(\sigma) (\mu(\phi) \text{ and } \mu(\psi) \text{ have been obtained}).$:

(a) The interpretation of $\neg, \land, \lor$ is given via truth functions that are the same as in standard classical logic:

$$\nu(\neg \phi) = \tau_{\neg}(\nu(\phi)),$$
$$\nu(\phi \land \psi) = \tau_{\land}(\nu(\phi), \nu(\psi)),$$
$$\nu(\phi \lor \psi) = \tau_{\lor}(\nu(\phi), \nu(\psi)).$$

(b) The interpretation of $\leftrightarrow$: (For this binary relation the truth assignment function can not be represented via truth functions.)

$$\nu(\phi \leftrightarrow \psi) = \begin{cases} 
T, & \text{if } \forall \mu \in V(\sigma) (\mu(\phi) = \mu(\psi)); \\
F, & \text{otherwise}. 
\end{cases}$$

(c) The interpretation of $\rightarrow$: (Again, for this binary relation the truth assignment function can not be represented via truth functions.)

$$\nu(\phi \rightarrow \psi) = \nu(\phi \land \psi \leftrightarrow \phi).$$

Remark 3.1.1. Classical interpretation of $\rightarrow$ is reduced to $\neg$ and $\lor$ since it takes for granted that $\phi \rightarrow \psi \equiv \neg \phi \lor \psi$ and just makes its truth assignment truth functional.

Definition 3.1.4 defines the implication relation semantically. The following axiomatization of the system defines the implication relation syntactically. This is done by starting with three axioms by Huntington (1933) for Boolean lattice, where only disjunction and negation are used.

Definition 3.1.5 (Deductive apparatus $\mathcal{D}$ of $\mathcal{L}(\sigma)$, with implication relation defined by syntactic equivalence). The deductive apparatus for $\mathcal{L}(\sigma)$, denoted by $\mathcal{D}$, consists of sets of definitions, axioms, and inference rules specified as follows.

Let $\phi, \psi$ and $\chi$ be arbitrary formulas of $\mathcal{L}(\sigma)$.

Definitions:
(EQ1) \(\vdash \varphi \leftrightarrow \varphi\). (definition of equivalence: reflexive)
(EQ2) \(\varphi \leftrightarrow \psi \vdash \psi \leftrightarrow \varphi\). (definition of equivalence: symmetric)
(EQ3) \(\varphi \leftrightarrow \psi, \psi \vdash \chi \leftrightarrow \varphi\). (definition of equivalence: transitive)

(CJ) \(\vdash \phi \wedge \psi \leftrightarrow \neg \psi \lor \neg \phi\). (definition of conjunction \(\wedge\))

(TP) \(\vdash \top \leftrightarrow \varphi \lor \neg \varphi\). (definition of top \(\top\))

(BT) \(\vdash \bot \leftrightarrow \neg \top\). (definition of bottom \(\bot\))

(IMP) \(\varphi \rightarrow \psi \vdash \psi \leftrightarrow \varphi\). (definition of implication relation \(\rightarrow\))

Axioms A1 – A3 are three axioms of Huntington (1933):

(A0) \(\vdash (\varphi) \leftrightarrow \varphi\). (parentheses)
(A1) \(\vdash \varphi \lor \psi \leftrightarrow \psi \lor \varphi\). (commutativity of \(\lor\))
(A2) \(\vdash (\varphi \lor \psi) \lor \chi \leftrightarrow \psi \lor (\varphi \lor \chi)\). (associativity of \(\lor\))
(A3) \(\vdash (\varphi \land \psi) \lor (\varphi \land \neg \psi) \leftrightarrow \varphi\). (Huntington equation)

Inference rules A substring of a formula is called a subformula if it is a formula. Let \(f[\varphi]\) denote a formula containing subformula \(\varphi\):

(R0) \(\varphi \rightarrow \psi, \psi \vdash \psi\). (modus polens)
(R1) \(\varphi \leftrightarrow \psi, f[\varphi] \vdash f[(\psi)]\). (replacement: equivalence)
(R2) \(f[(\varphi)] \vdash f[(\varphi)]\). (replacement: remove parentheses)
(R3) The precedence of the operators \((\neg, \land and \lor, \rightarrow and \leftrightarrow)\) can be used to add or remove pairs of parentheses. (replacement: parentheses)

3.2 Properties of the implication relation

Some useful properties of the implication relation defined in Definitions 3.1.1 and 3.1.5 are listed as follows.

Remark 3.2.1. In proofs, we abide by the following conventions about metalanguage usage:

(1) Use “if \(\alpha\) then \(\beta\)” and “\(\alpha\) iff \(\beta\)” to mean \(\alpha \vdash \beta\)” and “\(\alpha \equiv \beta\)” respectively;

(2) Use “if \(\alpha\) is true, then \(\beta\) is true” and “\(\alpha\) is true, iff \(\beta\) is true” to mean \(\alpha \vdash \beta\)” and “\(\alpha \equiv \beta\)” respectively.

Propostion 3.2.1 (Complemented distributive lattice). The structure \((L(\sigma), \lor, \land, \bot, \top, \neg, \rightarrow)\)

derived from the system \((L(\sigma), D)\) defined by Definition 3.1.4 is a complemented distributive lattice with its partial order being \(\rightarrow\).
Proof. Omitted. (We can just verify that the structure satisfies axiomatic conditions of complemented distributive lattice: associativity, commutativity, distributivity, unit elements, and complements.) □

Remark 3.2.2. A complemented distributive lattice is isomorphic to a Boolean lattice with its partial order \((\leq)\) defined by \((a \leq b \iff a \wedge b = a)\). This implies that all theorems in Boolean lattice, which excludes the problematic rule of replacement \((a \leq b \iff a \vee b)\), have their counterparts in the structure \((L(\sigma), \vee, \wedge, \bot, \top, \neg, \rightarrow)\), which can be applied to proofs in system \((L(\sigma), D)\).

Propostion 3.2.2 (Criteria for implication relation). For any formulas \(\varphi\) and \(\psi\) in \(L(\sigma)\) of the system \((L(\sigma), D)\), it holds that

\[
\varphi \rightarrow \psi \dashv \vdash \varphi \wedge \psi \iff \varphi \rightarrow \psi \dashv \vdash \varphi \wedge \neg \psi \iff \bot \iff \neg \varphi \vee \psi \iff \top.
\]

Proof. From Remark 3.2.2, we can apply any theorems of standard classical logic in proofs, except for those based on the rule of replacement for “material implication” \(\varphi \rightarrow \psi \dashv \vdash \neg \varphi \vee \psi\). Let us take the proof of \(\varphi \wedge \psi \iff \neg \varphi \wedge \neg \psi \iff \bot\) as an example. Rules of replacement based on \(\iff\) are applied throughout.

If \(\varphi \rightarrow \psi\), then \(\varphi \leftrightarrow \varphi \wedge \psi\), so \(\varphi \wedge \neg \psi \leftrightarrow (\varphi \wedge \psi) \wedge \neg \psi \leftrightarrow \varphi \wedge (\psi \wedge \neg \psi) \iff \bot\).

If \(\varphi \wedge \neg \psi \iff \bot\), then \(\varphi \wedge \psi \leftrightarrow (\varphi \wedge \psi) \vee \bot \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \neg \psi) \leftrightarrow \varphi \wedge (\psi \vee \neg \psi) \leftrightarrow \varphi\).

Remark 3.2.3. It should be noted that \(\varphi \rightarrow \psi \dashv \vdash \neg \varphi \vee \psi \iff \top\) by Proposition 3.2.2, while \(\varphi \rightarrow \psi \dashv \vdash \neg \varphi \wedge \psi\) by classical definition of implication.

Propostion 3.2.3 (Rules of inference about implication relation). For the system \((L(\sigma), D)\) defined in Definitions 3.1.1 and 3.1.5, we have

1. \(\forall \varphi, \psi \in L(\sigma)\) \((\varphi \rightarrow \psi \vdash \neg \varphi \vee \psi)\),

2. \(\exists \varphi, \psi \in L(\sigma)\) \((\neg \varphi \vee \psi \nvdash \varphi \rightarrow \psi)\).

Proof.

1. We always have \(\top\). If \(\varphi \rightarrow \psi\), then \(\varphi \wedge \psi \leftrightarrow \varphi\), so \(\bot \iff \neg \varphi \vee \top \iff \neg \varphi \vee (\neg \psi \vee \psi) \iff (\neg \varphi \vee \neg \psi) \vee \psi \iff (\neg (\varphi \wedge \psi) \vee \psi) \iff \neg \varphi \vee \psi\). Therefore, if \(\varphi \rightarrow \psi\) then \(\neg \varphi \vee \psi\).

2. Choose \(\psi \leftrightarrow \top\) and \(\varphi \leftrightarrow \neg \psi\), then we have \(\neg \varphi \vee \psi \leftrightarrow \psi \vee \psi \leftrightarrow \psi \leftrightarrow \top\), so \(\neg \varphi \vee \psi \leftrightarrow \top\), and it follows that \(\varphi \rightarrow \psi\) from Proposition 3.2.2. Thus, there exist some \(\varphi\) and \(\psi\) such that if \(\neg \varphi \vee \psi\) then \(\varphi \rightarrow \psi\).

Remark 3.2.4.

1. The rule of inference \(\varphi \rightarrow \psi \vdash \neg \varphi \vee \psi\) can be used safely in any cases.

2. The traditional rule of replacement \(\varphi \rightarrow \psi \vdash \neg \varphi \vee \psi\) can not be used.
We have revealed some properties of implication relation in the syntactic aspect. In the following we will give some semantic properties of implication relation.

**Definition 3.2.1 (T-level set).** Let $V(\sigma)$ be the interpretation space for $\mathcal{L}(\sigma)$. For a formula $\varphi \in \mathcal{L}(\sigma)$, its T-level set is defined as

$$V_\varphi \overset{\text{def}}{=} \{ \nu \in V(\sigma) \mid \nu(\varphi) = T \} \subseteq V(\sigma).$$

**Proposition 3.2.4 (Properties of T-level set).** The T-level set has the following properties. Suppose $\varphi, \psi \in \mathcal{L}(\sigma)$:

1. $\models \varphi$ iff $V_\varphi = V(\sigma)$, or $\not\models \varphi$ iff $V_\varphi \neq V(\sigma)$.
2. $V_\bot = \emptyset$, $V_\top = V(\sigma)$.
3. $V_{\neg \varphi} = V_\varphi^C = V(\sigma) \setminus V_\varphi$.
4. $V_{\varphi \lor \psi} = V_\varphi \cup V_\psi$, $V_{\varphi \land \psi} = V_\varphi \cap V_\psi$.
5. $V_{\varphi \rightarrow \psi} = \begin{cases} V(\sigma), & \text{if } V_\varphi = V_\psi \\ \emptyset, & \text{if } V_\varphi \neq V_\psi \end{cases}$.
6. $V_{\varphi \rightarrow \top} = \begin{cases} V(\sigma), & \text{if } V_\varphi \subseteq V_\psi \\ \emptyset, & \text{if } V_\varphi \nsubseteq V_\psi \end{cases}$.

**Proof.** Omitted. (These can be verified by Definitions 3.1.4 and 3.2.1.)

**Corollary 3.2.1 (More properties of T-level set).** Let $V(\sigma)$ be the interpretation space for $\mathcal{L}(\sigma)$. For any $\varphi \in \mathcal{L}(\sigma)$:

1. $V_{\bot \rightarrow \varphi} = V(\sigma)$.
2. $V_{\varphi \rightarrow \bot} = V(\sigma)$ iff $V_{\varphi \rightarrow \bot} \neq \emptyset$ iff $V_\varphi = \emptyset$.
3. $V_{\varphi \rightarrow \top} = V(\sigma)$.
4. $V_{\top \rightarrow \varphi} = V(\sigma)$ iff $V_{\top \rightarrow \varphi} \neq \emptyset$ iff $V_\varphi = V(\sigma)$.
5. $V_{\varphi \rightarrow \neg \varphi} = V(\sigma)$ iff $V_{\varphi \rightarrow \neg \varphi} \neq \emptyset$ iff $V_\varphi = \emptyset$.

**Proof.** Omitted. (They follow from item (6) in Proposition 3.2.4.)
3.3 Checking “paradoxes”: examples

Consider the most common “paradoxes” of material implication mentioned in Introduction and re-list them here for convenience.

(1) \( \vdash \neg p \rightarrow (p \rightarrow q) \) (a false proposition implies any proposition),
(2) \( \vdash p \rightarrow (q \rightarrow p) \) (a true proposition is implied by any proposition), and
(3) \( \vdash (p \rightarrow q) \lor q \rightarrow p \) (for two propositions, at least one implies the other).

Let us check these “paradoxes” under the implication relation defined in 3.1.

**Proposition 3.3.1** (No paradoxes). Let \( \varphi \) and \( \psi \) are formulas in \( L(\sigma) \), the implication relation “\( \rightarrow \)” is specified by Definitions 3.1.4 and 3.1.5. Then we have:

1. \( \nvdash \neg \varphi \rightarrow (\varphi \rightarrow \psi) \),
2. \( \nvdash \varphi \rightarrow (\psi \rightarrow \varphi) \),
3. \( \nvdash (\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi) \).

**Proof.** Let \( V(\sigma) \) be the interpretation space for \( L(\sigma) \). The following proofs are based on Definition 3.2.1, Proposition 3.2.4, and Corollary 3.2.1.

1. Choose \( \varphi \leftrightarrow \bot, \varphi \leftrightarrow \top, \psi \leftrightarrow \bot \). Then \( V_{\neg \varphi} \neq \emptyset \); on the other hand, we have \( V_{\varphi} \neq \emptyset \), so \( V_{\varphi \rightarrow \psi} = V_{\varphi \rightarrow \bot} = \emptyset \). Thus \( \emptyset \neq V_{\neg \varphi} \not\subseteq V_{\varphi \rightarrow \psi} = \emptyset \), therefore \( \neg \varphi \nrightarrow (\varphi \rightarrow \psi) \).

2. Choose \( \varphi \leftrightarrow \bot, \varphi \leftrightarrow \top, \psi \leftrightarrow \top \). Then \( V_{\varphi} \neq \emptyset \); on the other hand, we have \( V_{\varphi} \neq V(\sigma) \), so \( V_{\psi \rightarrow \varphi} = V_{\top \rightarrow \varphi} = \emptyset \). Thus \( \emptyset \neq V_{\varphi} \not\subseteq V_{\psi \rightarrow \varphi} = \emptyset \), therefore \( \varphi \nrightarrow (\psi \rightarrow \varphi) \).

3. Choose \( \varphi \leftrightarrow \bot, \varphi \leftrightarrow \top, \psi \leftrightarrow \neg \psi \). Then \( V_{\varphi} \neq \emptyset, V_{\psi} \neq \emptyset \), so \( V_{\varphi \rightarrow \psi} = V_{\varphi \rightarrow \neg \psi} = \emptyset, V_{\psi \rightarrow \varphi} = V_{\neg \psi \rightarrow \varphi} = \emptyset \); thus \( V_{(\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)} = V_{\varphi \rightarrow \psi} \cup V_{\psi \rightarrow \varphi} = \emptyset \neq V(\sigma) \), therefore, \( \nvdash (\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi) \).

This means that common “paradoxes” of traditional material implication do not exist under the implication relation defined in 3.1.

4 Concluding remarks

Classical logic such as standard propositional logic is simple and useful except for its problematic definition of “material implication” that unfortunately makes it defective. A binary relation such as implication and equivalence can not be represented by binary operations (functions) such as conjunction and disjunction. So it is not surprising that the semantic equivalence \( \varphi \rightarrow \psi \vDash \neg \varphi \lor \psi \)
does not hold, hence the syntactic rule of replacement $\varphi \rightarrow \psi \vdash \neg \varphi \lor \psi$, which must stem from its semantic counterpart, is not reasonable.

We define the implication relation in this work both semantically and syntactically. In each of the two definitions we define the equivalence relation first, and define the implication relation based on the equivalence relation. This is beneficial in that it is easy to verify that the system is (isomorphic to) a Boolean lattice, which is a very clear structure with its partial order well (correctly) defined. This definition of implication avoids naturally common “paradoxes” of traditional “material implication”. It becomes clear that (1) the two expressions $\varphi \rightarrow \psi$ and $\neg \varphi \lor \psi$ are not equivalent both semantically and syntactically; (2) both the semantic inference $\varphi \rightarrow \psi \models \neg \varphi \lor \psi$ and the syntactic inference rule $\varphi \rightarrow \psi \vdash \neg \varphi \lor \psi$ can be used safely in any cases.

More notes about the implication relation defined in this work are listed as follows.

1. The statement “$\varphi \rightarrow \psi$ is true” is not equivalent to “$\neg \varphi \lor \psi$ is true”, but instead it is equivalent to “$\neg \varphi \lor \psi \leftrightarrow \top$ is true”.

2. The expression $\neg \varphi \lor \psi \leftrightarrow \top$ enforces that whenever $\neg \varphi$ is false, i.e. $\varphi$ is true, $\psi$ must be true, and this is just the meaning of $\varphi \rightarrow \psi$. Ajdukiewicz (1978) stated this as “...an alternative sentence expresses not only 1◦ our knowledge of the fact that one at least of the alternants is true and 2◦ our ignorance as to which of them is true, but moreover 3◦ our readiness to infer one alternant from the negation of the other.”

3. The tautologies $\top \rightarrow \varphi$ and $\varphi \rightarrow \top$ are properties of Boolean lattice that is bounded. They are not paradoxical since they can be interpreted to intuition like “if an always-false-thing is true, then anything is true” and “an always-true-thing is true without any premises”, as similarly mentioned, e.g., in (Ceniza, 1988).

4. Since the traditional rule of replacement $\varphi \rightarrow \psi \vdash \neg \varphi \lor \psi$ is incorrect, binary relation symbols $\rightarrow$ and $\leftrightarrow$ cannot be removed from a logical system. That is, either implication relation symbol ($\rightarrow$) or equivalence relation symbol ($\leftrightarrow$) must be in the alphabet of the logical language. On the other hand, defining binary operations such as $\land$ and $\lor$ by binary relations $\rightarrow$ or $\leftrightarrow$ (together with, e.g. $\neg$ or $\bot$), as commonly doing in classical logic, is problematic either.

5. By definitions in this work, the concept of implication relation is compatible with traditional “logical implication”. Thus it might be ideal to abandon the problematic “material implication” and make unique the concept of “implication” in classical logics.

6. When adopting a unique definition of implication relation, two symbols can be used for it in two types of expressions: (a) general or conditional expressions, e.g., use “$\rightarrow$” to denote “implies” where it is not certain this implication being true, such as in $p \rightarrow q$; (b) tautological expressions, e.g.,
use “⇒” in a tautology where the implication is definitely true, such as in the expression \((p \rightarrow q) \land p \Rightarrow q\). This is just like the “equality” relation in numbers: the concept is unambiguous but two symbols are used, e.g., for real numbers the concept of equality is unique, but “=” and “≡” can be used in different expressions such as \(x = y\) and \((x + y)^2 \equiv x^2 + 2xy + y^2\).

The consistency, soundness and completeness of the system \((L(\sigma), M(\sigma), D)\) with the implication relation defined in 3.1, are not investigated in this work.

References


