Adaptive-Robust Control of a Class of Uncertain Nonlinear Systems Utilizing Time-Delayed Input and Position Feedback

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Abstract

In this paper, the tracking control problem of a class of Euler-Lagrange systems subjected to unknown uncertainties is addressed and an adaptive-robust control strategy, christened as Time-Delayed Adaptive Robust Control (TARC) is presented. The proposed control strategy approximates the unknown dynamics through time-delayed logic, and the switching logic provides robustness against the approximation error. The novel adaptation law for the switching gain, in contrast to the conventional adaptive-robust control methodologies, does not require either nominal modelling or predefined bounds of the uncertainties. Also, the proposed adaptive law circumvents the overestimation-underestimation problem of switching gain. The state derivatives in the proposed control law is estimated from past data of the state to alleviate the measurement error when state derivatives are not available directly. Moreover, a new stability notion for time-delayed control is proposed which in turn provides a selection criterion for controller gain and sampling interval. Experimental result of the proposed methodology using a nonholonomic wheeled mobile robot (WMR) is presented and improved tracking accuracy of the proposed control law is noted compared to time-delayed control and adaptive sliding mode control.

Index Terms

Adaptive-robust control, Euler-Lagrange system, time-delayed control, state derivative estimation, wheeled mobile robot.

I. INTRODUCTION

A. Background and Motivation

Design of an efficient controller for nonlinear systems subjected to parametric and nonparametric uncertainties has always been a challenging task. Among many other approaches, Adaptive control and Robust control are the two popular control strategies that researchers have extensively employed while dealing with uncertain nonlinear systems. In general, adaptive control uses predefined parameter adaptation laws and equivalence principle based control law which adjusts the parameters of the controller on the fly according to the pertaining uncertainties [1]. However, this approach has poor transient performance and online calculation of the unknown system parameters and controller gains for complex systems is computationally intensive [2]. Whereas, robust control aims at tackling the uncertainties of the system within an uncertainty bound defined a priori. It reduces computation complexity to a great extent for complex systems compared to adaptive control as exclusive online estimation of uncertain parameters is not required [3]. However, nominal modelling of the uncertainties is necessary to decide upon their bounds, which is not always possible. Again, to increase the operating region of the controller, often higher uncertainty bounds are assumed. This in turn leads to problems like higher controller gain and consequent possibility of chattering for the switching law based robust controller like Sliding Mode Control (SMC). This in effect reduces controller accuracy [4]. Higher order sliding mode [5] can alleviate the chattering problem but prerequisite of uncertainty bound still exists.

Time-Delayed Control (TDC) is utilized in [6] to implement state derivative feedback for enhancing stability margin of SISO linear time invariant (LTI) systems. In [7], [8], [26]-[27], [34], [35], TDC is used to provide robustness against uncertainties. In this process, all the uncertain terms are represented by a single function which is then approximated using control input and state information of the immediate past time instant. The advantage of this robust control approach in uncertain systems is that it reduces the burden of tedious modelling of complex system to a great extent. In spite of this, the unattended approximation error, commonly termed as time-delayed error (TDE) causes detrimental effect to the performance of the closed system and its stability. In this front, a few work have been carried out to tackle TDE which includes internal model [9], gradient estimator [10], ideal velocity feedback [11], nonlinear damping [12] and sliding mode based approach [13]-[14]. The stability of the closed loop system [9]-[11], [26]-[27], depends on the boundedness of TDE as shown in [7]. This method approximates the continuous time closed loop system in a discrete form without considering the effect of discretization error. Again, the stability criterion mentioned in [7] restricts the allowable range of perturbation and thus limits controller working range. Stability of the system in [13] is established in frequency domain, which makes the approach inapplicable to the nonlinear systems. Moreover, the controllers designed in [12], and [14], [35] require nominal modelling and upper bound of the TDE respectively which is not always possible in practical circumstances. Also, to the best knowledge of the authors, controller design issues such as...
selection of controller gains and sampling interval to achieve efficient performance is still an open problem. In contrast to TDC, works reported in [28]-[30] use low pass filter to approximate the unknown uncertainties and disturbances. However, frequency range of system dynamics and external disturbances are required to determine the time constant of the filter. Furthermore, the order of the low pass filter needs to be adjusted according to order of the disturbance to maintain stability of the controller.

Considering the individual limitations of adaptive and robust control, recently global research is reoriented towards adaptive-robust control (ARC) where switching gain of the controller is adjusted online. The series of publications [2], [15]-[20] regarding ARC, estimates the uncertain terms online based on predefined projection function, but predefined bound on uncertainties is still a requirement. The work reported in [21], [33] attempts to estimate the maximum uncertainty bound but the integral adaptive law makes the controller susceptible to very high switching gain and consequent chattering [34]. The adaptive sliding mode control (ASMC) as presented in [22]-[23] proposed two laws for the switching gain to adapt itself online according to the incurred error. In the first adaptive law, the switching gain decreases or increases depending on a predefined threshold value. However, until the threshold value is achieved, the switching gain may still be increasing (resp. decreasing) even if tracking error decreases (resp. increases) and thus creates overestimation (resp. underestimation) problem of switching gain. However, until the threshold value is achieved, the switching gain may still be increasing (resp. decreasing) even if tracking error decreases (resp. increases) and thus creates overestimation (resp. underestimation) problem of switching gain. Moreover, to decide the threshold value the maximum bound of the uncertainty is required. For the second adaptive law, the threshold value changes online according to switching gain. Yet, nominal model of the uncertainties is needed for defining the control law. This limits the adaptive nature of the control law and applicability of the controller.

B. Problem Definitions and Contributions

In this paper, three specific related problems on TDC have been dealt with and the corresponding solutions to the same which are also the contributions of this paper are summarized below:

- **Problem 1:** The stability analysis of TDC, as provided in [7], [20]-[27], [34], [35], approximates the continuous time system in discrete time domain without considering the effects of discretization error. Again, choice of the delay time and its relation with the controller gains is still an open problem.

  In this paper, a new stability analysis for TDC, based on the Lyapunov-Krasvoskii method, is provided in continuous time domain. Furthermore, through the proposed stability approach, a relation between the sampling interval and controller gain is established.

- **Problem 2:** The TDC reported in [7], [20]-[27], [34], [35], velocity and acceleration feedback are necessary to compute the control law. While in [6], only velocity feedback is required and acceleration term is approximated numerically using time delay. However, in many applications velocity and acceleration feedback are not available explicitly and numerical approximation of these terms invokes measurement error.

  As a second contribution of this paper, Filtered Time-Delayed Control (F-TDC) control law is formulated where only position feedback is sufficient while velocity and acceleration terms are estimated using past and present position information to curb the effect measurement error. Stability analysis of the proposed F-TDC is provided which also maintains the relation between controller gains and sampling interval.

- **Problem 3:** Robustness property against TDE is essential to achieve good tracking accuracy. The robust controllers reported in literature, either requires nominal model of the uncertainties ([22],[23]) or its predefined bound ([2], [12], [14], [15]-[20]). So, it is required to devise a control law which would avoid any prior knowledge of the uncertainties while providing robustness against TDE.

  Towards the last contribution of this article, an adaptive-robust control strategy, Time-Delayed Adaptive Robust Control (TARC) has been formulated for a class of uncertain Euler-Lagrange systems. The proposed control law approximates uncertainties by time-delayed logic and provides robustness against the TDE, arising from time-delayed logic based estimation, by switching control. The novel adaptive law, presented here, aims at overcoming the overestimation-underestimation problem of the switching gain without any prior knowledge of uncertainties. The proposed adaptive law provides flexibility to the control designer to select any suitable error function according to the application requirement while maintaining similar system stability notion.

  As a proof of concept, experimental validation of the proposed control methodology is provided using the "PIONEER-3" nonholonomic WMR in comparison to TDC [7] and ASMC [22]-[23].

C. Organization

The article is organized as follows: a new stability analysis of TDC along with its design issues is first discussed in Section II. This is followed by the proposed adaptive-robust control methodology and its detail analysis. Section III presents the experimental results of the proposed controller and its comparison with TDC and ASMC. Section IV concludes the entire work.

D. Notations

The following notations are assumed for the entirety of the paper: any variable $\mu$ delayed by an amount $h$ as $\mu(t-h)$, is denoted as $\mu_h$; $\lambda_{\min}(\cdot)$ and $\|\cdot\|$ represent minimum eigen value and Euclidean norm of the argument respectively; $I$ represents identity matrix.
II. CONTROLLER DESIGN

A. Time-Delayed Control: Revisited

In general, an Euler-Lagrange system with second order dynamics, devoid of any delay, can be written as,

$$M(q)\ddot{q} + N(q, \dot{q}) = \tau(t),$$

(1)

where, $q(t) \in \mathbb{R}^{n}$ is the state system, $\tau(t) \in \mathbb{R}^{n}$ is the control input, $M(q) \in \mathbb{R}^{n \times n}$ is the mass/inertia matrix and $N(q, \dot{q}) \in \mathbb{R}^{n}$ denotes combination of other system dynamics terms based on system properties. In practice, it can be assumed that unmodelled dynamics and disturbances is subsumed by $N$. The control input is defined to be,

$$\tau = \dot{\hat{M}}u + \dot{N},$$

(2)

where, $u$ is the auxiliary control input, $\dot{M}$ and $\dot{N}$ are the nominal values of $M$ and $N$ respectively. To reduce the modelling effort of the complex systems, $\dot{N}$ can be approximated from the input-output data of previous instances using the time-delayed logic ([7], [26]-[27]) and the system definition (1) as,

$$\dot{N}(q, \dot{q}) \equiv N(q_{h}, \dot{q}_{h}) = \tau_{h} - \dot{\hat{M}}(q_{h})\dot{q}_{h},$$

(3)

where, $h > 0$ is a fixed small delay time. Substituting (2) and (3) in (1), the system dynamics is converted into an input as well state delayed dynamics as,

$$\dot{M}(q)\ddot{q} + \dot{N}(q, \dot{q}, \dot{q}_{h}) = \tau_{h},$$

(4)

where $\dot{N} = (M - \dot{M})\ddot{q} + \dot{M}\dot{q}_{h} - \dot{M}u + \dot{N}$.

Let, $q^{d}(t)$ be the desired trajectory to be tracked and $e_{1}(t) = q(t) - q^{d}(t)$ is the tracking error. The auxiliary control input $u$ is defined in the following way,

$$u(t) = q^{d}(t) - K_{2}\dot{e}_{1}(t) - K_{1}e_{1}(t),$$

(5)

where, $K_{1}$ and $K_{2}$ are two positive definite matrices with appropriate dimensions. Putting (5) and (2) in (4), following error dynamics is obtained,

$$\ddot{e}_{1} = -K_{2}\dot{e}_{1h} - K_{1}e_{1h} + \sigma_{1},$$

(6)

where, $\sigma_{1} = (\dot{M}^{-1}\dot{M}_{h} - I)u_{h} + \dot{M}^{-1}(\dot{N}_{h} - \dot{N}) + \ddot{q}^{d} - \ddot{q}$ and can be treated as overall uncertainty. Further, (6) can be written in state space form as,

$$\dot{e} = A_{1}e + B_{1}e_{h} + B\sigma_{1},$$

(7)

where, $e = \begin{bmatrix} e_{1} \\ \dot{e}_{1} \end{bmatrix}$, $A_{1} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$, $B_{1} = \begin{bmatrix} 0 & 0 \\ -K_{1} & -K_{2} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ I \end{bmatrix}$. Noting that, $e(t - h) = e(t) - \int_{-h}^{0} \dot{e}(t + \theta)d\theta$, where the derivative inside the integral is with respect to $\theta$, the error dynamics (7) is modified as,

$$\dot{e}(t) = Ae(t) - B_{1}\int_{-h}^{0} \dot{e}(t + \theta)d\theta + B\sigma_{1},$$

(8)

where, $A = A_{1} + B_{1}$. It is assumed that the choice controller gains $K_{1}$ and $K_{2}$ makes $A$ Hurwitz which is always possible. Also, it is assumed that the unknown uncertainties are bounded. In this paper, a new stability criterion, based on the Lyapunov-Krasvoskii method, is presented through Theorem 1 which addresses the issues defined in Problem 1.

Theorem 1. The system (4) employing the control input (2), having auxiliary control input (5) is UUB if the controller gains and delay time is selected such that the following condition holds:

$$\Psi = \begin{bmatrix} Q - E - (1 + \xi)\frac{h^{2}}{\beta}D & 0 \\ 0 & (\xi - 1)\frac{h^{2}}{\beta}D \end{bmatrix} > 0$$

(9)

where, $E = \beta PB_{1}(A_{1}D^{-1}A_{1}^{T} + B_{1}D^{-1}B_{1}^{T} + D^{-1})B_{1}^{T}P$, $\xi > 1$ and $\beta > 0$ are scalar, and $P > 0$ is the solution of the Lyapunov equation $A^{T}P + PA = -Q$ for some $Q > 0$.

Proof. Let us consider the following Lyapunov function:

$$V(e) = V_{1}(e) + V_{2}(e) + V_{3}(e) + V_{4}(e),$$

(10)
where,
\[
V_1(e) = e^T Pe
\]  
\[
V_2(e) = \frac{h}{\beta} \int_{-h}^{0} \int_{t+h}^{t} e^T (\psi) De(\psi) d\psi d\theta
\]  
\[
V_3(e) = \frac{h}{\beta} \int_{-h}^{0} \int_{t+h}^{t} e^T (\psi - h) De(\psi - h) d\psi d\theta
\]  
\[
V_4(e) = \xi \frac{h^2}{\beta} \int_{-h}^{0} e^T (\psi) De(\psi) d\psi
\]  

Using [8], the time derivative of \(V_1(e)\) yields,
\[
\dot{V}_1(e) = -2e^T Qe - 2e^T PB_1 \int_{-h}^{0} \dot{e}(t + \theta) d\theta + 2s^T \sigma_1
\]  
where, \(s = B^T Pe\). Again using [7],
\[
-2e^T PB_1 \int_{-h}^{0} \dot{e}(t + \theta) d\theta = -2e^T PB_1 \int_{-h}^{0} [A_1 e(t + \theta) + B_1 e(t - h + \theta) + B e(t + \theta)] d\theta,
\]
For any two non zero vectors \(z_1\) and \(z_2\), there exists a scalar \(\beta > 0\) and matrix \(D > 0\) such that the following inequality holds,
\[
\pm 2z_1^T z_2 \leq \beta z_1^T D^{-1} z_1 + (1/\beta) z_2^T D z_2.
\]
Again, using Jensen’s inequality the following inequality holds [32],
\[
\int_{-h}^{0} e^T(\psi) De(\psi) d\psi \geq \frac{1}{h} \int_{-h}^{0} e^T(\psi) De\int_{-h}^{0} e(\psi) d\psi.
\]
Applying [17] and [18] to [16] the following inequalities are obtained,
\[
-2e^T PB_1 A_1 \int_{-h}^{0} e(t + \theta) d\theta \leq \beta e^T PB_1 A_1 D^{-1} A_1^T B_1^T Pe + \frac{1}{h} \int_{-h}^{0} e^T(t + \theta) d\theta D \int_{-h}^{0} e(t + \theta) d\theta
\]
\[
\leq \beta e^T [PB_1 A_1 D^{-1} A_1^T B_1^T P] e + \frac{h}{\beta} \int_{-h}^{0} e^T(t + \theta) D e(t + \theta) d\theta
\]
\[
-2e^T PB_1 B_1 \int_{-h}^{0} e(t - h + \theta) d\theta \leq \beta e^T PB_1 B_1 D^{-1} B_1^T B_1^T Pe + \frac{1}{h} \int_{-h}^{0} e^T(t - h + \theta) d\theta D \int_{-h}^{0} e(t - h + \theta) d\theta
\]
\[
\leq \beta e^T [PB_1 B_1 D^{-1} B_1^T B_1^T P] e + \frac{h}{\beta} \int_{-h}^{0} e^T(t - h + \theta) D e(t - h + \theta) d\theta
\]
\[
-2e^T PB_1 \int_{-h}^{0} [B e(t + \theta)] d\theta \leq \beta e^T PB_1 D^{-1} B_1^T Pe + \frac{1}{h} \int_{-h}^{0} (B e(t + \theta))^T D e(t + \theta) D \int_{-h}^{0} B e(t + \theta) d\theta
\]
\[
\leq \beta e^T [PB_1 D^{-1} B_1^T P] e + \frac{h}{\beta} \int_{-h}^{0} (B e(t + \theta))^T D e(t + \theta) D \int_{-h}^{0} B e(t + \theta) d\theta
\]
\[
\text{Since } D > 0, \text{ we can write } D = \tilde{D}^T \tilde{D} \text{ for some } \tilde{D} > 0. \text{ Then, assuming the uncertainties to be square integrable within the delay, let there exists a scalar } \Gamma_1 > 0 \text{ such that the following inequality holds:}
\]
\[
\left\| \int_{-h}^{0} [(B e(t + \theta))^T \tilde{D}^T \tilde{D} B e(t + \theta)] d\theta \right\| \leq \Gamma_1.
\]  
Again,
\[
\dot{V}_2(e) = \frac{h^2}{\beta} e^T De - \frac{h}{\beta} \int_{-h}^{0} e^T(t + \theta) D e(t + \theta) d\theta
\]  
\[
\dot{V}_3(e) = \frac{h^2}{\beta} e^T De - \frac{h}{\beta} \int_{-h}^{0} e^T(t - h + \theta) D e(t - h + \theta) d\theta
\]  
\[
\dot{V}_4(e) = \xi \frac{h^2}{\beta} (e^T De - e^T De) \]
Substituting [19]-[22] into [15] and adding it with [23]-[25] yields,
\[
\dot{V}(e) \leq -e^T \Psi e + \Gamma_1 + 2s^T \sigma_1,
\]
Applying Schur’s complement to (33), (30) can be achieved. Integrating (31) from $-\varsigma$ gives,

$$\varsigma > \lim_{t \to 0} \frac{\|\dot{\varsigma}\|}{\varsigma} \text{ for some } \varsigma > 0,$$

where $\varsigma = \frac{\|\dot{\varsigma}\|}{\varsigma}$. Let $\Xi$ denote the smallest level surface of $V$ containing the ball $B_{\omega_0}$ with radius $\omega_0$ centred at $\dot{\varsigma} = 0$. For initial time $t_0$, if $\dot{\varsigma}(t_0) \in \Xi$ then the solution remains in $\Xi$. If $\dot{\varsigma}(t_0) \notin \Xi$ then $V$ decreases as long as $\dot{\varsigma}(t) \notin \Xi$. The time required to reach $\omega_0$ is zero when $\dot{\varsigma}(t_0) \in \Xi$, otherwise, while $\dot{\varsigma}(t_0) \notin \Xi$ the finite time $t_{r_0}$ to reach $\omega_0$, for some $c_0 > 0$, is given by

$$t_{r_0} - t_0 \leq (\|\dot{\varsigma}(t_0)\| - \omega_0)/c_0 \quad \text{where} \quad \dot{V}(t) \leq -c_0$$

Remark 1: Since $E$ depends on the controller gains, (9) provides a selection criterion for the choice of delay $\varsigma$ for given controller gains and $Q$. This design issue was previously unaddressed in the literature. Moreover, the approximation error $(N - N)$, as in (4), would reduce for small values of $\varsigma$. However, $\varsigma$ cannot be selected smaller than the sampling interval because, the input output data is only available at sampling intervals. So, the lowest possible selection of $\varsigma$ is the sampling interval. Again, choice of sampling interval is governed by the corresponding hardware response time, computation time etc. Hence, the proposed stability approach provides a necessary step for the selection of sampling interval for given controller gains or vice-versa.

B. Filtered Time-Delayed Control (F-TDC)

It can be noticed from (5) and (6) that state derivatives are necessary to compute the control law of TDC. However, in many circumstances, only $q$ is available amongst $\dot{q}, \dot{\dot{q}}, \ddot{q}$. Under this scenario, a new control strategy F-TDC is proposed, which estimates the state derivatives from the state information of past instances (31). Before proposing the control structure of F-TDC, the following two Lemmas are stated which are instrumental for formulation as well as stability analysis of F-TDC.

Lemma 1 (31): For time $t \geq \varsigma$, the $j$-th order time derivative of the $\Lambda$-th degree polynomial $q$ in (4) can be computed in the following way,

$$\dot{q}(\varsigma)(t) = \int_{-\varsigma}^{0} \Omega_j(\varsigma, \psi)q(t + \psi)d\psi$$

where, $\varsigma > 0$ is a prespecified scalar and

$$\Omega_j(\varsigma, \psi) = \frac{(\Lambda + 1 + j)!}{\varsigma^{j+1}(\Lambda - j)!} \sum_{k=0}^{\Lambda} \frac{(-1)^k(\Lambda + 1 + k)!}{(j + k + 1)(\Lambda - k)!(k)!} \left(\frac{-\psi}{\varsigma}\right)^k.$$  

Proof. A sketch of the proof is provided in Appendix A.

Lemma 2. For any non zero vector $\vartheta(\psi)$, constant matrix $F > 0$ the following relation holds,

$$\int_{-h}^{0} \int_{-\varsigma}^{0} \vartheta^T(\psi)F\vartheta(\psi)d\psi d\theta \geq \frac{1}{h\varsigma} \left\{ \int_{-h}^{0} \int_{-\varsigma}^{0} \vartheta^T(\psi)F\vartheta(\psi)d\psi d\theta \right\} \geq 0$$

Proof. Since $F > 0$, we have

$$\left[ \frac{\vartheta^T(\psi)F\vartheta(\psi)}{\vartheta^T(\psi)F^{-1}\vartheta(\psi)} \right] \geq 0$$

Integrating (31) from $-\varsigma$ to 0 gives,

$$\left[ \int_{-\varsigma}^{0} \vartheta^T(\psi)F\vartheta(\psi)d\psi \int_{-\varsigma}^{0} \vartheta^T(\psi)d\psi \int_{-\varsigma}^{0} \vartheta(\psi)d\psi \right] \geq 0$$

Again, integrating (32) from $-h$ to 0 gives,

$$\left[ \int_{-h}^{0} \int_{-\varsigma}^{0} \vartheta^T(\psi)F\vartheta(\psi)d\psi d\theta \int_{-h}^{0} \vartheta^T(\psi)d\psi \int_{-h}^{0} \vartheta(\psi)d\psi \right] \geq 0$$

Applying Schur’s complement to (33), (30) can be achieved.
The structure of F-TDC is similar to (2), except, the auxiliary control input $u$ and $\hat{N}$ in (2) selected in the following way,

$$u(t) = \hat{q}^d(t) - K_1 e_1(t) - K_2 \dot{e}_1(t)$$

(34)

$$\hat{N}(t) = N_h = u_h - \hat{N}_h \hat{q}_h,$$

(35)

where, $\dot{e}_1 = \hat{q} - \hat{q}^d$, $\hat{q}$ and $\hat{q}$ are evaluated from (28) and (29). The stability of the system (4) employing F-TDC is derived in the sense of Uniformly Ultimately Bounded (UUB) notion as stated in Theorem 2.

**Theorem 2.** The system (4) employing the control input (2), having the auxiliary input (34) and (35) is UUB if

$$\begin{bmatrix} Q - \bar{E} - (1 + \xi) \frac{h}{2} D & P \bar{B} & P \bar{B} \\ \bar{B}^T P & (\xi - 1) \frac{h^2}{2} D - \bar{F} & 0 \\ \bar{B}^T P & 0 & L \end{bmatrix} = \Theta > 0$$

(36)

where, $\bar{E} = \beta PB_1(A_1 D^{-1} A_1^T + B_1 D^{-1} B_1^T + D^{-1} + \bar{B} D^{-1} \bar{B}^T)B_1^T P$, $\bar{F} = (\frac{h^2}{2} D + L)\varsigma \int_{\varsigma}^{0} A_2^T(\psi) d\psi$, $L > 0$, $A_2(\psi) = \Omega_1(\varsigma, \psi)$, $\bar{B} = B [K_2 \ \ 0]$, $\bar{B} = B [0 \ \ K_2]$.

**Proof.** The proof is provided in Appendix B.

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**C. Adaptive-Robust Control: Related Work**

It can be observed from (27) and (80) that TDE degrades tracking performance of both the TDC and F-TDC in the face of uncertainties. The control methods that attempt to counter uncertainties, as reported in [2], [12], [14], [15]-[20], requires predefined bound on the uncertainties which is not always possible in practical circumstances. To circumvent this situation Adaptive Sliding Mode Control (ASMC) was proposed in [22]-[23]. The control input of ASMC is given by,

$$\tau = \Sigma^{-1}(-\kappa_n + \Delta u_s),$$

(37)

where, $\Sigma_n$ and $\kappa_n$ is the nominal values of $\Sigma$ and $\kappa$, and $\Delta u_s$ is the switching control input. For a choice of sliding surface $\bar{s}$, $\Sigma$ and $\kappa$ is defined as follows:

$$\dot{\bar{s}} = \Sigma + \kappa \Delta u_s,$$

(38)

The switching control $\Delta u_s$ is calculated as

$$\Delta u_s = -\hat{c} \frac{\bar{s}}{||\bar{s}||}$$

(39)

$$\hat{c} = \begin{cases} \hat{c} ||\bar{s}|| \text{sgn}(||\bar{s}|| - \rho) & \hat{c} > \gamma, \\ \gamma & \hat{c} \leq \gamma, \end{cases}$$

(40)

where, $\hat{c}$ is the switching gain, $\hat{c} > 0$ is a scalar adaptive gain, $\rho > 0$ is a threshold value, $\gamma > 0$ is small scalar to always keep $\hat{c}$ positive. Evaluation of $\rho$ can be done in two ways [22]:

$$\rho = \varrho \quad \text{or},$$

(41)

$$\rho(t) = 4\hat{c}(t)t_s,$$

(42)

where, $\varrho > 0$ is a scalar, $t_s$ is the sampling interval. The choice (41) requires predefined bound of uncertainties. It can be noted from (40) that even if $||\bar{s}||$ decreases (resp. increases), unless it falls below (resp. goes above) $\rho$ switching gain does not decrease (resp. increase). This causes overestimation (resp. underestimation) of switching gain and controller accuracy is compromised. Again, improper and low choice of $\rho$ may lead to very high switching gain and consequent chattering. On the other hand, method (42) assumes that the nominal value of the uncertainties are always greater than the perturbations. This assumption may not hold due to the effect of unmodelled dynamics and thus, necessitates rigorous nominal modelling of the uncertainties in $N$ to design the control law. Either of the two situations, i.e. bound estimation or uncertainty modelling, is not always feasible in practical circumstances and consequently compromises the adaptive nature of the controller.
D. Time-Delayed Adaptive Robust Control

Considering the limitations of the existing controllers that aim at negotiating the uncertainties, as discussed earlier, a novel adaptive-robust control law, named Time-Delayed Adaptive Robust Control (TARC) is proposed in this endeavour, which neither requires the nominal model nor any predefined bound of the uncertainties as well as eliminates the overestimation-underestimation problem of switching gain. The structure of the control input of TARC is similar to (2) and \( \hat{N} \) is also evaluated according to (35). However, the auxiliary control input \( u \) is selected as below,

\[
u = \dot{u} + \Delta u.
\]

\( \dot{u} \) is the nominal control input and selected as similar to (34). \( \Delta u \) is the switching control law which is responsible for negotiating the TDE and it is defined as below,

\[
\Delta u = \begin{cases} 
-\alpha \dot{c}(e, t) \frac{s}{\|s\|} & \text{if } \|s\| \geq \epsilon, \\
-\alpha \dot{c}(e, t) \frac{\epsilon}{e} & \text{if } \|s\| < \epsilon, 
\end{cases}
\]

where, \( s = B^T P [\epsilon_1, \dot{\epsilon}_1]^T \) and \( \epsilon > 0 \) is a small scalar. The following novel adaptive control law for evaluation of \( \dot{c} \) is proposed:

\[
\dot{\dot{c}} = \begin{cases} 
\|s\| & \dot{\dot{c}} > \gamma, f(e) > 0 \\
-\|s\| & \dot{\dot{c}} > \gamma, f(e) \leq 0 \\
\gamma & \dot{\dot{c}} \leq \gamma,
\end{cases}
\]

where, \( \alpha > 0 \) is a scalar adaptive gain and \( \epsilon > 0 \) represents a small scalar, \( f(e) \) is a suitable function of error defined by the designer. Here, it is selected as \( f(e) = \|s(t)\| - \|s_h\| \). According to the adaptive law (45) and present choice of \( f(e) \), \( \dot{\dot{c}} \) increases (resp. decreases) whenever error trajectories move away from (resp. close to ) \( ||s|| = 0 \) The advantages of the proposed TARC can be summarized as follows:

- TARC reduces complex system modelling effort as only the knowledge of \( \hat{M} \) suffices the controller design since \( \hat{N} \) along with the uncertainties is approximated using the time-delayed logic as in (35). This in turn reduces the tedious modelling effort of complex nonlinear systems.
- Evaluation of switching gain does not require either of the nominal model or predefined bound of the uncertainties and also removes the overestimation-underestimation problem.
- State derivatives are not required to compute the control law explicitly, as they are evaluated from the past state information using (28) and (29).

The stability of the system (4) employing TARC is analysed in the sense of UUB as stated in Theorem 3.

**Assumption 1.** Let, \( ||\sigma_1|| \leq c \). Here, \( c \) is an unknown scalar quantity. Knowledge of \( c \), however, is only required for stability analysis but not to compute control law.

**Theorem 3.** The system (4) employing (2), (43) and having the adaptive law (45) is UUB, provided the selection of \( K_1, K_2, h \) and \( \varsigma \) holds condition (36).

**Proof.** : Let us define the Lyapunov functional as,

\[
V_f(e) = V_f(e) + (\dot{\dot{c}} - c)^2,
\]

where, \( V_f(e) \) is defined in (71). Again, putting (43) in (4) the the error dynamics becomes,

\[
\dot{e} = A_1 e + B_1 e_h - \hat{B} \int_{-\tau}^{0} A d(\psi) e(t - h + \psi) d\psi + B \sigma,
\]

where, \( \sigma = \Delta u_h + \sigma_1 \). Also following similar steps while proving Theorem 2 (provided in Appendix B) we have,

\[
V_f(e) \leq -e_f^T \Theta e_f + \Gamma + 2\hat{s}^T (\Delta u + \sigma_1) + 2\hat{s}^T \Upsilon,
\]

where, \( e_f \) is defined in Appendix B, \( \Gamma \geq \frac{\beta}{\delta} \left\| \int_{-h}^{0} [(B \sigma(t + \theta))^T \hat{D}^T \hat{D} \sigma(t + \theta)] d\theta \right\| \) is a positive scalar, \( \Upsilon = \Delta u_h - \Delta u \). Let us define the following,

\[
\hat{s} = s + \Delta s \quad \text{where} \quad \Delta s = B^T P [0 \ (\dot{\epsilon}_1 - \dot{\epsilon}_1)^T]^T.
\]

Evaluating the structure of \( s \) and \( \Delta s \) one can find two positive scalars \( \epsilon_2, \epsilon_3 \) such that \( ||s|| \leq \epsilon_2 ||e_f||, ||\Delta s|| \leq \epsilon_3 ||e_f|| \). Using (46) the stability analysis for (4) employing TARC is carried out for the following various cases.
Case (i): $f(e) > 0$, $\hat{c} \geq \gamma$, $||s|| \geq \epsilon$.
Utilizing (44), (45) and (48) we have,
\[
\dot{V}_r(e) \leq -e_f^T \Theta e_f + \Gamma + 2\hat{s}^T ( -\alpha \hat{s} + \sigma_1 ) + 2s^T \dot{Y} + 2(\hat{c} - c)||s||
\]
\[
= -e_f^T \Theta e_f + \Gamma - 2\alpha \hat{s}_s^T \hat{s} - 2\hat{\alpha} \Delta s^T \hat{s} - 2s^T \sigma_1 + 2s^T \dot{Y} + 2(\hat{c} - c)||s||
\]
\[
\leq -\lambda_{\min} (\Theta)||e_f||^2 - (\alpha - 1)\hat{c}||s|| + \Gamma + 2||\dot{Y}||||s|| + 2(\alpha \hat{c} + c + ||\dot{Y}||)||\Delta s||
\]  
(50)
So, for $\alpha > 1$, $\dot{V}_r(e) < 0$ would be established if $\lambda_{\min} (\Theta)||e_f||^2 \geq \Gamma + 2||s||||\dot{Y}|| + 2(\alpha \hat{c} + c + ||\dot{Y}||)||\Delta s||$. Thus, using the relation $||s|| \leq \upsilon_2 ||e_f||$, $\Delta s \leq \upsilon_3 ||e_f||$, the system would be UUB with the following ultimate bound
\[
||e_f|| = \mu_1 + \sqrt{\frac{\Gamma}{\lambda_{\min} (\Theta)} + \mu_1^2} = \varpi_1.
\]  
(51)
where, $\mu_1 = \upsilon_2 ||\dot{Y}|| + \upsilon_2 (\alpha \hat{c} + c + ||\dot{Y}||)$.

Case (ii): $f(e) \leq 0$, $\hat{c} > \gamma$, $||s|| \geq \epsilon$.
Again, utilizing (45) for Case (ii),
\[
\dot{V}_r(e) \leq -e_f^T \Theta e_f + \Gamma + 2\hat{s}^T ( -\alpha \hat{s} + \sigma_1 ) + 2s^T \dot{Y} - 2(\hat{c} - c)||s||
\]
\[
\leq -\lambda_{\min} (\Theta)||e_f||^2 + (4c - 2(\alpha + 1)\hat{c} + 2||\dot{Y}||)||s|| + \Gamma + 2(\alpha \hat{c} + c + ||\dot{Y}||)||\Delta s||.
\]  
(52)
\dot{V}_r(e) < 0 would be achieved if $\lambda_{\min} (\Theta)||e_f||^2 \geq \Gamma + (4c - 2(\alpha + 1)\hat{c} + 2||\dot{Y}||)||s|| + 2(\alpha \hat{c} + c + ||\dot{Y}||)||\Delta s||$ and system would be UUB having following ultimate bound,
\[
||e_f|| = \mu_2 + \sqrt{\frac{\Gamma}{\lambda_{\min} (\Theta)} + \mu_2^2} = \varpi_2.
\]  
(53)
where, $\mu_2 = \upsilon_2 (2c - (\alpha + 1)\hat{c} + ||\dot{Y}|| + \upsilon_3 (\alpha \hat{c} + c + ||\dot{Y}||)$.

Case (iii): $\hat{c} \leq \gamma$, $||s|| \geq \epsilon$.
Since $\hat{c} \leq \gamma$ we have $(\hat{c} - c)\gamma \leq \gamma^2 - c\gamma \leq \gamma^2$. Using the adaptive law (45), for Case (iii) we have,
\[
\dot{V}_r(e) \leq -e_f^T \Theta e_f + \Gamma + 2\hat{s}^T ( -\alpha \hat{s} + \sigma_1 ) + 2(\hat{c} - c)\gamma + 2s^T \dot{Y}
\]
\[
\leq -\lambda_{\min} (\Theta)||e_f||^2 + \Gamma + 2(\alpha \hat{c} + c + ||\dot{Y}||)||s|| + 2(\alpha \hat{c} + c + ||\dot{Y}||)||\Delta s|| + 2\gamma^2.
\]  
(54)
Similarly, as argued earlier the system would be UUB with the following ultimate bound,
\[
||e_f|| = \mu_3 + \sqrt{\frac{(\Gamma + 2\gamma^2)}{\lambda_{\min} (\Theta)} + \mu_3^2} = \varpi_3.
\]  
(55)
where, $\mu_3 = \upsilon_2 (2c - (\alpha \hat{c} + ||\dot{Y}||) + \upsilon_3 (\alpha \hat{c} + c + ||\dot{Y}||)$.

Case (iv): $f(e) > 0$, $\hat{c} > \gamma$, $||s|| < \epsilon$.
Utilizing (45) time derivative of (46) yields,
\[
\dot{V}_r(e) \leq -e_f^T \Theta e_f + \Gamma + 2\hat{s}^T ( -\alpha \hat{s} + \sigma_1 ) + 2(\hat{c} - c)||s|| + 2s^T \dot{Y}
\]
\[
= -e_f^T \Theta e_f + \Gamma - 2\alpha \hat{s}_s^T \hat{s} - 2\hat{\alpha} \Delta s^T \hat{s} - 2s^T \sigma_1 + 2s^T \dot{Y} + 2(\hat{c} - c)||s||
\]
\[
\leq -\lambda_{\min} (\Theta)||e_f||^2 - 2\alpha \hat{s}||s||^2 - 2\alpha \hat{c}||\Delta s|| \frac{||s||}{\epsilon} + 2\hat{\alpha} \frac{||s||}{\epsilon} + 2(\hat{c} + ||\dot{Y}||)||s|| + 2(\hat{c} + ||\dot{Y}||)||\Delta s|| + \Gamma.
\]  
(56)
Since $||s|| < \epsilon$, from (56) we get,
\[
\dot{V}_r(e) \leq -\lambda_{\min} (\Theta)||e_f||^2 - 2\alpha \hat{c}||s||^2 + 2(\hat{c} + ||\dot{Y}||)||s|| + 2(\hat{c} + ||\dot{Y}||)||\Delta s|| + \Gamma.
\]  
(57)
The combination of second and third term of (57) takes the maximum value $\mu_4 = \frac{\alpha \hat{c}}{2\epsilon} \left( \frac{\epsilon(\hat{c} + ||\dot{Y}||)}{\alpha \epsilon} \right)^2$ for $||s|| = \frac{\epsilon(\hat{c} + ||\dot{Y}||)}{2\alpha \epsilon}$. So, the system would be UUB with the ultimate bound determined to be,
\[
||e_f|| = \mu_4 + \sqrt{\frac{(\Gamma + \mu_4)}{\lambda_{\min} (\Theta)} + \mu_4^2} = \varpi_4.
\]  
(58)
where, $\mu_4 = \frac{\epsilon_2(\alpha \hat{c} + c ||Y||)}{\lambda_{min}(\Theta)}$.

**Case (v):** $f(\epsilon) \leq 0$, $\hat{c} > \gamma$, $||s|| < \epsilon$.

For this case the following is obtained:

$$
\dot{V}_r(e) \leq -e_j^T \Theta e_f + \Gamma + 2\hat{s}^T(-\alpha \hat{c} \frac{\hat{s}}{\epsilon} + \sigma_1) - 2(\hat{c} - c)||s|| + 2\hat{s}^T \Upsilon
$$

$$
= -\lambda_{min}(\Theta)||e_f||^2 + \Gamma - 2\alpha \epsilon||s||^2 + 2(\hat{c} - c)||s|| + 2||Y||||s|| + 2(\alpha \hat{c} + c + ||Y||)||\Delta s||.
$$

The combination of third, fourth and fifth term of (59) takes the maximum value $\mu_{51} = \frac{(2\hat{c} - c + ||Y||)\epsilon}{2\alpha \epsilon}$ for $||s|| = \frac{(2\hat{c} - c + ||Y||)\epsilon}{2\alpha \epsilon}$.

Therefore, for Case (v) the system would be UUB with the ultimate bound obtained as,

$$
||e_f|| = \mu_4 + \sqrt{\frac{(\Gamma + \mu_{51})}{\lambda_{min}(\Theta)}} + \mu_4^2 = \varpi_5.
$$

**Case (vi):** $\dot{c} \leq \gamma$, $||s|| < \epsilon$.

Again, time derivative of (46) for Case (vi) is evaluated as,

$$
\dot{V}_r(e) \leq -e_j^T \Theta e_f + \Gamma + 2\hat{s}^T(-\alpha \hat{c} \frac{\hat{s}}{\epsilon} + \sigma_1) + 2\dot{c} \gamma + 2\hat{s}^T \Upsilon
$$

$$
= -\lambda_{min}(\Theta)||e_f||^2 + \Gamma - 2\alpha \epsilon||s||^2 + 2(\hat{c} + ||Y||)||s|| + 2(\alpha \hat{c} + c + ||Y||)||\Delta s|| + 2\gamma^2.
$$

The combination of third and fourth term of (61) takes the maximum value $\mu_{61} = \frac{(\epsilon(\alpha + ||Y||))\epsilon}{2\epsilon \alpha}$ for $||s|| = \frac{(\epsilon(\alpha + ||Y||))\epsilon}{2\epsilon \alpha}$. So, the system would be UUB with the ultimate bound as,

$$
||e_f|| = \mu_4 + \sqrt{\frac{(\Gamma + 2\gamma^2 + \mu_{61})}{\lambda_{min}(\Theta)}} + \mu_4^2 = \varpi_6.
$$

The finite time $t_r$ to reach each error bound $\varpi_i$ is obtained to be (25),

$$
t_r = t_0 + (||e_f(t_0)|| - \varpi_i)/c_1, \quad \forall i = 1, \cdots, 6,
$$

where, $\dot{V}_r(t) \leq -c_1, c_1 > 0$.

**Remark 2.** The performance of TARC can be characterized by the various error bounds under various conditions. It can be noticed that low value of $h$ and high value of $\alpha$ would result in better accuracy. However, too large $\alpha$ may result in high control input. Also, one may choose different values of $\alpha$ for $||s|| > ||s_h||$ and $||s|| \leq ||s_h||$. Moreover, it is to be noticed that the stability notion of TARC is invariant to the choice of $f(\epsilon)$ and thus provides the designer the flexibility to select a suitable $f(\epsilon)$ according to the application requirement.

Since, the adaptive sliding mode control (ASMC), reported in [22]-[23], also provides robustness against the uncertain system without having any knowledge of the uncertainty bounds, it would be prudent to compare the performance of the ASMC with the proposed controllers. The adaptive law of ASMC can be computed either using (41) or (42). Henceforth, the ASMC law that follows (41) and (42) would be termed as ASMC 1 and ASMC 2 respectively for the remainder of the paper. However, the switching input in the form (59) induces chattering. Hence, it is modified as below

$$
\Delta u_s = \begin{cases} 
-\hat{c} \frac{\hat{s}}{||s||} & \text{if } ||s|| \geq \epsilon, \\
-\hat{c} \frac{\hat{s}}{\epsilon} & \text{if } ||s|| < \epsilon,
\end{cases}
$$

The control structure of the various controllers, which would further be studied in Section III, are provided in Table I.

<table>
<thead>
<tr>
<th>Controller</th>
<th>Control structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>TDC ([7]), ([26])-[([27])</td>
<td>(2), (4)</td>
</tr>
<tr>
<td>F-TDC (proposed)</td>
<td>(2), (34), (35)</td>
</tr>
<tr>
<td>ASMC 1 ([22])</td>
<td>(37), (38), (40), (41), (64)</td>
</tr>
<tr>
<td>ASMC 2 ([22], [23])</td>
<td>(37), (38), (40), (42), (64)</td>
</tr>
<tr>
<td>TARC (proposed)</td>
<td>(2), (34), (35), (43), (44), (45)</td>
</tr>
</tbody>
</table>
III. APPLICATION: NONHOLONOMIC WMR

In this section, we have focused on Nonholonomic WMR which provides a unique platform to test the proposed control law since under practical circumstances, a WMR is always subjected to uncertainties like friction, slip, skid etc. These terms are difficult to model and in many cases they are not considered while modelling. The dynamic equation of a WMR after solving the Lagrange multiplier as in \[14\] can be written as follows,

$$\dot{M}(q)\ddot{q} + \ddot{V}(q, \dot{q}) = Gu,$$  \hspace{1cm} (65)

where,

$$\dot{M} = \begin{bmatrix} m & 0 & K\sin\phi & k_1 & k_2 \\ 0 & m & -K\cos\phi & k_3 & k_4 \\ k_1 & k_3 & I & -k_5 & k_6 \\ k_2 & k_4 & 0 & I_w & 0 \\ k_5 & 0 & 0 & 0 & I_{w} \end{bmatrix}, \quad \ddot{V} = \begin{bmatrix} m\ddot{\phi}\cos\phi + m\dot{r}\dot{\theta}\sin\phi(\dot{\theta}^2 - \dot{\theta}_f^2)/2b \\ m\ddot{\phi}\cos\phi - m\dot{r}\dot{\theta}\sin\phi(\dot{\theta}^2 - \dot{\theta}_f^2)/2b \\ K_r\ddot{\theta}_f^2/2 \\ -K_r\ddot{\phi}^2/2 \\ -K_r\phi^2/2 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad u = \begin{bmatrix} u_r \\ u_l \end{bmatrix}.$$

Here \( q \in \mathbb{R}^5 = \{x_c, y_c, \phi, \theta_r, \theta_l\} \) is the generalized coordinate vector of the system. The position of the WMR can be specified by three generalized coordinates \((x_c, y_c, \phi)\) where, \((x_c, y_c)\) are the coordinates of the center of mass of the system and \(\phi\) is the heading angle. \((\theta_r, \theta_l)\) and \((u_r, u_l)\) are rotation and torque inputs of the right and left wheels respectively. \(m\) and \(I\) represents the mass and inertia of the overall system. Definition of other system parameters are detailed in [14].

A. Experimental Results and Comparison

TARC is employed in "PIONEER-3" WMR while the robot is directed to track a predefined circular path. The control architecture of the proposed TARC is depicted in Fig. 1. The 'State derivative estimator' block has a memory which can store state feedback history up to \((t - \zeta)\) for computing the state derivatives. Tracking performance of TARC is compared with TDC, ASMC 1, ASMC 2 and the proposed F-TDC. The necessary controller parameters are selected as \(K_1 = 4I, K_2 = Q = D = I, L = 0.3I, \xi = 11, \zeta = 3s, \beta = 0.1, \Lambda = 2\). The delay time and thus the sampling interval is selected as \(h = 30ms\) which satisfy both the conditions [25] and [36].

The threshold value \((\rho)\) for ASMC 1 is selected based on maximum uncertainty bound. There is no mean to determine exactly its value. However, \(\rho = 0.05\) is selected for ASMC 1. Other design parameters are defined as \(\alpha = \bar{\alpha} = 2, \bar{s} = \bar{s}, \epsilon = 0.1, \gamma = 0.001\). The desired trajectories for circular path is defined as,

\[
\begin{align*}
x_c^d &= 1.25\sin(0.35t) + .1, \\
y_c^d &= 1.25\cos(0.35t) + 1.35, \\
\phi^d &= 0.25t, \quad \theta_r^d = 3t, \quad \theta_l^d = 2t.
\end{align*}
\]

To create a dynamic parametric variation, a further \(3.5kg\) payload is added and kept for \(5sec\) and then removed. This process is carried out for the entire duration of experimentation. A time gap \(5sec\) is maintained between two successive instances of
addition of the payload. However, the payload was added randomly at different places on the robotic platform every time to create variation in center of mass and inertia.

The trajectory tracking performance of TARC is depicted in Fig. 2 while following the desired circular path. Tracking performance comparison of TARC against TDC and F-TDC is illustrated in Fig. 3 and Fig. 4 in terms of $x_c$ and $y_c$ position respectively. All the error plots are in absolute value. According to Lemma 1, $\dot{q}$, $\ddot{q}$ cannot be computed using (28)-(29) for $t < \varsigma$. So, they are computed numerically for $t < \varsigma$. Hence, it can be observed from the error plots that tracking performance of TDC and F-TDC is similar during this time period. However, from $t \geq \varsigma$, immediate improvement in controller performance for F-TDC over TDC can be noticed due to the inclusion of state derivative estimator for F-TDC. On the other hand, F-TDC is not embedded with any robust term to counter the approximation errors. TARC, on this front, provides robustness against all the uncertainties with its adaptive-robust control law, beside utilizing (28)-(29) for state derivative estimation. Thus, TARC provides better tracking accuracy than TDC and F-TDC. Yet again, tracking performance of TARC in comparison to ASMC 1 and ASMC 2 for $x_c$ and $y_c$ position is demonstrated through Fig. 5 and Fig. 6 respectively. Superior performance of TARC over ASMC 1 and ASMC 2 can easily be comprehended from the error plots. It is easier to infer the performance of the individual controllers in terms of average $x_c$ position error (AE-$x_c$) and average $y_c$ position error (AE-$y_c$) which are evaluated from the absolute value of the error. These data are represented in Table II where the percentage error is calculated with respect to the diameter of the circular path. Due to the robustness property, ASMC 1 and ASMC 2 provides better accuracy over TDC and F-TDC through their switching control logic.

Another important attribute for an adaptive-robust control law is the evaluation of switching gain which tackles the uncertainties. The switching gain for both the wheels are provided in Fig. 7, Fig. 8 and Fig. 9 for ASMC 2, ASMC 1 and TARC respectively. Switching gain for ASMC 1 depends on the choice of $\rho$. In the current form, the nature of switching gain for ASMC 1 is monotonically increasing. This may cause actuator saturation or even chattering due to high control input in longer run. Choice of a high threshold value can circumvent this situation but too high a value will further enhance the overestimation.
Figure 4. $y_c$ position tracking error performance comparison of TARC with TDC and F-TDC.

Figure 5. $x_c$ position tracking error performance comparison of TARC with ASMC 1 and ASMC 2.

Figure 6. $y_c$ position tracking error performance comparison of TARC with ASMC 1 and ASMC 2.

Table II

<table>
<thead>
<tr>
<th>Controller</th>
<th>AE-$x_c$</th>
<th>% AE-$x_c$</th>
<th>AE-$y_c$</th>
<th>% AE-$y_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>TDC ([7], [26]-[27])</td>
<td>105.09</td>
<td>4.20</td>
<td>99.55</td>
<td>3.98</td>
</tr>
<tr>
<td>F-TDC (proposed)</td>
<td>75.99</td>
<td>3.04</td>
<td>70.80</td>
<td>2.83</td>
</tr>
<tr>
<td>ASMC 1 ([22])</td>
<td>65.78</td>
<td>2.63</td>
<td>63.97</td>
<td>2.56</td>
</tr>
<tr>
<td>ASMC 2 ([22]-[23])</td>
<td>58.15</td>
<td>2.33</td>
<td>56.89</td>
<td>2.28</td>
</tr>
<tr>
<td>TARC (proposed)</td>
<td>37.57</td>
<td>1.50</td>
<td>13.00</td>
<td>0.52</td>
</tr>
</tbody>
</table>

problem of switching gain and reduce accuracy. Though ASMC 2 does not suffer from this problem, however, it assumes that
the perturbations in any parameter is always less than its nominal values. This further necessitates nominal modelling of the uncertainties in $N(q, \dot{q}, \ddot{q})$. Modelling of the uncertainties is tedious and thus ignored here. Thus the performance of ASMC 2 gets degraded. TARC is devoid of any such issues and this augments its superior performance. The required control input for ASMC 1, ASMC 2 and TARC is illustrated through the Figs. 7-9, respectively.

**IV. CONCLUSION**

Selection of the controller gain and sampling interval is crucial for the performance of TDC and this design issue is addressed in this paper through a new stability approach. A bound on the delay is derived to select a suitable sampling interval. A new control approach, F-TDC is devised where the state derivatives are estimated from the previous state information. Moreover, a novel adaptive-robust control law, TARC has been proposed for a class of uncertain nonlinear systems subjected to unknown uncertainties. The proposed controller approximates unknown dynamics through time-delayed law and negotiates...
the approximation error, that surfaces due to the time-delayed approximation of uncertainties and state derivatives, by switching logic. The adaptive law eliminates the overestimation-underestimation problem for online evaluation of switching gain without any prior knowledge of uncertainties. Experimentation with a WMR shows improved path tracking performance of TARC compared to TDC and conventional ASMC. The proposed framework can also be extended for other systems such as Autonomous Underwater Vehicle, Unmanned Aerial Vehicle, Robotic manipulator etc.
APPENDIX A

PROOF OF LEMMA 1

Proof. Let us consider a linear time invariant system,

\[
\begin{align*}
\dot{x}(t) &= \Sigma x(t) \\
y(t) &= Cx(t)
\end{align*}
\]

where, \( x \in \mathbb{R}^{A+1}, \Sigma \in \mathbb{R}^{(A+1) \times (A+1)}, y \in \mathbb{R} \). If \((\Sigma, C)\) is observable, then from the definition of observability, we can detect \( x(t_1) \) from the measurement \( y(t) \) in \( v \in [t_0, t_1] \). So,

\[
y(v) = C\Upsilon(v, t_1)x(t_1),
\]

where, \( \Upsilon(v, t_1) \) is state transition matrix. Again, from (68) we have

\[
\int_{t_0}^{t_1} \Upsilon^T(v, t_1)C^T y(v)dv = \int_{t_0}^{t_1} \Upsilon^T(v, t_1)C^T \times
\]

\[
\times C\Upsilon(v, t_1)dv x(t_1).
\]

An estimate of \( x(t) \) can be obtained from (69) as,

\[
\dot{x}(t) = \omega_r^{-1}(t_0, t_1) \int_{t_0}^{t_1} \Upsilon^T(v, t_1)C^T y(v)dv,
\]

where, \( \omega_r(t_0, t_1) = \left( \int_{t_0}^{t_1} \Upsilon^T(v, t_1)C^T \Upsilon(v, t_1)dv \right) \) is the observability Gramian. As, \((\Sigma, C)\) is observable, \( \omega_r^{-1} \) would always exist. Now, taking \( x = [y, \dot{y}, \ldots, y^{(j)}, \ldots, y^{(A)}]^T, C = [1 \ 0 \ \cdots \ 0], \Sigma = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, t_0 = t - \varsigma, t_1 = t, \) (70) can be obtained as (28)-(29).

APPENDIX B

PROOF OF THEOREM 2

Proof. Let us define the Lyapunov functional as,

\[
V_f(e) = V(e) + V_{f1}(e) + V_{f2}(e) + V_{f3}(e),
\]

\[
V_{f1}(e) = \frac{h\varsigma}{\beta} \int_{-h}^{0} \int_{-\varsigma}^{0} \int_{t-h+\psi}^{t-h} e^T(\eta + \theta) D A_2^2(\psi) e(\eta + \theta) d\eta d\psi d\theta,
\]

\[
V_{f2}(e) = \frac{h\varsigma}{\beta} \int_{-h}^{0} \int_{-\varsigma}^{0} \int_{t+\theta}^{t} e^T(\eta - h) D A_2^2(\psi) e(\eta - h) d\eta d\psi d\theta,
\]

\[
V_{f3}(e) = \varsigma \int_{-\varsigma}^{0} \int_{t+\psi}^{t} e^T(\eta - h) L A_2^2(\psi) e(\eta - h) d\eta d\psi.
\]

where, \( V \) is given in (10) and putting (34) into (4) we obtain,

\[
\dot{e} = A_1 e + B_1 \dot{e}_h - \bar{B} \int_{-\varsigma}^{0} A_d(\psi) e(t - h + \psi) d\psi + B\sigma_2,
\]

where, \( \sigma_2 = K_2 \dot{e}_1 h + \sigma_1 \). Further the error dynamics (72) can be written as,

\[
\dot{e} = A e - B_1 \int_{-h}^{0} \dot{e}(t + \theta) d\theta - \bar{B} \int_{-\varsigma}^{0} A_d(\psi) e(t - h + \psi) d\psi + B\sigma_2.
\]

Applying (17) and Lemma 2 to the term \(-2e^T PB_1 \bar{B} \int_{-h}^{0} \int_{-\varsigma}^{0} A_d(\psi) e(t - h + \theta + \psi) d\psi \) we get,

\[
-2e^T PB_1 \bar{B} \int_{-h}^{0} \int_{-\varsigma}^{0} A_d(\psi) e(t - h + \theta + \psi) d\psi \leq \beta e^T PB_1 \bar{B} D^{-1} \bar{B}^T B_1^T P e + \frac{1}{\beta} \int_{-h}^{0} \int_{-\varsigma}^{0} e^T(t - h + \theta + \psi) A_d(\psi) d\psi D \times
\]

\[
\times \int_{-h}^{0} \int_{-\varsigma}^{0} A_d(\psi) e(t - h + \theta + \psi) d\psi 
\]

\[
\leq \beta e^T PB_1 \bar{B} D^{-1} \bar{B}^T B_1^T P e + \frac{h\varsigma}{\beta} \int_{-h}^{0} \int_{-\varsigma}^{0} e^T(t - h + \theta + \psi) D \times
\]

\[
\times A_2^2(\psi) e(t - h + \theta + \psi) d\psi
\]
Following the similar procedure for proving Theorem 1 and utilizing \((74)\) the time derivative of \(V_{f1}, V_{f3}\) and \(V_{f3}\) yields
\[
\dot{V}(e) \leq -e^T \left[ Q - \dot{E} - (1 + \xi) \frac{h^2}{\beta} D \right] e + \Gamma_2 + \frac{h\xi}{\beta} \int_{-h}^{0} \int_{-h}^{0} e^T (t-h+\theta+\psi) DA_2^2(\psi)e(t-h+\theta+\psi)d\psi d\theta
\]
\[
+ 2s^T \sigma_2 - 2e^T PB \int_{-\varsigma}^{0} A_d(\psi)e(t-h+\psi)d\psi,
\]
where, \(\frac{h}{\beta} \left\| \int_{-h}^{0} \left[ (B \sigma_2(t+\theta))^T \bar{D}^T \bar{D} \sigma_2(t+\theta) \right] d\theta \right\| \leq \Gamma_2.
\]
\[
\dot{V}_{f1} \leq \frac{h\xi}{\beta} \int_{-h}^{0} \int_{-h}^{0} e^T (t-h+\theta) DA_2^2(\psi)e(t-h+\theta)d\psi d\theta - \frac{h\xi}{\beta} \int_{-h}^{0} \int_{-h}^{0} e^T (t-h+\theta+\psi) DA_2^2(\psi)e(t-h+\theta+\psi)d\psi d\theta
\]
\[
\dot{V}_{f2} \leq \frac{h^2}{\beta} e^T (t-h) D \int_{-\varsigma}^{0} A_2^2(\psi)d\psi e(t-h) - \frac{h\xi}{\beta} \int_{-h}^{0} \int_{-h}^{0} e^T (t-h+\theta) DA_2^2(\psi)e(t-h+\theta)d\psi d\theta
\]
\[
\dot{V}_{f3}(e) \leq s e^T (t-h) \int_{-\varsigma}^{0} L A_2^2(\psi)d\psi e(t-h) - \int_{-\varsigma}^{0} A_d(\psi)e^T (t-h+\psi)d\psi L \int_{-\varsigma}^{0} A_d(\psi)e(t-h+\psi)d\psi
\]
Now, \(2s^T K e_{1h} = 2e^T PB e_h.\) Combining \((75)-(78)\)
\[
\dot{V}_f(e) \leq -e^T \Theta e_f + \Gamma_2 + 2s^T \sigma_1,
\]
where \(e_f = \left[ e^T \int_{-\varsigma}^{0} A_d(\psi)e^T (t-h+\psi)d\psi \right]^T.\) Let choice of \(K_1, K_2, h, \varsigma\) make \(\Theta > 0.\) Again, a positive scalar \(\iota_1\) can be found such that \(||\dot{s}|| \leq \iota_1 ||e_f||.\) So, \(\dot{V}_f(e) < 0\) would be achieved if \(\lambda_{min}(\Theta)||e_f||^2 > \Gamma_2 + 2\iota_1 ||\sigma_1||||e_f||.\)
Thus, the system would be UUB with the following ultimate bound
\[
||e_f|| = \gamma_2 + \sqrt{\frac{\Gamma_2}{\lambda_{min}(\Theta)}} + \frac{\gamma_2^2}{2} = \varpi.
\]
where, \(\gamma_2 = \frac{\iota_1 ||\sigma_1||}{\lambda_{min}(\Theta)}\). The time \(t_r\) to reach \(\varpi\), for some \(c_2 > 0\), is given by
\[
t_r - t_0 \leq \left( ||e_f(t_0)|| - \varpi \right)/c_2 \quad \text{where} \quad \dot{V}_f(e) \leq -c_2.
\]

**References**


