Symmetries of Canal Surfaces and Dupin Cyclides

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Abstract

We develop a characterization for the existence of symmetries of canal surfaces defined by a rational spine curve and rational radius function. This characterization leads to a method for constructing rational canal surfaces with prescribed symmetries, and it inspires an algorithm for computing the symmetries of such canal surfaces. For Dupin cyclides in canonical form, we apply the characterization to derive an intrinsic description of their symmetries and symmetry groups, which gives rise to a method for computing the symmetries of a Dupin cyclide not necessarily in canonical form.

1. Introduction

Canal surfaces are envelopes of 1-parameter families of spheres, whose radii are parametrized by a radius function and whose centers form a parametric spine curve. These surfaces have been studied extensively during the last 20 years \([9,10,12,13,18,19,22,27]\) because of their applications in Computer Aided Geometric Design in general and, in particular, in operations like joining and blending \([12,13,19]\).

Dupin cyclides form a special family of rational canal surfaces, and they have also received much attention since their introduction \([15]\). In this case, the source of interest is two-fold: from a theoretical point of view, due to the fact that they are the only surfaces that are canal surfaces in more than one way \([20]\) and their remarkable geometric properties \([8]\); from a practical point of view, because of their applicability in joining and blending \([16]\).

Canal surfaces can be defined by a spine curve and a radius function, and the Dupin cyclides are the canal surfaces for which these entities are not unique. Under the assumption that the spine curve and radius function are rational, we study in this paper the symmetries of the surface and provide a simple algorithm for finding them. Moreover, our results also provide a recipe for constructing symmetric canal surfaces.

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In order to conduct our study, we make use of ideas regarding symmetries and similarities of rational curves developed by two of the authors [3–7]. In particular, we relate the symmetries of the canal surface to symmetries of the spine curves, and to isometries (if any) between different spine curves, in the case of Dupin cyclides.

When applied to Dupin cyclides, our results provide a classification of the symmetries of these surfaces, together with their symmetry groups. Certainly, some results regarding the symmetries of Dupin cyclides are already known. For instance, it is well known — and easy to see from the implicit equations of Dupin cyclides in canonical form — that these surfaces are symmetric with respect to the planes containing the spine curves, and therefore with respect to the line intersecting these planes as well. Nevertheless, it is not always easy to derive all the symmetries of a surface from its implicit equation. As an illustration of our characterisation of symmetric canal surfaces and Dupin cyclides, we prove that in the generic situation the aforementioned symmetries are the only symmetries exhibited by Dupin cyclides. Furthermore, we identify the special cases where extra symmetries appear and then determine these symmetries, thus leading to a complete classification.

The computations in this paper have been implemented in a Sage worksheet, which can be tried out online following a link on the website of the last author [21].

2. Preliminaries on canal surfaces, Dupin cyclides and symmetries

2.1. Background on canal surfaces

A canal surface $S = S_{c,r} \subset \mathbb{R}^3$ is the envelope of a 1-parameter family $\Sigma = \Sigma_{c,r}$ of spheres, centered at a spine curve $c$ and with radius function $r$. The defining equations for $S$ are

\begin{align*}
\Sigma_{c,r}(t) : \|x - c(t)\|^2 - r(t)^2 &= 0, \\
\dot{\Sigma}_{c,r}(t) : \langle x - c(t), \dot{c}(t) \rangle + r(t)\dot{r}(t) &= 0.
\end{align*}

For fixed $t$, the first equation is a sphere and the second is a plane, intersecting in the characteristic circle $k(t) = k_{c,r}(t) := \Sigma_{c,r}(t) \cap \dot{\Sigma}_{c,r}(t)$. In order to have a well-defined envelope surface, we require that $\|\dot{c}(t)\|^2 > \dot{r}(t)^2$ holds for all parameters $t$ for which $c(t)$ and $r(t)$ are defined. For this statement and other general results on canal surfaces that we recall in this section, we refer the interested reader to [12][13][19][22].

The family $\Sigma$ of spheres can be identified with a curve in the 4-dimensional Minkowski space $\mathbb{R}^{3,1}$, where the point $(c(t); r(t))$ corresponds to a sphere $\Sigma(t)$ with center $c(t)$ and radius $r(t)$. The sign of the radius gives us the orientation of the sphere (towards the centre when $r(t) > 0$, and outwards when $r(t) < 0$), which is inherited by the canal surface. Thus an oriented canal surface corresponds to a curve $(c; r)$ in $\mathbb{R}^{3,1}$. 
The canal surface $S$ admits parametrizations of the form

$$F(t, s) = F_{c, r}(t, s) := c(t) + r(t)N(t, s).$$

Note that while (1) and (2) remain valid upon replacing $r$ by $-r$, the change in sign of the radius results in a change of orientation of the canal surface, i.e., a reversal of the direction of the unit normal vector field described by $N(t, s)$. Thus the sign of the second term of the parametrization in (3) remains positive.

The isoparametric curve $F_t(s)$ parametrizes the characteristic circle $k(t)$. Correspondingly, by (1) and (2), $N_t(s)$ traces a circle on the unit sphere. More precisely, for a fixed parameter $t$ with nonzero radius $r(t)$, Equations (1) and (2) become

$$\Sigma(t) : \|N(t, s)\|^2 = 1,$$

$$\Sigma(t) : \langle N(t, s), \dot{c}(t) \rangle + \dot{r}(t) = 0.$$

In fact, $N(t, s)$ describes the unit normal vector field of $S$. Also, for each characteristic circle, the normal lines to $S$ along the circle intersect at one point $c(t)$ belonging to the spine curve. Moreover, the derivative $\dot{c}(t)$ at this point is normal to the plane containing the characteristic circle.

If $r$ is a constant, then we have a special type of canal surface, called a pipe surface. Furthermore, if $c$ is a line, then $S$ is a surface of revolution.

In this paper we assume that $c$ and $r$ are real and rational and known. Additionally, we assume that $c$ is proper, i.e., birational or equivalently injective except for perhaps finitely many parameter values. Since $c$ and $r$ are rational, finding a rational parametrization of type (3) reduces to finding a rational parametrization of $N$. This was first accomplished in [22], and minimal degree rational parametrizations were obtained in [19]. As a result, the surface $S$ is also rational, and therefore irreducible. Observe that the spine curve of $S$ is irreducible as well, since it is rationally parametrized by $c$.

2.2. Background on Dupin cyclides

One can wonder if $c$ and $r$ are unique for a given canal surface $S$, or if there are surfaces that are canal surfaces in at least two different ways. This question was answered by Maxwell [20], who showed that Dupin cyclides are the only canal surfaces with the latter property. In fact, these surfaces are the envelope of exactly two different 1-parameter families of spheres with distinct spine curves $c_1, c_2$ and radius functions $r_1, r_2$. Dupin cyclides, now called Dupin cyclides, as surfaces whose lines of curvatures are circles. Since then, a variety of alternative definitions has arisen [8], which is the underlying reason for their value in a variety of applications.

Maxwell [20] also showed that these spine curves must be conics lying in perpendicular planes and passing through each other’s foci, yielding three different cases corresponding to the nature of the spine curves. For each of these cases one can provide [14 §23.2] a canonical description of the spine curves, radius functions and implicit forms, shown in Table 1. This form depends on certain
parameters \(a, b, c, f, g\) satisfying \(f^2 = a^2 - b^2\). Notice that \(a, c \neq 0\) in Type I, \(a, b \neq 0\) and \(f > 0\) in Type II, and \(g \neq 0\) in Type III. Note also that Type II degenerates to Type I in the limit \(f \to 0\). We say that a Dupin cyclide is in canonical form, if it is the zero set of \(F^I, F^II\) or \(F^III\) in Table 1 for certain parameters \(a, b, c, f, g\) satisfying the above constraints.

As Dupin cyclides have rational spine curves and radius functions, eliminating the parameter \(t\) from \([1] \) and \([2] \) yields a description as the zeroset of a polynomial. The implicit forms in Table 1 show that the Dupin cyclides of Types I and II are quartics, while the Dupin cyclides of Type III are cubics. Furthermore, the Dupin cyclides of Type I are tori, and these surfaces are surfaces of revolution, since the characteristic circles are rotationally invariant about the axis traced out by \(c_2\).

For the Dupin cyclides in this paper we will assume that the spine curves \(c_1, c_2\) and the radius functions \(r_1, r_2\) are known, yielding alternative representations \(\mathcal{S}_{c_1, r_1}\) and \(\mathcal{S}_{c_2, r_2}\).

2.3. Symmetries of curves and surfaces

An (affine) isometry \(f\) of \(\mathbb{R}^n\) takes the form \(f(x) = Qx + b\), with \(b \in \mathbb{R}^n\) a vector and \(Q \in \mathbb{R}^{n \times n}\) an orthogonal matrix, i.e., \(QQ^T = I\). If \(f\) leaves a curve \(\mathcal{C} \subset \mathbb{R}^3\) (respectively a surface \(\mathcal{S} \subset \mathbb{R}^3\)) invariant, in the sense that \(f(\mathcal{C}) = \mathcal{C}\) (respectively \(f(\mathcal{S}) = \mathcal{S}\)), then \(f\) is called a symmetry of \(\mathcal{C}\) (respectively \(\mathcal{S}\)).
identity map \( f = \text{id}_{\mathbb{R}^n} \) is referred to as the trivial isometry/symmetry. A curve or surface is called symmetric if it has a nontrivial symmetry. For a description of the types of nontrivial symmetries of Euclidean space, see [11] and [5, §2].

It is well known that the birational functions on the line are the Möbius transformations [24], i.e., rational functions
\[
\varphi : \mathbb{R} \to \mathbb{R}, \quad \varphi(t) = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \alpha \delta - \beta \gamma \neq 0.
\]
(9)
The identity map \( \varphi = \text{id}_{\mathbb{R}} \) is referred to as the trivial Möbius transformation.

The following result is provided for the case \( n = 2, 3 \) in [6], but the proof extends to any integer \( n \geq 1 \).

**Theorem 1.** Let \( c : \mathbb{R} \to \mathbb{C} \subset \mathbb{R}^n \) be a properly parametrized curve. The curve \( C \) is symmetric if and only if there exists a nontrivial isometry \( f \) and nontrivial Möbius transformation \( \varphi \) for which we have a commutative diagram
\[
\begin{array}{ccc}
C & \xrightarrow{f} & C \\
\uparrow & & \uparrow \\
\mathbb{R} & \xrightarrow{\varphi} & \mathbb{R}
\end{array}
\]
(10)
Moreover, for each isometry \( f \) there exists a unique Möbius transformation \( \varphi \) that makes this diagram commute.

We will say that the Möbius transformation in Theorem 1 is associated with the isometry \( f \). Theorem 1 is used in [5] to reduce the computation of the symmetries of rational space curves to the computation of their associated Möbius transformations, using the classical differential invariants (curvature and torsion) of space curves.

Suppose \( C \) is a rational space curve that is non-linear, i.e., not a straight line. Then, for all but finitely many parameters \( t \), the unit tangent, principal normal, and bi-normal vectors of \( c \) at \( t \),
\[
t_c = t_c(t) := \frac{\dot{c}(t)}{\|\dot{c}(t)\|}, \quad n_c = n_c(t) := \frac{\dot{t}_c(t)}{\|\dot{t}_c(t)\|}, \quad b_c = b_c(t) := t_c(t) \times n_c(t)
\]
are well defined and together form the Frenet frame \((t_c, n_c, b_c)\) of the curve at \( t \).

**Lemma 2.** Let \( f(x) = Qx + b \) be an isometry of \( \mathbb{R}^3 \) and \( c \) a parametrized curve with Frenet frame \((t_c, n_c, b_c)\). Then the curve \( f \circ c \) has Frenet frame
\[
(t_{f \circ c}, n_{f \circ c}, b_{f \circ c}) = (Qt_c, Qn_c, \det(Q)Qb_c).
\]
Proof. One has
\[
\begin{align*}
t_{foc} &= \frac{df \circ c}{dt} \left\| \frac{df \circ c}{dt} \right\| = Q \frac{dc}{dt} \left\| \frac{dc}{dt} \right\| = Qt_c, \\
n_{foc} &= \frac{dt_{foc}}{dt} \left\| \frac{dt_{foc}}{dt} \right\| = Q \frac{dt_c}{dt} \left\| \frac{dt_c}{dt} \right\| = Qn_c, \\
b_{foc} &= t_{foc} \times n_{foc} = (Qt_c) \times (Qn_c) = \det(Q)Q(t_c \times n_c) = \det(Q)Qb_c,
\end{align*}
\]
where we used the identity
\[
(Mx) \times (My) = \det M \cdot M^{-T}(x \times y),
\]
which holds for any invertible matrix $M \in \mathbb{R}^{3 \times 3}$ and vectors $x,y \in \mathbb{R}^3$. \hfill \square

The following result on real rational functions is necessary for establishing the characterization of symmetries of canal surfaces stated in Theorem 7. Since we were unable to find an appropriate reference, we include a short proof.

**Lemma 3.** Let $g,h$ be two real rational functions. If $|g(t)| = |h(t)|$ holds for every $t \in \mathbb{R}$ for which both sides are defined, then either $g = h$ or $g = -h$.

**Proof.** Since $g,h$ are real rational functions, there exists an open interval $I \subset \mathbb{R}$ where $g,h$ are both defined and have constant sign. Therefore, for any $t \in I$, $|g(t)| = |h(t)|$ is defined, and $g(t) - h(t) = 0$ or $g(t) + h(t) = 0$ holds identically for $t \in I$. Assume that $g(t) - h(t) = 0$ holds identically for $t \in I$; the second possibility is analogous. Since the restriction of $g - h$ to $I$ is rational and well defined, it is analytic on its domain $I$. Therefore, if this restriction is zero in $I$, the Identity Theorem implies that $g = h$. \hfill \square

3. Symmetries of canal surfaces with a unique spine curve: characterization and algorithm

In this section we characterize the existence of symmetries of a canal surface $S$ that is not a Dupin cyclide. As an application of this characterization, we develop an algorithm for computing these symmetries.

If $S$ has a linear spine curve $c$, then it is a surface of revolution. Surfaces of revolution can be detected by using the methods in [1,26], and their symmetries essentially follow from those of the directrix curve [2, §2.2.4]. Hence from now on we will assume that $c$ is non-linear.

In order to solve our problem (in this and the next section) we recall that whenever $c$ is non-linear, the Frenet frame of $c$ is well defined and forms the basis of the following convenient (but in general nonrational) parametrization of the surface normals [12, Equation (3.12)],
\[
N_{c,r}(t,s) = \frac{\dot{r}(t)}{||\dot{e}(t)||} t_c + \sqrt{1 - \left( \frac{\dot{r}(t)}{||\dot{e}(t)||} \right)^2 \left( \frac{1 - s^2}{1 + s^2} n_c + \frac{2s}{1 + s^2} b_c \right)}.
\]
3.1. General lemmas

In this section we consider the effect of a symmetry $f$ of the canal surface $S_{c,r}$ on its spine curve $c$. Allowing for the case that $S_{c,r}$ is a Dupin cyclide, these results will be applied both in this section and in Section 4. In particular, the following lemma shows that $\tilde{c} := f \circ c$ is also a spine curve of $S$, and the subsequent lemmas describe its relation to $c$.

**Lemma 4.** Let $f$ be a symmetry of $S_{c,r}$. Then $S_{f \circ c, \tilde{r}} = S_{c,r}$, where either $\tilde{r} = r$ or $\tilde{r} = -r$. In particular, $f \circ c$ is also a spine curve of $S_{c,r}$.

**Proof.** Let $\Sigma_{c,r}$ be the 1-parameter family of spheres (1) corresponding to the pair $(c, r)$. For every $t$, the isometry $f$ maps $\Sigma_{c,r}(t) \rightarrow \Sigma_{f \circ c, \tilde{r}}(t)$, where $\tilde{r}^2 = r^2$, so by Lemma 3 we have either $\tilde{r} = r$ or $\tilde{r} = -r$. The envelope of the spheres $\Sigma_{f \circ c, \tilde{r}}(t)$ defines a canal surface $S_{f \circ c, \tilde{r}}$, and $f$ maps characteristic circles of $S_{c,r}$ to characteristic circles of $S_{f \circ c, \tilde{r}}$.

Moreover, since $f$ is a symmetry of $S_{c,r}$, the latter characteristic circles are contained in $S_{c,r}$. Since $S_{c,r}$ is irreducible, it follows that $S_{c,r} = S_{f \circ c, \tilde{r}}$. In particular $f$ maps each spine curve of $S_{c,r}$ to a spine curve of $S_{c,r}$.

**Lemma 5.** The spine curves $c$ and $\tilde{c}$ have identical speed.

**Proof.** Since $Q$ is orthogonal, for case $i = 1, 2$,

$$\left\| \dot{\tilde{c}}(t) \right\| = \left\| \frac{d}{dt} (f \circ c)(t) \right\| = \left\| Q \dot{c}(t) \right\| = \left\| \dot{c}(t) \right\|. \quad (13)$$

**Lemma 6.** The spine curves $c$ and $\tilde{c}$, together with the radius function $r$, have surface normal parametrizations (12) related by

$$QN_{c,r}(t, s) = N_{\tilde{c}, \tilde{r}}(t, \det(Q)s). \quad (14)$$

**Proof.** The result follows after multiplying (12) by $Q$, using Lemmas 2 and 5.

3.2. Characterization

Now we can state the characterization theorem for the existence of symmetries of canal surfaces with a single spine curve.

**Theorem 7.** Let $S_{c,r}$ be a canal surface, not a Dupin cyclide, with non-linear spine curve $c$. The isometry $f(x) = Qx + b$ is a symmetry of $S_{c,r}$ if and only if there exists a Möbius transformation $\varphi$ such that

C1: the spine curve satisfies $f \circ c = c \circ \varphi$;
C2: the radius function satisfies \( r^2 = (r \circ \varphi)^2 \).

Proof. “\( \Rightarrow \)” By Lemma 4, \( f \) must be a symmetry of the spine curve \( c \). By Theorem 1, this is equivalent to the existence of a Möbius transformation \( \varphi \) for which \( f \circ c = c \circ \varphi \), establishing C1. Using, in order, Condition C1, Lemma 4 with \( \tilde{r} = \pm r \), and that \( \varphi \) is a birational map on the real line, one obtains
\[
S_{c \circ \varphi, \tilde{r}} = S_{f \circ c, \tilde{r}} = S_{c, r} = S_{c \circ \varphi, r \circ \varphi}
\]
and deduces \( r^2 = \tilde{r}^2 = (r \circ \varphi)^2 \), establishing C2.

“\( \Leftarrow \)” Let \( F_{c, r}(t, s) \) be the parametrization (3) of \( S_{c, r} \) with normals \( N_{c, r}(t, s) \) as in (12). Since the radius function \( r \) and \( \tilde{r} := r \circ \varphi \) are real rational functions satisfying \( r^2 = \tilde{r}^2 \), Lemma 3 implies \( r = \pm \tilde{r} \). As a change of sign of the radius function results in a change of orientation of the canal surface and leaves the geometric shape unchanged, we may safely assume that \( r = \tilde{r} \). Then using Lemma 6 with \( \tilde{c} := c \circ \varphi = f \circ c \), the isometry \( f \) maps any point \( F_{c, r}(t, s) \) on \( S_{c, r} \) to
\[
f \circ F_{c, r}(t, s) = (f \circ c)(t) + r(t)Q N_{c, r}(t, s) = \tilde{c}(t) + \tilde{r}(t) N_{c, \tilde{r}}(t, \det(Q) s) = F_{\tilde{c}, \tilde{r}}(t, \det(Q) s).
\]
It follows that \( f \circ F_{c, r}(t, s) \) is a point on the canal surface with spine curve \( \tilde{c} \), radius function \( \tilde{r} \) and corresponding normal \( N_{\tilde{c}, \tilde{r}} \) as in (12), which is just a reparametrization of \( S_{c, r} \). Therefore \( f(S_{c, r}) = S_{c, r} \), and \( f \) is a symmetry of \( S_{c, r} \).

In the specific case of a pipe surface, Theorem 7 takes the following form.

Corollary 8. Let \( S \) be a pipe surface, not a Dupin cyclide, with a non-linear spine curve \( c \). An isometry is a symmetry of \( S \) if and only if it is a symmetry of \( c \).

Theorem 7 can be applied to construct canal surfaces with prescribed symmetries, by choosing a symmetric spine curve (Condition C1) and a radius function that respects the symmetries of the spine curve (Condition C2). This is illustrated in the following example.

Example 1. Suppose we would like to construct a canal surface for which the half-turn \( f \) around the \( y \)-axis is a symmetry. Clearly the twisted cubic curve
\[
c(t) = (t, t^2, t^3)
\]
has such a symmetry. Moreover, Condition C1,
\[
(\ -t, t^2, -t^3) \ = \ f \circ c(t) \ = \ c \circ \varphi(t) \ = \ (\varphi(t), \varphi^2(t), \varphi^3(t)),
\]
is only satisfied by \( \varphi(t) = -t \). To satisfy Condition C2, we require a rational radius function \( r \) satisfying \( r^2(t) = r^2(-t) \). By Lemma 3, this happens precisely when \( r \) is an odd or even rational function. The choice \( r(t) = t/2 \) yields the canal surface shown in Figure 1 left.
3.3. Algorithm

Let us cast the conditions in Theorem 7 into a computer algebra setting. Each Möbius transformation \( \varphi \) can be described by introducing an auxiliary variable \( u \) satisfying \( u - \varphi(t) = 0 \). Clearing denominators, we arrive at the M"{o}bius-like polynomial \( F(t, u) := u(\gamma t + \delta) - (\alpha t + \beta) \), which is zero precisely when \( u = \varphi(t) \). Note that the Möbius-like polynomials are the irreducible bilinear polynomials, since \( \alpha \delta - \beta \gamma \neq 0 \). The Möbius-like polynomial \( F(t, u) \) is called trivial, as the associated Möbius transformation is the identity.

Under the condition \( u = \varphi(t) \), Condition C2 of Theorem 7 takes the form \( r^2(t) - r^2(\varphi(t)) = 0 \). Writing \( r(t) = A(t)/B(t) \), with \( A(t), B(t) \) coprime, and clearing denominators yields the polynomial condition

\[
R(t, u) = A^2(t)B^2(u) - A^2(u)B^2(t) = 0. 
\]  

(16)

The following result shows how Condition C2 in Theorem 7 can be tested by checking for Möbius-like factors of \( R \).

**Proposition 9.** Let \( r = A/B \) be a real rational function, with corresponding bivariate polynomial \( R \) as in (16), and let \( \varphi \) be a Möbius transformation with corresponding Möbius-like polynomial \( F \). Then Condition C2 holds precisely when \( F \) divides \( R \).

**Proof.** The equation \( r^2(t) - r^2(\varphi(t)) = 0 \) holds identically iff \( R(t, \varphi(t)) = 0 \) holds identically. In that case, the zero set of \( R \) contains the graph of \( \varphi \), which is equal to the zero set of \( F \). Since \( F \) is irreducible, it follows that this happens precisely when \( F \) divides \( R \).

Combining Theorem 7 and Proposition 9 we observe that we can determine the symmetries of a canal surface by:

(i) testing Condition C2 on the radius function, by finding the Möbius-like factors \( F \) of \( R \) and corresponding tentative Möbius transformations \( \varphi \);
Algorithm 1 SymCanal

**Require:** A real, rational, properly parametrized, non-linear spine curve \( c \), and a real, rational radius function \( r \), defining a parametrization \( S \) of a canal surface \( S \) that is not a Dupin cyclide.

**Ensure:** The symmetries of \( S \).

1: if \( r \) is constant then
2: return the symmetries of the spine curve by using the algorithm in [5]
3: else
4: find the Möbius-like factors of \( R \) and associated Möbius transformations \( \varphi \)
5: for each Möbius transformation \( \varphi \) do
6: find the isometry \( f \) associated with \( \varphi \) using the method in [5, §4]
7: if \( f \circ c = c \circ \varphi \), return the isometry \( f \)
8: end for
9: end if

(ii) testing Condition C1 on the spine curve for each such \( \varphi \), by determining whether \( \varphi \) corresponds to a symmetry \( f \) of the spine curve \( c \).

In step (i), since \( \varphi \) does not necessarily have rational coefficients, we need to compute the factors of \( R(t,u) \) with real algebraic coefficients. This can be done for instance by using the `AFactor` command in Maple 18, which works fine for moderate inputs. An alternative is to use the method of [5, §3.2] to find only the Möbius-like factors of \( R(t,u) \), instead of a complete factorization.

In step (ii), one can apply the method in [3, §4] and [2] to determine an isometry \( f \) associated with \( \varphi \). Then one verifies Condition C1 by direct substitution.

Thus we arrive at Algorithm SymCanal for computing the symmetries of a canal surface with a unique non-linear spine curve. Notice that in this algorithm we distinguish pipe surfaces as a special case, since in that case \( R \) is identically zero.

**Example 2.** Consider again the canal surface from Example 1 shown in Figure 1 to the left, with twisted cubic spine curve (15) and linear radius function \( r(t) = t/2 \). Let us apply Algorithm SymCanal to determine whether it has other symmetries besides the prescribed symmetry.

One obtains \( R(t,u) = 4(t-u)(t+u) \), whose Möbius-like factors \( F_{\pm}(t,u) = t \mp u \) correspond to Möbius transformations \( \varphi_{\pm}(t) := \pm t \). Now assume that Condition C1 holds with \( \varphi = \varphi_{\pm} \), i.e.,

\[ Qc(t) + b = f \circ c(t) = c \circ \varphi_{\pm}(t) = c(\pm t). \] (17)

Substituting \( t = 0 \) we immediately determine \( b = 0 \). Taking, for \( n = 1, 2, 3 \), the \( n \)-th derivative of (17) and evaluating at \( t = 0 \), we get

\[ Q \cdot [c'(0), c''(0), c'''(0)] = [-c'(0), c''(0), -c'''(0)]. \]
Thus we determine $Q$ from its action on $\{c'(0), c''(0), c'''(0)\}$, and we obtain

$$f_{\pm}(x) = Q_{\pm}x, \quad Q_{\pm} = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix},$$

corresponding to the trivial symmetry $f_+$ and the half-turn $f_-$ around the $y$-axis. In particular

$$f_- \circ c(t) = (-t, t^2, -t^3) = c \circ \varphi_-(t), \quad r^2(t) = t^2/4 = r^2(\varphi_-(t)),$$

certifying that conditions C1 and C2 hold, and therefore that $f_-$ is the unique nontrivial symmetry of the canal surface defined by $(c, r)$.

**Example 3.** Consider the crunode spine curve and radius function

$$c(t) = \left(\frac{t}{t^4 + 1}, \frac{t^2}{t^4 + 1}, \frac{t^3}{t^4 + 1}\right), \quad r(t) = \frac{t^2}{t^4 + 1},$$

with corresponding canal surface shown in Figure 1, right. Then

$$R(t, u) = (u - t)(u + t)(ut - 1)(ut + 1)(u^2t^2 + 1)(u^2 + t^2),$$

whose Möbius-like factors correspond to Möbius transformations

$$\varphi_1(t) = t, \quad \varphi_2(t) = -t, \quad \varphi_3(t) = 1/t, \quad \varphi_4(t) = -1/t.$$ 

Suppose Condition C1 holds with $\varphi = \varphi_1, \varphi_2$, i.e.,

$$Qc(t) + b = f \circ c(t) = c \circ \varphi_i(t) = c((-1)^{i+1}t), \quad i = 1, 2. \quad (18)$$

Substituting $t = 0$ we immediately determine $b = 0$. Taking, for $n = 1, 2, 3$, the $n$-th derivative of (18) and evaluating at $t = 0$, we again get $Q$ from its action on $\{c'(0), c''(0), c'''(0)\}$, and

$$f_{i}(x) = Q_{i}x, \quad Q_{i} = \begin{bmatrix} (-1)^{i+1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^{i+1} \end{bmatrix}, \quad i = 1, 2,$$

corresponding to the trivial symmetry and the half-turn around the $y$-axis. One directly verifies $f_{i} \circ c(t) = c \circ \varphi_i(t)$ for $i = 1, 2$, confirming that Condition C1 holds for $f_1, f_2$; so $f_1, f_2$ are symmetries of the canal surface defined by $(c, r)$.

Next suppose Condition C1 holds with $\varphi = \varphi_3, \varphi_4$. With the above procedure, we obtain isometries

$$f_{i}(x) = Q_{i}x, \quad Q_{i} = \begin{bmatrix} 0 & 0 & (-1)^{i+1} \\ 0 & 1 & 0 \\ (-1)^{i+1} & 0 & 0 \end{bmatrix}, \quad i = 3, 4,$$

which are reflections in the planes $x + (-1)^i z = 0$. Verifying $f_{i} \circ c(t) = c \circ \varphi_i(t)$ for $i = 3, 4$, we confirm that $f_3$ and $f_4$ are also symmetries of the canal surface defined by $(c, r)$. 

11
4. Symmetries of Dupin cyclides: characterization, classification and algorithm

In this section we consider the remaining case of a Dupin cyclide $S$ with two distinct spine curves $c_1, c_2$ and corresponding radius functions $r_1, r_2$. First we provide a characterization theorem for an isometry to be a symmetry of a Dupin cyclide. Based on this theorem, we provide a complete classification of the symmetries of the three types of Dupin cyclides, together with the symmetry group in each case. Finally, based on this classification, we present an algorithm for computing the symmetries of a Dupin cyclide represented by pairs $(c_i, r_i)$, with $i = 1, 2$, not necessarily in canonical form.

4.1. Characterization

Using Lemmas 4–6, we establish the following characterization theorem for the symmetries of Dupin cyclides.

**Theorem 10.** Let $S$ be a Dupin cyclide with non-linear spine curves $c_1, c_2$ and radius functions $r_1, r_2$. The isometry $f(x) = Qx + b$ is a symmetry of $S$ if and only if there exist Möbius transformations $\varphi_1, \varphi_2$ such that either

- **A1:** the spine curves satisfy $f \circ c_1 = c_1 \circ \varphi_1$ and $f \circ c_2 = c_2 \circ \varphi_2$;
- **A2:** the radius functions satisfy $r_1^2 = (r_1 \circ \varphi_1)^2$ and $r_2^2 = (r_2 \circ \varphi_2)^2$,

or

- **B1:** the spine curves satisfy $f \circ c_1 = c_2 \circ \varphi_1$ and $f \circ c_2 = c_1 \circ \varphi_2$;
- **B2:** the radius functions satisfy $r_1^2 = (r_2 \circ \varphi_1)^2$ and $r_2^2 = (r_1 \circ \varphi_2)^2$.

**Proof.** “$\Longrightarrow$”: By Lemma 4 for $i = 1, 2$ we have that $f \circ c_i$ must also be a spine curve of $S$. Suppose $f$ maps one of the spine curves of $S$, say $c_1$, to itself. Since $f$ is a bijection, it follows that $f$ also maps $c_2$ to itself. Thus $f$ is a symmetry of both $c_1$ and $c_2$, and by Theorem 1 this is equivalent to the existence of Möbius transformations $\varphi_1$ and $\varphi_2$ for which $f \circ c_i = c_i \circ \varphi_i$, $i = 1, 2$, establishing A1. Then, for $i = 1, 2$, A2 is established as in the implication “$\Longrightarrow$” of Theorem 7.

Now let $C_1, C_2$ be the curves defined by $c_1, c_2$ and suppose that $f$ maps $C_1$ to $C_2$. Then, since $f$ is a bijection, it also maps $C_2$ to $C_1$ by Lemma 4. In particular $f^2$ is a symmetry of both $C_1$ and $C_2$. By Theorem 2 in [5] and Theorem 9 in [4] there exist Möbius transformations $\varphi_1, \varphi_2$ such that $f \circ c_1 = c_2 \circ \varphi_1$ and $f \circ c_2 = c_1 \circ \varphi_2$, establishing B1. Using, in order, Condition B1, Lemma 4 with $r = r_1$ and $\tilde{r} = \tilde{r}_1 = \pm r_1$, that $(c_1, r_1)$ and $(c_2, r_2)$ both define $S$, and that $\varphi_1$ is a birational map on the real line, one obtains

$$S_{c_2 \circ \varphi_1, \tilde{r}_1} = S_{f \circ c_1, \tilde{r}_1} = S_{c_1, r_1} = S_{c_2, r_2} = S_{c_2 \circ \varphi_1, r_2 \circ \varphi_1},$$

and deduces $r_1^2 = \tilde{r}_1^2 = (r_2 \circ \varphi_1)^2$, establishing B2.

“$\Longleftarrow$”: Let $i, j \in \{1, 2\}$ and suppose $i = j$ (resp. $i \neq j$). Let $F_{c_i, r_i}(t, s)$ be the parametrization of $S$ with normals $N_{c_i, r_i}(t, s)$ as in [12]. Since the radius function $r_i$ and $\tilde{r}_j := r_j \circ \varphi_i$ are real rational functions satisfying
\( r_i^2 = \tilde{r}_j^2 \) by A2 (resp. B2), Lemma 4 implies \( r_i = \pm \tilde{r}_j \). As a change of sign of the radius function results in a change of orientation of the canal surface and leaves the geometric shape unchanged, we may safely assume that \( r_i = \tilde{r}_j \). Then using Lemma 6 with \( c = c_i, \tilde{c} = c_j := c_j \circ \varphi_1 = f \circ c_i, r = r_i = \tilde{r}_j \), we get \( QN_{e_i, r_i}(t, s) = N_{\tilde{e}_j, \tilde{r}_j}(t, \det(Q)s) \), and the proof proceeds as in the implication "\( \iff \)" of Theorem 7.

**Remark 1.** Notice that \( \varphi_1 \) is typically not equal to \( \varphi_2 \). By Theorem 11, Case A of Theorem 10 implies that \( f \) is a symmetry of each spine curve of \( S_t \); in particular, in this case \( f \) maps each spine curve to itself. Case B implies that the spine curves \( C_1 \) and \( C_2 \) are mapped to each other by an isometry \( f \) that is not a symmetry of either \( C_1 \) or \( C_2 \). Nevertheless, \( f^2 \) is a symmetry of both \( C_1 \) and \( C_2 \) with associated Möbius transformations \( \varphi_2 \circ \varphi_1 \) and \( \varphi_1 \circ \varphi_2 \).

Analogous to the method in Section 3.3, we can use Conditions A2 and B2 to compute the symmetries of the surface. Again introducing the auxiliary variable \( u \) satisfying \( u - \varphi(t) = 0 \), Conditions A2 and B2 of Theorem 10 take the form \( r_i^2(t) - r_j^2(u) = 0 \), with \((i, j) = (1, 1), (2, 2) \) for Condition A2 and \((i, j) = (1, 2), (2, 1) \) for Condition B2. Clearing denominators yields corresponding polynomial conditions

\[
R_{ij}(t, u) := A_i^2(t)B_j^2(u) - A_i^2(u)B_j^2(t) = 0, \quad i, j = 1, 2, \quad (19)
\]

where \( r_1 = A_1/B_1 \) and \( r_2 = A_2/B_2 \), with \((A_1, B_1) \) and \((A_2, B_2) \) pairs of coprime univariate polynomials. The following proposition, which has a proof analogous to that of Proposition 9, shows how the tentative Möbius transformations can be determined in the form of Möbius-like factors of the polynomials \( R_{ij} \).

**Proposition 11.** For \( i, j = 1, 2 \), let \( r_i = A_i/B_i \) be real rational functions, with corresponding bivariate polynomials \( R_{ij} \) as in (19), and let \( \varphi_1, \varphi_2 \) be Möbius transformations with corresponding Möbius-like polynomials \( F_1, F_2 \). Then

A2 holds precisely when \( F_1 \) divides \( R_{11} \) and \( F_2 \) divides \( R_{22} \);

B2 holds precisely when \( F_1 \) divides \( R_{12} \) and \( F_2 \) divides \( R_{21} \).

**Example 4.** In the proof of Theorem 14, it is derived that the Möbius transformations \( \varphi_\pm(t) := \pm t \) satisfy Condition B2 in Theorem 10 for a Dupin cyclide of Type III with \( c = 0 \). Let us determine the associated symmetries in Case B using the method in [5].

Let \( f(x) = Qx + b \), and suppose that \( f \circ c^{III}_1 = c^{III}_2 \circ \varphi_\pm \). Then

\[
Qf \begin{bmatrix} t^2 - \frac{1}{2} \\ 2t \\ 0 \end{bmatrix} + b = Qc^{III}_1(t) + b = f \circ c^{III}_1(t) = c^{III}_2(\pm t) = f \begin{bmatrix} \frac{1}{2} - t^2 \\ 0 \\ \pm 2t \end{bmatrix}. \quad (20)
\]

Differentiating once and twice, evaluating at \( t = 0 \), and taking the cross product,

\[
Q \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}, \quad Q \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad Q \begin{bmatrix} 0 \\ 0 \\ \pm \det Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (21)
\]
where we used the identity (11). Applying the rules of matrix block multiplication to (21), we obtain

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & \pm \det Q \\ 0 & \pm 1 & 0 \end{bmatrix}.$$  

Substituting this $Q$ and $t = 0$ in (20) yields $b = 0$, and it follows that $f(x, y, z) = (-x, \pm \det Qz, \pm y)$. Moreover, $f \circ c_{II}^I = c_{I}^{III} \circ \varphi_{\pm \det Q}$, so that Conditions B1 and B2 are satisfied for the Cases (i)–(l) in Table 2.

4.2. Classification

In this subsection we apply the preceding results and ideas to classify the symmetries of Dupin cyclides, providing the symmetry groups in each case. For this purpose, we assume each Dupin cyclide to be in canonical form, i.e., with spine curves $c_{\alpha 1}, c_{\alpha 2}$ and radius functions $r_{\alpha 1}, r_{\alpha 2}$ for one of the Types $\alpha = I, II$ or III as in Table 1, and corresponding implicit equation $F_{\alpha}$. The results are summarized in Tables 2 and 3.

4.2.1. Type I

Let us address first the symmetries of the Dupin cyclides of Type I, i.e., tori. This can be deduced from results on symmetries of surfaces of revolution [2, §2.2.4]. However, we will show that the results in this paper can also be used to easily derive these symmetries.

Even though one of the spine curves of the Dupin cyclide of Type I is a line, the following theorem shows that Theorem 10 extends to these surfaces as well and explicitly describes the symmetries and associated Möbius transformations.

**Theorem 12.** The isometry $f(x) = Qx + b$ is a symmetry of the Dupin cyclide $S$ of Type I in Table 1 if and only if there exist Möbius transformations $\varphi_1, \varphi_2$...
such that Conditions A1 and A2 in Theorem 10 hold. Moreover, with signs \( \varepsilon_1, \varepsilon_2 \in \{-1, 1\} \simeq \mathbb{Z}_2 \) and angle \( \theta \in [0, 2\pi) \simeq S^1 \), these Möbius transformations take the form

\[
\varphi_1(t) = -\begin{pmatrix} \cos(\theta/2)t + \sin(\theta/2) & \varepsilon_2 \\ \sin(\theta/2)t - \cos(\theta/2) & \varepsilon_2 \end{pmatrix}, \quad \varphi_2(t) = \varepsilon_1 t, \tag{22}
\]

with associated symmetries

\[
f(x) = Qx, \quad Q = \begin{bmatrix} \varepsilon_2 \cos \theta & -\varepsilon_2 \sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & \varepsilon_1 \end{bmatrix} \tag{23}
\]

forming a symmetry group isomorphic to \( \mathbb{Z}_2^2 \times S^1 \).

**Proof.** We first determine the isometries \( f \) and associated Möbius transformations \( \varphi_1, \varphi_2 \) satisfying Conditions A1, A2. Substituting \( r_1^2 \) into (19) yields

\[
R_{22}^1(u, t) = 4a(u + t)(u - t)(t^2u^2a + t^2u^2c - t^2c - u^2c - a + c).
\]

Since \( a, c \neq 0 \), the right-most factor in \( R_{22}^1 \) does not split, and the only Möbius-like factors are \( u - t \) and \( u + t \), corresponding to Möbius transformations \( \varphi_2 = \pm t \) satisfying Condition A2 by Proposition 11. The relation \( f \circ \varphi_2 = \varphi_2 \circ \varphi_2 \) implies that \( f \) satisfies \( f(0, 0, z) = (0, 0, \pm z) \) for any point \( (0, 0, z) \) on the line \( c_2 \), i.e.,

\[
f(x, y, z) = \begin{bmatrix} \hat{Q} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \pm 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \hat{Q}\hat{Q}^T = I.
\]

On the other hand, since \( r_1^2 \) is constant, Condition A2 does not determine (or even restrict) \( \varphi_1 \). However, it is well known that any orthogonal matrix \( \hat{Q} \in \mathbb{R}^{2,2} \) maps the circle to itself. This can be shown explicitly using the trigonometric reparameterization \( c_1^1 \simeq (\tan(\phi/2)) = a(\cos \phi, \sin \phi, 0) \) and representation

\[
\hat{Q} \in \left\{ \begin{bmatrix} \varepsilon_2 \cos \theta & -\varepsilon_2 \sin \theta \\ \sin \theta & \cos \theta \\ 0 & 0 \end{bmatrix} : \varepsilon_2 \in \{\pm 1\}, \theta \in [0, 2\pi) \right\}.
\]

Thus \( f \) necessarily takes the form (23) and corresponds to the reparameterizations \( \phi \mapsto \phi + \theta \) and \( \phi \mapsto \pi - \phi - \theta \), or equivalently to the Möbius transformations (22). A direct calculation shows that such \( \varphi_1, \varphi_2 \) and \( f \) satisfy Conditions A1 and A2.

\[ \text{“} \Rightarrow \text{”: Using in addition that the circle } c_1^1 \text{ and line } c_2^2 \text{ are not related by an isometry, this is established as in Theorem 10.} \]

\[ \text{“} \Leftarrow \text{”: Using the above explicit form of } f, \text{ a straightforward calculation shows that these indeed are symmetries of } S. \]

\[ \square \]

**Remark 2.** Notice that the matrix \( Q \) in (23) can be decomposed as a product

\[
\begin{bmatrix} \varepsilon_2 \cos \theta & -\varepsilon_2 \sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & \varepsilon_1 \end{bmatrix} = \begin{bmatrix} \varepsilon_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon_1 \end{bmatrix}.
\]
<table>
<thead>
<tr>
<th>Case A</th>
<th>((\varphi_1, \varphi_2))</th>
<th>(f(x, y, z))</th>
<th>description of the symmetry</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>((+t, +t))</td>
<td>((+x, +y, +z))</td>
<td>trivial symmetry</td>
<td>II, III</td>
</tr>
<tr>
<td>(b)</td>
<td>((+t, -t))</td>
<td>((+x, +y, -z))</td>
<td>reflection in the plane (\Pi_1)</td>
<td>II, III</td>
</tr>
<tr>
<td>(c)</td>
<td>((-t, +t))</td>
<td>((+x, -y, +z))</td>
<td>reflection in the plane (\Pi_2)</td>
<td>II, III</td>
</tr>
<tr>
<td>(d)</td>
<td>((-t, -t))</td>
<td>((+x, -y, -z))</td>
<td>half-turn about the line (\Pi_1 \cap \Pi_2)</td>
<td>II, III</td>
</tr>
<tr>
<td>(e)</td>
<td>((\frac{1}{2}t, -\frac{1}{2}t))</td>
<td>((-x, +y, +z))</td>
<td>reflection in the plane (\Pi_0)</td>
<td>II, III</td>
</tr>
<tr>
<td>(f)</td>
<td>((\frac{1}{2}t, +\frac{1}{2}t))</td>
<td>((-x, +y, -z))</td>
<td>half-turn about the line (\Pi_0 \cap \Pi_1)</td>
<td>II, III</td>
</tr>
<tr>
<td>(g)</td>
<td>((-\frac{1}{2}t, -\frac{1}{2}t))</td>
<td>((-x, -y, +z))</td>
<td>half-turn about the line (\Pi_0 \cap \Pi_2)</td>
<td>II, III</td>
</tr>
<tr>
<td>(h)</td>
<td>((\frac{1}{2}t, +\frac{1}{2}t))</td>
<td>((-x, -y, -z))</td>
<td>central inversion about (O)</td>
<td>II, III</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case B</th>
<th>((\varphi_1, \varphi_2))</th>
<th>(f(x, y, z))</th>
<th>description of the symmetry</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>((+t, +t))</td>
<td>((-x, +z, +y))</td>
<td>half-turn about the line (\Pi_0 \cap \Pi_3)</td>
<td>III, c = 0</td>
</tr>
<tr>
<td>(j)</td>
<td>((-t, -t))</td>
<td>((-x, -z, -y))</td>
<td>half-turn about the line (\Pi_0 \cap \Pi_4)</td>
<td>III, c = 0</td>
</tr>
<tr>
<td>(k)</td>
<td>((+t, -t))</td>
<td>((-x, -z, +y))</td>
<td>composition of a reflection in the plane (\Pi_0) and a quarter-turn about the line (\Pi_1 \cap \Pi_2)</td>
<td>III, c = 0</td>
</tr>
<tr>
<td>(l)</td>
<td>((-t, +t))</td>
<td>((-x, +z, -y))</td>
<td>composition of a reflection in the plane (\Pi_0) and a quarter-turn about the line (\Pi_1 \cap \Pi_2)</td>
<td>III, c = 0</td>
</tr>
</tbody>
</table>

Table 2: For the Types II and III of Dupin cyclides in Table 1, the table lists the discrete symmetries \(f\) and Möbius transformations \((\varphi_1, \varphi_2)\) associated with \(f\) via Case A and Case B in Theorem 10. Here \(\Pi_0 : x = 0\) is the plane through \(O := (0, 0, 0)\), the centroid of the ellipse and hyperbola and average of the foci (which are, as a set, also the vertices) of the parabolas, and perpendicular to the planes \(\Pi_1, \Pi_2\) containing the spine curves \(c_1, c_2\); \(\Pi_3 : z - y = 0, \Pi_4 : z + y = 0\) are the ‘midplanes’ of \(\Pi_1, \Pi_2\).

This shows that the symmetries of cyclides of Type I are the rotations about the line \(c^1\), together with the composition of these rotations with the reflection in the plane containing the circle \(c^1\) and/or with the reflection in any plane containing the line \(c^0\).

### 4.2.2. Type II

Now we apply Theorem 10 to derive the symmetries of Dupin cyclides of Type II. Every such Dupin cyclide has an ellipse and hyperbola as its spine curves, which are not related by an isometry. It follows that for these Dupin cyclides Case B in Theorem 10 cannot happen. We distinguish cases according to whether the parameter \(c\) vanishes; see Figure 3.

**Theorem 13.** For any Dupin cyclide \(S\) of Type II in Table 1, Conditions A1 and A2 in Theorem 10 are satisfied if and only if:

- \(c \neq 0\) and \((\varphi_1, \varphi_2)\) and \(f\) are given by Cases (a)–(d) in Table 2, forming a symmetry group isomorphic to \(\mathbb{Z}_2^2\), the Klein four group.
<table>
<thead>
<tr>
<th>Type</th>
<th>graphic</th>
<th>parameters</th>
<th>symmetries</th>
<th>group</th>
</tr>
</thead>
<tbody>
<tr>
<td>I: Quartic</td>
<td>![Graphic]</td>
<td>all</td>
<td>{(a),(b)} × \mathbb{Z}_2 × SS^1</td>
<td>\mathbb{Z}_2^2 × SS^1</td>
</tr>
<tr>
<td>II: Quartic</td>
<td>![Graphic]</td>
<td>(c \neq 0)</td>
<td>(a)–(d)</td>
<td>(\mathbb{Z}_2^2)</td>
</tr>
<tr>
<td></td>
<td>![Graphic]</td>
<td>(c = 0)</td>
<td>(a)–(h)</td>
<td>(\mathbb{Z}_2^3)</td>
</tr>
<tr>
<td>III: Cubic</td>
<td>![Graphic]</td>
<td>(c \neq 0)</td>
<td>(a)–(d)</td>
<td>(\mathbb{Z}_2^2)</td>
</tr>
<tr>
<td></td>
<td>![Graphic]</td>
<td>(c = 0)</td>
<td>(a)–(d), (i)–(l)</td>
<td>(D_4)</td>
</tr>
</tbody>
</table>

Table 3: The symmetries and symmetry groups for the Dupin cyclides of Types I, II, III in Table 1 with parameters \(a, b, c, f, g\). Here \(\mathbb{Z}_2 \times SS^1\) is identified with the subgroup of \(O(3)\) of rotations about the \(z\)-axis and reflections in planes containing the \(z\)-axis.

- \(c = 0\) and \((\varphi_1, \varphi_2)\) and \(f\) are given by Cases (a)–(h) in Table 2, forming a symmetry group isomorphic to \(\mathbb{Z}_2^3\), the elementary abelian group of order 8.

Proof. Inserting the radius functions \(r_{11}^{II}, r_{22}^{II}\) from Table 1 into (19) yields

\[
R_{11}^{II}(t, u) = -4f(u - t)(u + t)(cu^2t^2 + fu^2t^2 + cu^2 + ct^2 - f + c),
\]

\[
R_{22}^{II}(t, u) = +4a(u - t)(u + t)(au^2t^2 + cu^2t^2 - cu^2 - ct^2 - a + c),
\]

each of which has the Möbius-like factors \(F_1(t, u) = u - t\) and \(F_2(t, u) = u + t\). In each case the remaining factor has degree two, and whether it splits into two additional Möbius-like factors

\[
F_3(t, u) = u(\gamma t + \delta) - (\alpha t + \beta), \quad F_4(t, u) = u(\gamma' t + \delta') - (\alpha' t + \beta')
\]
depends on the parameters \(a, c, f\).

In particular for the polynomial \(R_{11}^{II}\), if the remaining factor satisfies

\[
cu^2t^2 + fu^2t^2 + cu^2 + ct^2 - f + c = u^2[(c + f)t^2 + c] + (ct^2 + c - f) = F_3(t, u)F_4(t, u),
\]

comparing the coefficients of \(u\) on each side yields

\[
(\gamma t + \delta)(\alpha' t + \beta') = -(\gamma' t + \delta')(\alpha t + \beta),
\]
implying that the Möbius transformations corresponding to \(F_3(t, u)\) and \(F_4(t, u)\) are opposite. Hence, after an appropriate scaling of \(F_3\) or \(F_4\), the remaining factor satisfies

\[
\left[(c + f)t^2 + c]\right] + (ct^2 + c - f) = u^2 \cdot (\gamma t + \delta)^2 - (\alpha t + \beta)^2.
\]

Comparing the coefficients of \(t\) and the coefficients of \(u^2t\) yields \(c = 0\), in which case

\[
R_{11}^{II}(t, u)_{c=0} = -4f^2(u - t)(u + t)(ut - 1)(ut + 1).
\]
A similar argument shows that the remaining factor of $R_{II}^{11}$ only factors when $c = 0$, in which case

$$R_{II}^{11}(t, u)|_{c=0} = 4a^2(u - t)(u + t)(ut - 1)(ut + 1).$$

Thus $R_{II}^{11}$ and $R_{II}^{22}$ each determine tentative Möbius transformations $\varphi_1(t) = t, \varphi_2(t) = -t$, and two additional Möbius transformations $\varphi_3(t) = 1/t, \varphi_4(t) = -1/t$ if and only if $c = 0$.

When $c \neq 0$, we therefore only get the Möbius transformations $\varphi_1(t) = t, \varphi_2(t) = -t$, for both $c_{II}^{11}$ and $c_{II}^{22}$, which combine in pairs associated with four potential symmetries. As in Example 4, the nature of these symmetries can be determined by using the techniques in [5], yielding Cases (a)–(d) in Table 2.

When $c = 0$, we get the Möbius transformations $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ for both $c_{II}^{11}$ and $c_{II}^{22}$, which combine in pairs associated with 16 potential symmetries. Proceeding as in Examples 2–4, one finds that only 8 of these correspond to a symmetry of $\mathcal{S}$, namely Cases (a)–(h) in Table 2.

Noting from the explicit representations in Table 2 that the symmetries (b)–(h) have order 2, the second part follows from comparing to a list of groups of small order [23, p. 85].

4.2.3. Type III

Next we apply Theorem 10 to derive the symmetries of Dupin cyclides of Type III. Every such Dupin cyclide has parabolas as its spine curves, which might be related by an isometry. Hence, for such Dupin cyclides, it is necessary to analyse Case B of Theorem 10 as well. We distinguish cases according to whether the parameter $c$ vanishes; see Figure 4.

**Theorem 14.** For any Dupin cyclide $\mathcal{S}$ of Type III in Table 1:

- Conditions A1 and A2 in Theorem 10 are satisfied if and only if $(\varphi_1, \varphi_2)$ and $f$ are given by Cases (a)–(d) in Table 2.

- Conditions B1 and B2 in Theorem 10 are satisfied if and only if $c = 0$ and $(\varphi_1, \varphi_2)$ and $f$ are given by Cases (i)–(l) in Table 2.
In particular:

- If $c \neq 0$, the symmetries of $S$ are (a)–(d), forming a group isomorphic to $\mathbb{Z}_2^2$, the Klein four group.

- If $c = 0$, the symmetries of $S$ are (a)–(d) and (i)–(l), forming a group isomorphic to $D_4$, the dihedral group of order eight.

Proof. Case A: Inserting the radius functions $r_{1\text{III}}, r_{2\text{III}}$ from Table 1 into (19) yields

\[
R_{11\text{III}}(t,u) = -16g(u + t)(u - t)(g(u^2 + t^2 + 1) + 2c), \\
R_{22\text{III}}(t,u) = -16g(u + t)(u - t)(g(u^2 + t^2 + 1) - 2c),
\]

each of which has the Möbius-like factors $F_1(t,u) = u - t$ and $F_2(t,u) = u + t$. As the remaining factor is irreducible in each case, we find that the tentative Möbius transformations are $\varphi_1(t) = t$ and $\varphi_2(t) = -t$ for both $c_{1\text{III}}$ and $c_{2\text{III}}$. These again combine to four potential symmetries, corresponding exactly to the Cases (a)–(d) in Table 2.

Case B: One computes

\[
R_{12\text{III}}(t,u) = (r_{1\text{III}}^2(t) - (r_{2\text{III}}^2)(u) = -16g(gu^2 + gt^2 - 2c)(u^2 + t^2 + 1).
\]

The factor $u^2 + t^2 + 1$ is irreducible. If $c \neq 0$, then the factor $-gu^2 + gt^2 + 2c$ defines a hyperbola, since $g \neq 0$, and is therefore also irreducible. It follows that $R_{12\text{III}}$ does not have Möbius-like factors, so that there are no symmetries corresponding to Case B of Theorem 10 when $c \neq 0$. However, if $c = 0$, then $R_{12\text{III}}$ has the Möbius-like factors $F_3(t,u) = u + t$, corresponding to the Möbius transformations $\varphi_3(t) = \pm t$. From Example 4 it follows that Conditions B1 and B2 are satisfied for the Cases (i)–(l) in Table 2.

Noting from the explicit representations in Table 2 that the symmetries (b), (c), (d), (i), (j) have order 2 and (k), (l) have order 4, the second part follows from comparing to a list of groups of small order [23, p. 85].

While it is well known that any cyclide of Type II or III is symmetric with respect to the planes containing each of the spine curves, and therefore also with respect to the intersection line of these two planes, the preceding results show that when $c \neq 0$, such a cyclide cannot have any other symmetry. In fact, we have proven that cyclides of Type II and III have either 4 or 8 symmetries; in this last case, we say that it is a super-symmetric cyclide (see Figure 3, left, and Figure 4, left).

4.3. Algorithm

We end with providing an algorithm for computing the symmetries of a Dupin cyclide $S$ defined by spine curves $c_1, c_2$ and corresponding radius functions $r_1, r_2$, not necessarily given in canonical form. Whether the Dupin cyclide is of Type I, II, or III follows from the nature of the conics; this is easily determined, for instance by implicitization or by computing the curvature, which is independent of position and orientation. Moreover, we have the following result.
Lemma 15. A Dupin cyclide $S = S_{c_1, r_1} = S_{c_2, r_2}$ is super-symmetric if and only if one of the following cases holds:

- $S$ is of Type II, and the radius function corresponding to the ellipse has minimum $r_{\text{min}}$ and maximum $r_{\text{max}}$ satisfying $r_{\text{min}} + r_{\text{max}} = 0$.
- $S$ is of Type III, and $r_1 + r_2 = 0$ holds identically.

Proof. Since the conditions $r_{\text{min}} + r_{\text{max}} = 0$ and $r_1 + r_2 = 0$ remain valid under reparametrization, we can assume that $(c_1, r_1)$ and $(c_2, r_2)$ take the canonical forms in Table 1. In the first case of the lemma,

$$r_{\text{min}} = r_{\text{II}}^1(0) = c - f, \quad r_{\text{max}} = \lim_{t \to \pm \infty} r_{\text{II}}^1(t) = c + f,$$

while in the second case

$$r_1 = r_{\text{III}}^1 = c + g \left( t^2 + \frac{1}{2} \right), \quad r_2 = r_{\text{III}}^2 = c - g \left( t^2 + \frac{1}{2} \right).$$

In either case the sum is zero precisely when $c = 0$, which proves the lemma.

We can now sketch a method for determining the symmetries of $S$.

1. Determine the Type of $S$ from the nature of its spine curves.
2. Determine the invariants of the conics referenced in Table 2.
   Find planes $\Pi_1, \Pi_2$ containing $c_1, c_2$. Next determine $O$, i.e., for
   - Type I, the center of the circle;
   - Type II, the centroid of the ellipse/hyperbola;
   - Type III, the average of the vertex and focal point, for each parabola.
   Let $\Pi_0$ be the plane passing through $O$ and perpendicular to $\Pi_1, \Pi_2$. For
   Type III, we also determine the ‘mid-planes’ of $\Pi_3, \Pi_4$.
3. For Types II and III, determine if $S$ is super-symmetric using Lemma 15.
4. Look up the symmetry groups and symmetries of $S$ in Tables 2 and 3.
5. Conclusion and final remarks

We have presented results that characterize the symmetries of a canal surface $S$ in terms of its spine curve and radius function. These results lead to an algorithm, which is easy to implement, for computing the symmetries of a canal surface with just one spine curve, and to a complete classification of the symmetries of Dupin cyclides. Moreover, the results lead to a method for constructing symmetric canal surfaces.

In this paper, we have assumed that $S$ is defined by a rational spine curve $c$ and a rational radius function $r$, in which case it is well known that $S$ admits a rational parametrization as well. Conversely, and according to Proposition 1.1 in [27], if $S$ is rational then $c$ and $r^2$ must be rational too. However, there are rational canal surfaces where $r$ is not rational, but the square root of a rational function; see Example 3.2 in [27]. The methods of Section 3 can be adapted to this situation.

Finally, in Section 2.1 we observed that $S$ can be identified with a curve in four-dimensional Minkowski space $\mathbb{R}^{3,1}$. We have not yet explored the possibility of using this setup for computing or studying the symmetries of $S$. In this sense, a potential idea is to use differential invariants of curves in $\mathbb{R}^{3,1}$; see for instance [25].

References


