

Weakly Supplement Extending Modules

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Article Info

Received
29/10/2019

Accepted
25/12/2019

Published
15/04/2020

ABSTRACT

In this paper, the extending property of modules is generalized by using weakly supplement submodules. We call a module M is weakly supplement extending, if each submodule of M is essential in a weakly supplement submodule of M . Many characterization of weakly supplement extending module are obtained, we show that M is weakly supplement extending if and only if each closed submodule is weakly supplement submodule of M . Moreover, we study the relation of weakly supplement extending module and among other known classes of module such as lifting module, weakly supplemented module, supplement extending module and others. Also, we study conditions under it a direct sum of weakly supplement extending module is weakly supplement extending.

KEYWORDS: Extending module; Supplement submodule; Supplement extending module; Weakly supplement submodule; Weakly supplement extending module.

الخلاصة

في هذا البحث، يتم تعميم خاصية تمديد الوحدات باستخدام وحدات فرعية مكاملة ضعيفة. يقال للمقاس M بأنه مقاس توسع المكمل الضعيف، إذا كان كل مقاس جزئي من M يكون جوهرى من مركبة مقاس جزئي مكمل ضعيف من M . تم الحصول على العديد من التشخيصات لمقاس التوسع المكمل الضعيف. نبرهن ان M هي مقاس توسع مكمل ضعيف، اذا فقط اذا كان كل مقاس جزئي مغلق هو مقاس جزئي مكمل ضعيف من M . اكثر من ذلك، ندرس العلاقة بين مقاس التوسع المكمل الضعيف واصناف معروفة من المقاسات مثل مقاسات الرفع، مقاسات المكاملة الضعيفة، مقاسات التوسع المكاملة واخرين. ايضا ندرس شروط تحت الجمع المباشر لمقاس التوسع المكمل الضعيف هو مقاس التوسع المكمل الضعيف

INTRODUCTION

In this paper, all rings are associative with identity and all modules are unitary left R -module. A submodule N of M is essential, if N has nonzero intersection with any nonzero submodule of M [2]. A submodule V of M is small (denoted by $V \ll M$), if there is a submodule W of M such that $M=V+W$ implies $W=M$ [2]. If a submodule N has no proper essential extension in M , then N is said to be closed of M [2]. A submodule A of M is supplement, if there is a submodule F of M such that $M=A+F$ and $A \cap F \ll A$ [9]. A module M is said to be supplemented if each submodule of M has a supplement submodule in M [9]. A submodule W of M is weakly supplement, if there is a submodule V in M such that $M=V+W$ and $W \cap V \ll M$. If every submodule J of M is weakly supplement of M , then a module M is said to be weakly supplemented [2]. A module M is uniform if each nonzero submodule of M is essential submodule in M [3]. A module M is \oplus -

supplemented, if for any submodule N of M has a supplement submodule that is direct summand [2]. Equivalent, each submodule of M has a weakly supplement which is direct summand. The singular submodule of a module M is $Z(M)=\{a \in M \mid Ea=0 \text{ for some essential left ideal } E \text{ of } R\}$. A module M is said to be singular if $Z(M)=M$, and M is said to be non-singular if $Z(M)=0$ [3]. A module M is said to be lifting, if each submodule L of M there is a direct summand W of M with $W \subseteq L$ such that $M=W \oplus W'$ and $W' \cap L \ll W'$ [2]. A module M is hollow, if each proper submodule is small in M [2]. A module M is said to be simple, if $M \neq (0)$ and the only submodule of M are (0) and M . A module M is extending if each submodule of M is essential in direct summand of M . The extending property and their generalizations are studied by different authors. Following [10], A module M is called supplement extending if each submodule N of M is essential in supplement submodule W of M .

In this work and motivated by the known fact that each direct summand is weakly supplement submodule [7] and above concept, we introduce and study the concept weakly supplement extending module as generalization of extending module. We get many equivalent statements for weakly supplement extending module. Also, we answer the question, when the direct sum of weakly supplement extending module is weakly supplement extending? Moreover, a lot of results and property of weakly supplement extending module are obtained.

WEAKLY SUPPLEMENT EXTENDING MODULES:

We introduce the following concept that is as a generalization of extending and weakly supplemented modules:

Definition (2.1): A module M is weakly supplement extending, if each submodule of M is essential in a weakly supplement submodule in M . A right (left) ring R is weakly supplement extending, if R weakly supplement extending right (left) R -module.

It is known that, a module M is extending if and only if each closed submodule of M is a direct summand [3]. So we obtained the following characterization of weakly supplement extending module.

Proposition (2.2): A module M is a weakly supplement extending if and only if each closed submodule of M is a weakly supplement.

Proof: Let A be a closed submodule of weakly supplement extending module M . Since M is weakly supplement extending, then there exists a weakly supplement submodule B in M such that A is essential in B . But A is closed then A has no proper essential extension so (by definition of closed submodule) $A=B$, so A is weakly supplement submodule. Conversely, let A be a submodule of M . There is a closed submodule B of M such that A is essential in B (from Zorn's lemma). By hypothesis, B is weakly supplement submodule. Then M is weakly supplement extending.

Remarks and Examples (2.3):

1. Every (weakly) supplemented module is weakly supplement extending, but it is not conversely. Indeed, Z is weakly supplement extending Z -module which it is not (weakly) supplemented

(since a submodule $2Z$ in Z has no a weakly supplement submodule in Z).

2. Every extending module is weakly supplement extending, but the converse is not true. $M=Z_8 \oplus Z_2$ is weakly supplement extending Z -module which it is not extending (since M is weakly supplemented module).

3. Every supplement extending module is a weakly supplement extending, while it is not conversely. In fact, $Q \oplus Z_2$ is weakly supplement extending Z -module, since it is weakly supplemented (since Q and Z_2 are weakly supplemented and every direct sum of weakly supplemented is weakly supplemented [8]). While it is not supplement extending [10, Example (2.1.18)].

4. Every uniform module is weakly supplement extending, while the converse is not true. In fact, Z_6 is weakly supplement extending Z -module which it is not uniform.

5. It is well known that the concepts of lifting module and extending module are different. Here, we have every lifting module is weakly supplement extending, while the converse is not true. Indeed, since Z is uniform and so by (4) Z is weakly supplement extending Z -module which it is not lifting see [6].

6. Every \oplus -supplemented module is weakly supplement extending, it is not conversely. For example, Z is weakly supplement extending Z -module but it is not \oplus -supplemented.

The next result is another characterization of weakly supplement extending module.

Proposition (2.4): The following statements are equivalent for any module M :

1. M is weakly supplement extending module.
2. Every closed submodule in M is weakly supplement submodule.
3. The intersection any direct summand of the injective hull of M with M is weakly supplement submodule of M .

Proof: (1) \Rightarrow (2), directly by proposition (2.2).

(2) \Rightarrow (3) Let W be a direct summand of injective hull of M , i.e $E(M)=W \oplus K$, where K is a submodule of injective hull of M . To easy check that $W \cap M$ is closed submodule of M and by (2) $W \cap M$ is weakly supplement submodule of M .

(3) \Rightarrow (1) Let C be a submodule of M and let J be a relative complement of C in M , i.e $C \cap J=0$, then $C \oplus J$ is essential in M , but M is essential in injective hull of M , so $C \oplus J$ is essential in

injective hull of M . Then $E(M) = E(C \oplus J) = E(C) \oplus E(J)$, and since $E(C)$ is direct summand of injective hull of M , then $E(C) \cap M$ is weakly supplement submodule in M . Since C is essential in injective hull of C and M is essential in M , then $C \cap M$ is essential in $E(C) \cap M$ which is weakly supplement in M . Hence M is weakly supplement extending.

We asserted that, weakly supplement extending module need not be a supplement extending. Here, we discuss when this implication is valid.

Following [2] A and K are submodule of a module M with $A \subseteq K \subseteq M$. If A is small of M , thus A need not be small in K .

Proposition (2.5): Every weakly supplement extending module M is supplement extending module if for each a submodule A of K , where $A \subseteq K \subseteq M$, and $A \ll M$ implies $A \ll K$.

Proof: Let M be a weakly supplement extending and let W be a submodule of M , so there is a weakly supplement submodule K in M such that W is essential in K . So $M = K + V$ and $K \cap V \ll M$, where V is a submodule of M . By assumption, $K \cap V \ll K$ and hence K is a supplement submodule of V in M . Hence M is supplement extending module.

It is well-known, every quotient of weakly supplemented module is weakly supplemented [2]. For weakly supplement extending module, we have:

Proposition (2.6): Let M be a weakly supplement extending module, then any nonsingular image of M is weakly supplement extending.

Proof: Let $f : M \rightarrow W$ be an epimorphism image and A be a closed submodule of W , then $H = f^{-1}(S)$ is closed in M by [11]. But M is weakly supplement extending, so H is weakly supplement submodule in M . Then there is a submodule V of M such that $M = H + V$ and $H \cap V \ll M$. Now, we have $W = f(M) = f(H + V) = f(H) + f(V) = S + f(V)$ and since $H = f^{-1}(S)$ and f is epimorphism then $f(H) = f(f^{-1}(S)) = S$, since f is epimorphism and $\ker f \subseteq H$ then by [5] $f(H \cap V) = f(H) \cap f(V)$ and by [6] we have $f(H \cap V) = f(H) \cap f(V) = S \cap f(V) \ll f(M) = W$. Then S is weakly supplement submodule in W and so that W is weakly supplement extending.

We introduce the next concept that is as a generalization of \oplus -supplemented module:

Definition (2.7): A module M is closed \oplus -supplemented if each closed submodule N of M has a supplement submodule that is direct summand of M .

One can easily show that, a module M is called closed \oplus -supplemented if and only if each closed submodule is a weakly supplement which is direct summand.

It is clear that, every \oplus -supplemented module is closed \oplus -supplemented. Also, every closed \oplus -supplemented module is weakly supplement extending module (by using proposition (2.2))

Recall that, a module M is refinable, if for each submodule W, V of M with $W + V = M$, there exists a direct summand W' of M such that $W' \subseteq W$ and $W' + V = M$ [11].

Proposition (2.8): Let M be a refinable module, then the following statements are equivalent:

1. M is weakly supplement extending module.
2. M is closed \oplus -supplemented module.

Proof: (1) \Rightarrow (2) Suppose that M is a weakly supplement extending module and let N be a closed submodule of M , then N is weakly supplement submodule of M by proposition (2.2). So we have $M = N + W$ and $N \cap W \ll M$ for a submodule W of M , but M is refinable then there is a direct summand W' of M such that $W' \subseteq W$, so, $M = W' + N$ and $W' \cap N \subseteq W \cap N \ll M$, since $W' \cap N \subseteq W' \subseteq M$ and since W' is direct summand then $W' \cap N \ll W'$ [6]. So, N has a supplement submodule W' which is direct summand of M then M is closed \oplus -supplemented.

(2) \Rightarrow (1) Let W be a closed submodule in a closed \oplus -supplemented module M , then W has a supplement submodule V in M that is direct summand (i.e) $M = W + V$ and $W \cap V \ll V$. So we have $W \cap V \ll M$. Hence M is weakly supplement extending.

In the following, we prove that the class of weakly supplement extending module is closed under isomorphic property.

Proposition (2.9): If a module M_1 is weakly supplement extending and $M_1 \cong M_2$, then M_2 is weakly supplement extending.

Proof: Let $f : M_1 \rightarrow M_2$ be an isomorphism and M_1 is a weakly supplement extending. Let N be a

submodule of M_2 so $f^{-1}(N)$ is a submodule of M_1 . Since M_1 is weakly supplement extending then $f^{-1}(N)$ is essential in a weakly supplement submodule S of M_1 , then $f(f^{-1}(N)) = N$ is an essential in $f(S)$ in M_2 . Since S is weakly supplement submodule in M_1 then there is H is a submodule of M_1 such that $M_1 = S + H$ and $S \cap H \ll M_1$. Then $f(M_1) = f(S) + f(H)$, So $M_2 = f(S) + f(H)$ (since $S \cap H \ll M_1$ and $f : M_1 \rightarrow M_2$ homomorphism, then $f(S \cap H) \ll M_2$) [6], since f is isomorphism then f is monomorphism and then $\ker f = 0$ [5], and we have $\ker f \subseteq H$, thus by [5]:

$f(S \cap H) = f(S) \cap f(H)$, and $f(S) \cap f(H) \ll M_2$,
then $f(S)$ is weakly supplement submodule of M_2 ,
hence M_2 is a weakly supplement extending.

Proposition (2.10): Let $f : M \rightarrow W$ be a small epimorphism (that is f is called small epimorphism if $\ker f \ll M$ [11]) and W be a weakly supplement extending module, if any nonzero closed submodule S of M with $\ker f \subseteq S$, then M is weakly supplement extending.

Proof: Suppose that W is a weakly supplement extending module, let $f : M \rightarrow W$ be a small epimorphism and S be a closed submodule of M , since $\ker f \subseteq S$ and f is epimorphism. Then by first isomorphism $f(S) \cong S / \ker f$ is closed in $M / \ker f \cong W$, then $M / \ker f$ is weakly supplement extending, so $f(S)$ is weakly supplement submodule of W , then by proposition (2.4) S is weakly supplement submodule of M . Then M is weakly supplement extending.

Following [11], let $f : R \rightarrow T$ be a ring homomorphism and M a right T -module. One can define M to be a right R -module by $ar = af(r)$ for all $a \in M$ and $r \in R$. Moreover, if M is a right R -module and f is an epimorphisms satisfy $\ker f \subseteq M r$, so also can define M to be a right T -module by $at = ar$, where $f(r) = t$. (denote by M_T, M_R) that M is a right T -module, right R -module, respectively.

Proposition (2.11): Let $f : R \rightarrow T$ be a ring epimorphism and M be a module (R -) T -module with $\ker f \subseteq r(M)$. Then M_R is weakly supplement extending if and only if M_T is weakly supplement extending.

Proof: Let M_R is weakly supplement extending and let N_T be a submodule of M_T , since f is homomorphism and by $rx = f(r)x$ for all $x \in N$, so N_R be a submodule of M_R , since M_R is a weakly

supplement extending then N_R is essential in weakly supplement submodule L_R in M_R , then we have $M_R = K_R + L_R$ and $K_R \cap L_R \ll M_R$ where K_R be a submodule in M_R , then K_T be a submodule in M_T (by f is homomorphism and by $rx = f(r)x$) such that $M_T = L_T + K_T$ and $K_T \cap L_T \ll M_T$, then L_T is weakly supplement submodule in M_T . To prove N_T is essential in L_T . Let $0 \neq x \in L$, there is $t \in T$, so we can define L_T to be a right R -module by f is homomorphism and $ry = f(r)y$ for each $y \in L$ and $r \in R$. Since N_R is essential in L_R . Then $ry \in N_R$. Also, N_R we can define to be a right T -module by f is an epimorphism such that $\ker(f) \subseteq r(M)$ and by $ty = ry$ where $f(r) = t$, so $ty \in N_T$. Hence N_T is essential in L_T . Conversely, suppose that M_T is weakly supplement extending and let N_R be a submodule in M_R . Since f is epimorphism and $\ker f \subseteq r(M)$, by $tx = rx$ where $x \in N$. Then N_T be a submodule in M_T . Since M_T is weakly supplement extending. Then N_T is essential in weakly supplement submodule L_T in M_T , so we have $M_T = L_T + K_T$ and $L_T \cap K_T \ll M_T$ where K_T is submodule in M_T . To prove N_R is essential in L_R . Let $x \in L$ and $r \in R$, so L_R we can define to be T -module by f is an epimorphism such that $\ker(f) \subseteq r(M)$ and by $tx = rx$ where $f(r) = t$, so $x \in L$ and $t \in T$. Since N_T is essential in L_T . Then $xt \in N_T$. Also, N_T we can define to be a right R -module by f is a homomorphism and by $rx = f(r)x$. Then $xr \in N_R$ and hence N_R is essential in L_R .

PROPOSITION (2.12): Let M be a refinable module in which for each submodule W of M there is a closed submodule F (depending on W) of M such that $W = F + H$ or $F = W + H$ where $H \ll M$. Then M is \oplus -supplemented if and only if M is weakly supplement extending.

Proof: (\Rightarrow) By Remarks and Examples (2.3, (6)). (\Leftarrow) Let W be a submodule of weakly supplement extending M . Then there is a closed submodule F in M such that $F = W + H$ where $H \ll M$. Since M is weakly supplement extending, then there is a submodule V in M such that $M = V + F$ and $V \cap F \ll M$, so $M = V + F = V + W + H = V + W$ (since $H \ll M$) and $V \cap W \subseteq V \cap F \ll M$ [5]. Then $V \cap W \ll M$. But M is refinable. Then there is a direct summand $U' \subseteq V$ such that $U' + W = M$. Then $U' \cap W \subseteq V \cap W \ll M$ so $U' \cap W \ll M$ but $U' \cap W \subseteq U' \subseteq M$, then $U' \cap W \ll U'$ (since $U' \cap W \ll M$ and U' is direct summand by [6]). M is \oplus -supplemented. Or let W be a submodule of weakly supplement extending M . There is a closed submodule F in M

such that $N=F+H$ where $H \ll M$, since M is weakly supplement extending, then by hypothesis, F is weakly supplement submodule (by using proposition (2.2)). So there exists a submodule V in M such that $M=F+V$ and $F \cap V \ll M$. But $H \ll M$, so $M=F+V+H=V+W$, but M is refinable. Then there is a direct summand U' in M and $U' \subseteq V$. Such that $M=W+U'$, since $W \cap U' \subseteq W \cap V \subseteq (F+H) \cap V$. Now to show that $(F+H) \cap V \ll M$. Suppose that $g: M \rightarrow (M/F) \oplus (M/V)$ is defined by $g(m) = (m+F, m+V)$ for each $m \in M$ and $h: (M/F) \oplus (M/V) \rightarrow (M/F+H) \oplus (M/V)$ is defined by $h(m+F, n+V) = (m+F+H, n+V)$ for each $n, m \in M$. Now by weakly supplement submodule of F (i.e) $M=V+F$, thus g is an epimorphism and $\ker g = V \cap F \ll M$, to easy check that $\ker h = ((F+H)/F) \oplus 0, H \ll M/F$. Since $p: M \rightarrow M/F$ is canonical epimorphism, thus h is a small epimorphism and we have hg is a small epimorphism and $\ker hg = (H+F) \cap V \ll M$. So $W \cap U' \ll M$. Thus $W \cap U' \subseteq U' \subseteq M$. So $W \cap U' \ll U'$. Then M is \oplus -supplemented.

Recall that [9], let R be a ring and W, L be a submodule of a module M . The residual of W by L is $[W:L] = \{x \in R : xL \subseteq W\}$ and the annihilator of M (denoted by $\text{ann}_R(M)$) is $\text{ann}_R(M) = [0:M]$. Also, a module M is called faithful if $\text{ann}_R(M) = 0$.

Proposition (2.13): Every finitely generated faithful multiplication module over weakly supplement extending commutative ring R is weakly supplement extending.

Proof: Let W be a closed submodule of a module M , then $W = [W:M]M$ (since M is multiplication μ) where $[W:M]$ is an ideal of R . Since L is closed submodule in M so, $[W:M]$ is closed ideal in R [1]. Thus, by hypothesis, $[W:M]$ is weakly supplement ideal in R . Then there exists an ideal I of R such that $[W:M]+I=R$ and $[W:M] \cap I \ll R$. Now $M=RM = ([W:M]+I)M = [W:M]M + IM$. Let $([W:M]M) \cap IM = KM = M$, since M is multiplication module so $([W:M] \cap I)M = M$ [4], since $M=RM = ([W:M] \cap I)M + KM$, then $R = ([W:M] \cap I) + K$ and since $[W:M] \cap I \ll R$ then we have $R=K$. Since $M=RM$ and $R=K$ thus, $KM=M$, so $([W:M]M \cap IM) \ll M$. And so, L is weakly supplement submodule in M . Then M is weakly supplement extending.

One can see that a submodule of a weakly supplement extending module need not be weakly

supplement extending. In fact, we can consider any injective hull of a module which is weakly supplement extending. $Z \oplus Z_2$ as Z -module is submodule of $E(Z \oplus Z_2)$ and every injective envelope is weakly supplement extending module while $Z \oplus Z_2$ it is not weakly supplement extending.

In the following results, we discuss when weakly supplement extending property is closed under submodule

Proposition (2.14): Let W be a submodule of weakly supplement extending module M , if the intersection of W with any weakly supplement submodule of M is weakly supplement submodule of W , then W is weakly supplement extending.

Proof: Let W be a submodule of weakly supplement extending module M and V be a submodule of W then there is a weakly supplement submodule S in M such that V is essential in S and so V is essential in $W \cap S$. By hypothesis, $W \cap S$ is a weakly supplement submodule of W . Then W is a weakly supplement extending.

We do not be whether, a weakly supplement submodule of weakly supplement extending module is weakly supplement extending.

Proposition (2.15): Let W be a weakly supplement submodule of weakly supplement extending module M such that the intersection of any two weakly supplement submodule in M is weakly supplement in W . Then W is weakly supplement extending.

Proof: Let W be a weakly supplement submodule in M and let K be a submodule in W . So we have K be a submodule in M . Since M is weakly supplement extending, so K is essential in weakly supplement submodule H in M . By hypothesis, $W \cap H$ is weakly supplement submodule in W since K is essential in H and W is essential in W . Then we have $K \cap W$ is essential in $H \cap W$ and we have $K \cap W = K$, so K is essential in $H \cap W$. Then W is weakly supplement extending.

An analogous to extending modules, the next result asserts that a direct summand of weakly supplement extending module is weakly supplement extending.

Proposition (2.16): Every direct summand of a weakly supplement extending module is weakly supplement extending.

Proof: Let W be a direct summand of weakly supplement extending module M and let S be a closed submodule in W , so S is closed submodule of M . Since M is a weakly supplement extending module, so by proposition (2.2), S is weakly supplement submodule of M so we have $M = S + H$ and $S \cap H \ll M$, where H is a submodule of M so $W \cap M = W \cap (S + H)$ then (by modular law) $W = S + (W \cap H)$ and $S \cap (W \cap H) = (S \cap H) \cap W$. Since $(S \cap H) \cap W$ be a submodule of $S \cap H$ and $S \cap H \ll M$, so, we have $S \cap (W \cap H) \ll M$ [6]. Also, since $S \cap (W \cap H)$ be a submodule in W and W is direct summand in M , then by [6], $S \cap (W \cap H) \ll W$. Thus, we have, S is a weakly supplement submodule in W . There for, by using proposition (2.2), W is a weakly supplement extending module.

Recall that, the radical submodule of a module M is the intersection of all maximal submodule of M (denoted by $Rad(M)$). Equivalent, the radical of M is the sum of all small submodule of M . [9].

Corollary (2.17): Let M be a module with zero radical, the next conditions are equivalent:

1. M is extending.
2. M is weakly supplement extending.

Proof: (1) \Rightarrow (2) It is clear.

(2) \Rightarrow (1) Let N be a submodule of weakly supplement extending module M . So there exists a weakly supplement submodule S in M such that N is essential in S , since S is weakly supplement submodule. Then $M = S + H$ where H be a submodule of M and $S \cap H \ll M$. But $Rad(M) = 0$ then $S \cap H = 0$. So, We have S is a direct summand of M . Then M is extending.

It is clear that, the direct sum of extending module need not be extending [2]. Also, we have here a direct sum of weakly supplement extending modules need not necessary weakly supplement extending. $M = Z[X] \oplus Z[X]$ as $Z[X]$ -module is not weakly supplement extending, because M is not extending [3] and $Rad(M) = 0$ [11].

Recall that, a module M is distributive if for all submodule A , S and N of M , $A \cap (S + N) = (A \cap S) + (A \cap N)$ [12].

The next result answer of the question: when the weakly supplement extending property is closed under the direct sum?

Proposition (2.18): Let $M = M_1 \oplus M_2$ where M , M_1 and M_2 are modules. Suppose that M is a distributive module, then M is weakly supplement

extending if and only if each M_i is weakly supplement extending ($i=1,2$).

Proof: Let M is weakly supplement extending. Since M_i is direct summand of M (where $i=1,2$). Then by proposition (2.15) M_i is weakly supplement extending. Conversely, let V be a closed submodule in M . To prove $V \cap M_i$ is closed in M_i , since M is distributive module, then we have $V = (V \cap M_1) \oplus (V \cap M_2)$. Hence $V \cap M_1$ is closed in M_1 and $V \cap M_2$ is closed in M_2 . But M_1 and M_2 are weakly supplement extending. Then there is a submodule S_1 of M_1 and S_2 of M_2 such that $S_1 + (V \cap M_1) = M_1$ and $S_2 + (V \cap M_2) = M_2$, $S_1 \cap (V \cap M_1) = (S_1 \cap V) \ll M_1$ and $S_2 \cap (V \cap M_2) = (S_2 \cap V) \ll M_2$. Now,

$$M = M_1 \oplus M_2 = (S_1 + (V \cap M_1)) \oplus (S_2 + (V \cap M_2)) = (S_1 \oplus S_2) + V.$$

Then $M = (S_1 \oplus S_2) + V$ and $(S_1 \oplus S_2) \cap V = (S_1 \cap V) \oplus (S_2 \cap V) \ll M_1 \oplus M_2 = M$. Hence M is weakly supplement extending.

Following [9], A submodule H of a module M is fully invariant μ , if $f(H) \subseteq H$ for each endomorphism f of M .

Proposition (2.19): Let $M = \bigoplus_{i \in I} M_i$ be a module, where each M_i is a submodule of M for every $i \in I$. Suppose that every closed submodule in M is fully invariant. Then M is weakly supplement extending module if and only if each M_i is weakly supplement extending module.

Proof: Let M be a weakly supplement extending module, so by proposition (2.15) M_i is weakly supplement extending. Conversely, let K be a closed module of M and $p: M \rightarrow M_i$ be the natural projection mapping on M_i (for every $i \in I$). Let $x \in K$, then $x = \sum_{i \in I} m_i$, where $m_i \in M_i$ and hence $\pi_i(x) = m_i$. Since $\pi_i(K) \subseteq K$ and $\pi_i(K) \subseteq M_i$. Then $\pi_i(K) \subseteq K \cap M_i$. Since K is fully invariant of $M = \bigoplus M_i$. Then $K = \bigoplus K \cap M_i$. Now since $K \cap M_i$ is direct summand of K , then $K \cap M_i$ is closed in K . But K is closed in M , thus $K \cap M_i$ is closed in M . Since $K \cap M_i \subseteq M_i \subseteq M$ then $K \cap M_i$ is closed in M_i . $K \cap M_i$ is weakly supplement submodule of M_i . Then $K = \bigoplus (K \cap M_i)$ is weakly supplement submodule of $M = \bigoplus M_i$. Then K is a weakly supplement submodule of M . Then by using proposition (2.2) M is weakly supplement extending.

Proposition (2.20): Let M be a module such that every weakly supplement submodule is direct summand. Then M is weakly supplement extending if and only if M is extending.

Proposition (2.21): Let M be a module such that each weakly supplement submodule is direct summand. Then M is weakly supplement extending if and only if M is supplement extending.

The condition in above propositions is necessary. For example, $M=Z_8\oplus Z_2$ is weakly supplement extending Z -module see (Remarks and Examples (2.3)) and supplement extending see [10] while M is not satisfy above condition, since $Z(2,1)$ is weakly supplement submodule but it is not direct summand. In the other direction, $Z\oplus Z_2$ as Z -module is satisfy above condition while M are not weakly supplement extending, extending and supplement extending.

Proposition (2.22): Let M be a module such that if $M\neq(0)$ and (0) the only small submodule in M . Then M is extending module if and only if M is weakly supplement extending module.

Proposition (2.23): Let M be a module such that if $M\neq(0)$ and (0) the only small submodule in M . Then M is supplement extending module if and only if M is weakly supplement extending.

It is clear that, the concept of weakly supplement extending module and above conditions are different. For example, $M=Z_8\oplus Z_2$ is weakly supplement extending Z -module and supplement extending see (Remarks and Examples (2.3)). But it is not satisfy above condition, since $Z(2,0)$ and $Z(4,0)$ are small submodule in M . In other direction, $M=Z\oplus Z_2$ as Z -module is satisfy above condition while M are not weakly supplement extending, extending and supplement extending.

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