Diagonal Lift in the Tangent Bundle of Order Two 
and its Applications* 

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Abstract

In this paper we define a diagonal lift $D_g$ of Riemannian metric $g$ of manifold $M_n$ to the tangent bundle of order two denoted by $T^2 M_n$ of $M_n$, we associate to $D_g$ its Levi-civita connection of $T^2 M$ and we investigate applications of the diagonal lifts in the killing vectors and geodesics.

Key Words: Tangent bundle of order two, Riemannian metric, Diagonal lift, Levi-civita connection, Killing vector field, Geodesic.

1. Introduction

Let $M_n$ be an $n$-dimensional differentiable manifold endowed with a linear connection $\nabla$. The tangent bundle of order two, $T^2 M_n$ of $M_n$ is the $3n$-dimensional manifold of 2-jets at $0 \in \mathbb{R}$ of differentiable curves $f : \mathbb{R} \rightarrow M_n$; $T^2 M_n$ has a natural bundle structure over $M_n$,

$$\pi_2 : T^2 M_n \rightarrow M_n$$

denoting the canonical projection.

The tangent bundle $TM_n$ is nothing by the manifold of 1-jets $j^1 f$ at $0 \in \mathbb{R}$ of the curves $f : \mathbb{R} \rightarrow M_n$.

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If we denote $\pi_{12} : T^2M_n \to TM_n$ be a canonical projection $\pi_{12}$, then $T^2M$ has a bundle structure over $TM_n$, with projection $\pi_{12}$.

For any coordinate neighborhood $(U, x^i)$ in $M$, $(\pi^{-1}(U), x^i, y^j)$ denotes the induced coordinate neighborhood in $TM_n$, that is, if $j^1f \in TU$ then
\[
x^i = f^i(0), \quad y^i = \frac{df^i}{dt}(0)
\]
and $((\hat{\pi})^{-1}(U), x^i, y^j, z^k)$ denotes the induced coordinate neighborhood in $T^2M_n$, that is, if $j^2f \in T^2U$ then
\[
x^i = f^i(0), \quad y^i = \frac{df^i}{dt}(0), \quad z^i = \frac{d^2f^i}{dt^2}(0)
\]
where $x^i = f^i(t)$ are the local expression of the curve $f$ in $U$.

Let $f : \mathbb{R} \to M_n$ be a curve in $M_n$, then the tangent vector $\dot{f}(0)$ to $f$ at $f(0)$ will be called the velocity of $f$ at $f(0)$ and the covariant derivative $(\nabla_{\dot{f}(0)} f)(0)$ of $\dot{f}$ at $f(0)$ with respect to $f(0)$ will be called the covariant acceleration of $f$ at $f(0)$.

If $(U, x^i)$ is a coordinate neighborhood in $M_n$ and $x^i = f^i(t)$ are the local expressions of $f$ in $U$, we have
\[
\dot{f}(0) = \frac{df^i}{dt} \frac{\partial}{\partial x^i}
\]
\[
(\nabla_{\dot{f}(0)} f)(0) = \left( \frac{d^2f^i}{dt^2} + \frac{df^j}{dt} \frac{df^k}{dt} \Gamma^i_{jk} \right) \frac{\partial}{\partial x^i},
\]
\(\Gamma^i_{jk}\) being the components of $\nabla$ in $U$.

2. **$\lambda$-lift from $M_n$ to $T^2M_n$**

For any $x \in M_n$, we define the map
\[
S_x : T^2_x M \to T_x M \oplus T_x M
\]
\[
j^2f \mapsto (\dot{f}(0), (\nabla_{\dot{f}(0)} f)(0)).
\]
Then, $S_x$ is bijective and permits one to define a vector space structure on $T_x^2 M_n$ such that $S_x$ is a vector space isomorphism. Therefore $T^2 M_n$ becomes a vector bundle over $M_n$ with fibre $\mathbb{R}^{2n}$ and projection $\pi_2$.

Indeed, if $(U, x^i)$ is a coordinate neighborhood in $M_n$, then $U$ can be considered as a vector bundle chart by defining the diffeomorphism

$$ j^2 f \rightarrow (f(0), \frac{df}{dt}(0), (\nabla_{f(0)} J)(0)), $$

or in the induced coordinates

$$ (x^i, y^i, z^i) \rightarrow (x^i, y^i, w^i), $$

where $w^i = z^i + y^i g^k \Gamma^i_{jk}$.

Moreover, let $TM_n \oplus TM_n$ be the Whitney sum of $TM_n$ with itself, then the map

$$ S : T^2 M_n \rightarrow TM_n \oplus TM_n $$

defined on each fibre $T_x^2 M_n$ as $S_x$, becomes a vector bundle isomorphism.

Thus, we have the following theorem.

**Theorem 1** The linear connection $\nabla$ on $M_n$ determines a vector bundle structure on $\pi_2 : T^2 M_n \rightarrow M_n$ and a vector bundle isomorphism $S : T^2 M_n \rightarrow TM_n \oplus TM_n$.

For any vector fields $X$ on $M_n$, we shall denote by $X^V$ (resp $X^H$) the vertical lift (resp the horizontal lift) with respect to $\nabla$ of $X$ to $TM_n([3])$.

If we have in $TU$

$$ X^H = \left( \frac{\partial}{\partial x^i} \right)^H = \frac{\partial}{\partial x^i} + y^i g^k \Gamma^i_{jk} \frac{\partial}{\partial y^j}, \quad X^V = \left( \frac{\partial}{\partial x^i} \right)^V = \frac{\partial}{\partial y^i}, $$

consequently $\{X^H, X^V\}$ is a 2n-frame which will be called the adapted frame to $\nabla$ in $TU$.

Now, for any vector field $X$ on $M_n$ we shall consider three vectors fields $X^0, X^I$ and $X^{II}$ on $T^2 M_n$ defined by

$$ X^0 = S^{-1}_x (X^H + X^H), $$
$$ X^I = S^{-1}_x (X^V + 0), $$
$$ X^{II} = S^{-1}_x (0 + X^V). $$

(1)
If we put in $T^2U$, then

\[ X^0 = (\frac{\partial}{\partial x^i})^0; \quad X^1 = (\frac{\partial}{\partial y^i})^i; \quad X^2 = (\frac{\partial}{\partial z^i})^i \]  

(2)

and

\[ y^h \Gamma^k_{ih} = \Gamma^k_i; \quad A^k_i = z^h \Gamma^k_{ih} + y^r \left( \frac{\partial \Gamma^k_i}{\partial x^r} + \Gamma^k_{ih} \Gamma^i_r - \Gamma^k_i \Gamma^r_h \right). \]

we thus obtain

\[ X^0 = \frac{\partial}{\partial x^i} - \Gamma^k_i \frac{\partial}{\partial y^k} - A^k_i \frac{\partial}{\partial z^k} \]
\[ X^1 = \frac{\partial}{\partial y^i} - 2 \Gamma^k_i \frac{\partial}{\partial z^k} \]
\[ X^2 = \frac{\partial}{\partial z^i}. \]

(3)

and therefore, $\{X^0, X^1, X^2\}$ is a 3n-frame which will be called the adapted frame to $\nabla$ in $T^2U([7], [8])$.

From (1), (2) and theorem 1 we easily obtain

\[ X^0 = S^{-1}_x (X^H, X^H), \quad X^1 = S^{-1}_x (X^V, 0), \quad X^2 = S^{-1}_x (0, X^V). \]

(4)

Now have the following definition.

**Definition 2** If $X$ is a vector field on $U$, $X^\lambda (\lambda = 1, 2, 3)$ is called the $\lambda$-lift of $X$ to $T^2U$.

$\lambda$-lift were studied in [8] and applied to the tangent bundle of higher order $T^r U$; and in the case of $r = 1$, we have $X^1 = X^V$ and $X^0 = X^H$.

**Proposition 3** For any $\lambda = 0, 1, 2$ we have

\[ (fX)^\lambda = f(X^\lambda) \]

for all $f \in C^\infty(M)$.

For any 1-form $w$ in $M_n$, there exists a unique 1-form $w^\lambda (\lambda = 0, 1, 2)$ in $T^2M_n$, which for any vectors field $X$ on $M_n$ we have

\[ w^\lambda (X^\lambda) = \delta^2 - \lambda w(X) \circ \frac{2}{7}. \]

(5)

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Definition 4 The 1-form \( w^\lambda \) in \( T^2M_n \) is called the \( \lambda \)-lift of \( w \).

If we put
\[
\bar{A}^k_i = 2(g^h_i g^m_j \Gamma^k_{im} + A^k_i),
\]
and by taking account of (5), we have

\[
\begin{align*}
\bar{dx}^0_i &= \bar{A}^k_i dx_k + 2\Gamma^k_i dy_k + dz_i \\
\bar{dx}^1_i &= \Gamma^k_i dx_k + dy_i \\
\bar{dx}^2_i &= dx_i
\end{align*}
\]

Let now \( M_n \) be a Riemannian manifold with nondegenerate metric \( g \) whose components is a coordinate neighborhood \( U \) are \( g_{ij} \) and denote by \( \Gamma^k_{ij} \) the Christoffel symbols formed with \( g_{ij} \).

3. Lift \( Dg \) of Riemannian \( g \) to \( T^2M_n \)

For any tensor field \( g \) of type (0, 2) in \( M_n \), there exist a unique tensor field \( Dg \in \mathcal{T}^0_2(T^2M_n) \) which for any vector fields \( X, Y \) on \( M_n \) and any \( i, j = 0, 1, 2 \), we have ([9])
\[
Dg(X^i, Y^j) = \delta^k_j g(X, Y) \circ \bar{x}^k,
\]
and locally in \( T^2M_n \) we have
\[
Dg = g_{ij} \bar{dx}^0_i \otimes \bar{dx}^0_j + g_{ij} \bar{dx}^1_i \otimes \bar{dx}^1_j + g_{ij} \bar{dx}^2_i \otimes \bar{dx}^2_j.
\]
Thus from (7) and (8), \( Dg \) has components of the form
\[
\left( \begin{array}{ccc} g_{ij} & 0 & 0 \\ 0 & g_{ij} & 0 \\ 0 & 0 & g_{ij} \end{array} \right)
\]
with respect to the adapted frame \( \{ X^0, X^1, X^2 \} \) in \( T^2M_n \).
and components

$$Dg = \begin{pmatrix}
g_{ij} + g_{lm} \Gamma^i_l \Gamma^j_m & g_{kj} \Gamma^i_k + 2g_{lm} \dot{A}^i_l \Gamma^j_m & g_{kj} \dot{A}^i_k \\
g_{kl} \Gamma^j_k & g_{ij} + 2g_{lm} (\Gamma^i_l + \Gamma^j_m) & 2g_{kl} \Gamma^j_k \\
g_{ki} \dot{A}^j_k & 2g_{kl} \Gamma^j_k & g_{ij}
\end{pmatrix}$$

(10)

with respect to the coordinates \((x^i, y^i, z^i)\).

From (9) it follows that if \(g\) is a Riemannian metric in \(M_n\), then \(Dg\) is a Riemannian metric in \(T^2M_n\):

**Definition 5**
The metric \(Dg\) is called the diagonal lift of the tensor field \(g\) to \(T^2M_n\) (see [9]).

In the case of \(TM_n\) we find diagonal lift studied by S.Sasaki ([13]).

4. Levi-Civita Connection of \(Dg\)

Let \(\nabla\) be a linear Levi-Civita connection on \(M_n\), and taking account that \(\nabla\) is torsion free we shall need the following identities:

$$\begin{align*}
[X^0, Y^0] &= [X, Y]^0 - \sum_{k=1}^{2} (R(X, Y)u)^k \\
[X^0, Y^j] &= (\nabla_X Y)^j \\
[X^i, Y^j] &= 0 \quad \forall i, j = 1, 2
\end{align*}$$

(11)

(for proof, see [7], [8], [12]).

And by koszule formula, the Levi-Civita Connection \(D\nabla\) of \((T^2M_n, Dg)\) is given as following

$$\begin{align*}
1/ D\nabla_{X^0} Y^0 &= (\nabla_X Y)^0 - \frac{1}{2} \sum_{k=1}^{2} (R(X, Y)u)^k \\
2/ D\nabla_{X^0} Y^1 &= (\nabla_X Y)^1 + \frac{1}{2} (R(u, Y)X)^0 \\
3/ D\nabla_{X^0} Y^2 &= (\nabla_X Y)^2 + \frac{1}{2} (R(u, Y)X)^0 \\
4/ D\nabla_{X^1} Y^0 &= D\nabla_{X^2} Y^0 = \frac{1}{2} (R(u, X)Y)^0 \\
5/ D\nabla_{X^1} Y^1 &= D\nabla_{X^2} Y^1 = D\nabla_{X^2} Y^2 = 0
\end{align*}$$

(12)
for any vectors fields $X$, $Y \in C^\infty(M_n)$ and for all $(p, u) \in TM_n$.

Thus, according to (8), (9) and (12), the components $D\Gamma^h_{\alpha\gamma}$ with respect to the adapted frame are given by

$$D\Gamma^h_{ij} = \Gamma^h_{ij}; \quad D\Gamma^h_{ij} = D\Gamma^h_{ij} = \frac{1}{2}y^k R^h_{kij}; \quad D\Gamma^h_{ij} = D\Gamma^h_{ij} = \frac{1}{2}y^k R^h_{kij}$$

$$D\Gamma^h_{ij} = D\Gamma^h_{ij} = D\Gamma^h_{ij} = D\Gamma^h_{ij} = 0$$

$$D\Gamma^h_{ij} = -y^k \Gamma^h_{ij} \Gamma^h_k - y^k \frac{1}{2} R^h_{ijk}; \quad D\Gamma^h_{ij} = D\Gamma^h_{ij} = -\frac{1}{2} y^k \Gamma^h_s R^s_{kij}$$

$$D\Gamma^h_{ij} = \Gamma^h_{ij} - \frac{1}{2} y^k \Gamma^h_s R^s_{kij}; \quad D\Gamma^h_{ij} = D\Gamma^h_{ij} = 0$$

$$D\Gamma^h_{ij} = -\frac{1}{2} y^k \Gamma^h_s R^s_{kij}; \quad D\Gamma^h_{ij} = D\Gamma^h_{ij} = 0$$

$$D\Gamma^h_{ij} = -\frac{1}{2} y^k \Gamma^h_s R^s_{kij}; \quad D\Gamma^h_{ij} = -\frac{1}{2} y^k \Gamma^h_s R^s_{kij} A^h_s$$

$$D\Gamma^h_{ij} = -\frac{1}{2} y^k \Gamma^h_s R^s_{kij} A^h_s; \quad D\Gamma^h_{ij} = -\frac{1}{2} y^k \Gamma^h_s R^s_{kij} A^h_s - 2\Gamma^h_{ij} \Gamma^h_s; \quad D\Gamma^h_{ij} = D\Gamma^h_{ij} = 0$$

$$D\Gamma^h_{ij} = -\frac{1}{2} y^k \Gamma^h_s R^s_{kij} A^h_s + \Gamma^h_{ij}; \quad D\Gamma^h_{ij} = D\Gamma^h_{ij} = 0.$$  \hspace{1cm} (13)

5. Killing Vector Fields

A vector fields $X$ is said to be infinitesimal isometry or a Killing vector field of a riemannian manifold with metric $g$, if

$$\mathcal{L}_X g = Xg(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]) = 0$$  \hspace{1cm} (14)

for all $X, Y \in C^\infty(M_n)$. In the terms of components $g_{ij}$ of $g$, $X$ is an infinitesimal isometry if and only if

$$X^h \partial_h g_{ij} + g_{ij} \partial_h X^h + g_{hk} \partial_j X^h = 0$$

where $X^h$ are components of $X$. (see [3])

We see by virtue of (8) that $\bar{X}$ is a Killing vector field in $T^2 M_n$ with metric $Dg$ if and only if

$$\mathcal{L}_{\bar{X}} Dg = \bar{X}g(\bar{Y}, \bar{Z}) - Dg([\bar{X}, \bar{Y}], \bar{Z}) - Dg(\bar{Y}, [\bar{X}, \bar{Z}]) = 0$$  \hspace{1cm} (15)
for all \( \tilde{Z}, \tilde{Y} \in C^\infty(T^2M_n) \).

Then by (11) we have

\[
\begin{align*}
\mathfrak{L}_{X^0} D g(Y^i, Z^i) &= X g(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0 \\
\mathfrak{L}_{X^0} D g(Y^0, Z^j) &= g(R(X, Y)u^j, Z^j) = g(R(X, Y)u, Z) \\
\mathfrak{L}_{X^0} D g(Y^j, Z^0) &= g(R(X, Z)u^j, Y^j) = g(R(X, Y)u, Z) \\
\mathfrak{L}_{X^0} D g(Y^1, Z^2) &= (\mathfrak{L}_{X^0} D g)(Y^2, Z^1) = 0
\end{align*}
\]

(16)

\[
\begin{align*}
\mathfrak{L}_{X^1} D g(Y^i, Z^i) &= 0 \\
\mathfrak{L}_{X^1} D g(Y^0, Z^1) &= g(\nabla_Y X, Z) \\
\mathfrak{L}_{X^1} D g(Y^1, Z^0) &= g(Y, \nabla_Z X) \\
\mathfrak{L}_{X^1} D g(Y^0, Z^2) &= (\mathfrak{L}_{X^1} D g)(Y^2, Z^0) = 0 \\
\mathfrak{L}_{X^1} D g(Y^1, Z^2) &= (\mathfrak{L}_{X^1} D g)(Y^2, Z^1) = 0
\end{align*}
\]

(17)

\[
\begin{align*}
\mathfrak{L}_{X^2} D g(Y^i, Z^i) &= 0 \\
\mathfrak{L}_{X^2} D g(Y^0, Z^1) &= (\mathfrak{L}_{X^2} D g)(Y^1, Z^0) = 0 \\
\mathfrak{L}_{X^2} D g(Y^0, Z^2) &= g(\nabla_Y X, Z) \\
\mathfrak{L}_{X^2} D g(Y^2, Z^0) &= g(Y, \nabla_Z X) \\
\mathfrak{L}_{X^2} D g(Y^1, Z^2) &= (\mathfrak{L}_{X^2} D g)(Y^2, Z^1) = 0
\end{align*}
\]

(18)

for all \( X \in C^\infty(M_n) \), \( j = 1, 2 \) and \( i = 0, 1, 2 \).

Since we have

\[
X g(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]) = X g(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) + g(\nabla_Y X, Z) + g(Y, \nabla_Z X),
\]

we conclude by means of (16), (17) and (18) that if \( \mathfrak{L}_{X^0} D g \) and \( \mathfrak{L}_{X^1} D g \) or \( \mathfrak{L}_{X^2} \), that \( D g \) vanishes implies that \( \mathfrak{L}_X g = 0 \).

We next have

\[
\begin{align*}
R(X, Y)u &= 0 \iff X^h R^h_{hij} = 0 \\
\nabla_Z X &= \nabla_X (Z) = 0
\end{align*}
\]

(20)

and \( \mathfrak{L}_X g = 0 \) imply that \( \mathfrak{L}_X D g = 0 \) for \( i = 0, 1, 2 \). Thus, we have.
Theorem 6 The vector field $X$ in $M_n$ is a killing vector field if its 0-lift and $\lambda$-lift ($\lambda = 1$ or 2) are killing vector fields in $T^2M_n$. Conversely, if $X$ is a killing vector field, parallel and $R(X,Y)u = 0$ vanishes for all $Y \in C^\infty(M_n)$ (i.e. $X^h R^h_{hij} = 0$), then $\lambda$-lift ($\lambda = 0, 1, 2$) of $X$ is a killing vector field in $T^2M_n$.

6. Geodesics in $T^2M_n$ with metric $^Dg$

Let $C$ be a curve in $M_n$ expressed locally by $x^i = x^i(t)$ and $y^i(t)$ be a vector field along $C$. Then, in the tangent bundle of order two $T^2M_n$ over the Riemannian manifold $M_n$ with metric $^Dg$, we define a curve $\tilde{C}$ by

$$x^i = x^i(t), \quad x^j = y^j(t), \quad x^\pi = z^i(t).$$

We consider now differential equations of the geodesics of the tangent bundle of order two $T^2M_n$ with the metric $^Dg$. If $t$ is the arc length of the curve $x^A = x^A(t)$ in $T^2M_n$, equations of geodesic in $T^2M_n$ have the usual form

$$\frac{d^2x^A}{dt^2} = D_{\Gamma^A_{CB}} \frac{dx^C}{dt} \frac{dx^B}{dt}$$

(21)

with respect to the induced coordinates $(x^i, x^j, x^\pi) = (x^i, y^i, z^i)$ in $T^2M_n$.

We find it more convenient to refer equations (22) to the adapted frame $\{dx^0, dx^1, dx^2\}$.

Using (6), we write

$$\begin{align*}
\theta^i &= dx_i \\
\theta^j &= \delta y_j = \Gamma^k_i dx_k + dy_i \\
\theta^\pi &= \delta z_i = \tilde{A}^k_i dx_k + 2\Gamma^k_i dy_k + dz_i
\end{align*}$$

and put

$$\begin{align*}
\frac{\theta^i}{dt} &= \frac{dx_i}{dt} \\
\frac{\theta^j}{dt} &= \frac{\delta y_j}{dt} = \frac{\theta^i}{dt} + \frac{\theta^\pi}{dt} \\
\frac{\theta^\pi}{dt} &= \frac{\delta z_i}{dt}
\end{align*}$$

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along a curve $x^A = x^A(t)$, i.e., $x^i = x^i(t)$, $x^i = x^i(t)$, $x^i = z^i(t)$ in $T^2M_n$.

If we write, therefore, down the form equivalent to (22), namely,

$$
\frac{d}{dt}\left(\frac{d\theta^\alpha}{dt}\right) + D\Gamma^\alpha_{\beta\gamma} \frac{d\theta^\beta}{dt} \frac{d\theta^\gamma}{dt} = 0
$$

with respect to the adapted frame and take account of (13), then the curve $x^A = x^A(t)$ in $T^2M_n$ with the metric $Dg$ is a geodesic in $T^2M_n$ if and only if

$$
\begin{cases}
\frac{\delta^2 z^i}{dt^2} + y^h R^i_{hjk} \frac{\delta y^j}{dt} \frac{dx^k}{dt} + y^h R^i_{hjk} \frac{\delta z^j}{dt} \frac{dx^k}{dt} = 0 \\
\frac{\delta^2 y^i}{dt^2} - \Gamma^i_{jk} \left( \frac{dx^j}{dt} \frac{dx^k}{dt} - y^h R^i_{hjk} \frac{\delta y^j}{dt} \frac{dx^k}{dt} - y^h R^i_{hjk} \frac{\delta z^j}{dt} \frac{dx^k}{dt} = 0 \\
\frac{\delta^2 y^j}{dt^2} - A^i \Gamma^j_{ik} \frac{dx^i}{dt} - 2\Gamma^i_{jk} \frac{\delta y^j}{dt} \frac{dx^k}{dt} - y^h R^i_{hjk} A^i \frac{\delta y^j}{dt} \frac{dx^k}{dt} = 0 \\
- y^h R^i_{hjk} A^i \frac{\delta z^j}{dt} \frac{dx^k}{dt} = 0
\end{cases}
$$

with

$$
\frac{\delta^2 z^i}{dt^2} = \frac{d}{dt}\left(\frac{\delta z^i}{dt}\right) + \Gamma^i_{\alpha\beta} \frac{\delta z^\alpha}{dt} \frac{dx^\beta}{dt}
$$

If a curve satisfying (22) lies on the a fiber given by $x^i = \text{const}$, $y^i = \text{const}$ in $TM$, then (22) reduce to

$$
\frac{d^2 z^i}{dt^2} = 0
$$

so that

$$
z^i = a^i t + b^i,
$$

$a^i$ and $b^i$ being constant. Thus, we have the following theorem.

**Theorem 7** If the geodesic $x^i = x^i(t)$, $y^i = y^i(t)$ and $z = z^i(t)$ lies in fiber of $T^2M_n$ with the metric $Dg$, the geodesic is expressed by linear equations $x^i = c^i$, $y^i = d^i$ and $z^i = a^i t + b^i$ with induced coordinates $(x^i, y^i, z^i)$, where $a^i, b^i, c^i$ and $d^i$ are constant.
If a curve satisfying (22) lies on the fiber given by \( x^i = \text{const} \), then (22) reduce to

\[
egin{align*}
\frac{d^2 y^k}{dt^2} &= \frac{d^2 y^i}{dt^2} = 0 \quad \text{(i)} \\
\frac{d^2 z^i}{dt^2} - 2\Gamma^h_{jk} \Gamma^i_h \frac{\delta z^j}{dt} \frac{\delta y^k}{dt} &= 0 \quad \text{(ii)}
\end{align*}
\]

so that from (26,i) \( y^i = a^i t + b^i \), \( a^i \) and \( b^i \) going constant.

From (23) we have

\[
\frac{d^2 z^i}{dt^2} = 2a^r a^l \Gamma^i_{rj} + \frac{d^2 z_j}{dt^2}
\]

and (26,ii) become

\[
\frac{d^2 z^i}{dt^2} - 2\Gamma^h_{jk} \Gamma^i_h \frac{\delta z^j}{dt} \frac{\delta y^k}{dt} = 2a^r a^l \Gamma^i_{rj} + \frac{d^2 z_j}{dt^2} - 2\Gamma^h_{jk} \Gamma^i_h (2a^l a^k + \frac{d z_j}{dt} a^k)
\]

\[
= \frac{d^2 z_j}{dt^2} - 2\Gamma^h_{jk} \Gamma^i_h a^k \frac{d z_j}{dt} - 4\Gamma^h_{jk} \Gamma^i_h \Gamma^j_l a^l a^k;
\]

then (26,ii) is given by

\[
\frac{d^2 z_j}{dt^2} - 2\Gamma^h_{jk} \Gamma^i_h a^k \frac{d z_j}{dt} + 4\Gamma^h_{jk} \Gamma^i_l \Gamma^j_l a^l a^k. \tag{27}
\]

Thus, we have the following theorem.

**Theorem 8** If the geodesic \( x^i = x^i(t) \), \( y^i = y^i(t) \) and \( z = z^i(t) \) lies in fiber of \( T^2 M_n \) with the metric \( Dg \), the geodesic is expressed by linear equations \( x^i = c^i \), \( y^i = a^i t + b^i \) and \( z^i \) solution of differential system (27) with induced coordinates \( (x^i, y^i, z^i) \), where \( a^i, b^i, c^i \) and \( d^i \) are constant.

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