Incomplete Information Games with Smooth Ambiguity Preferences

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Abstract

We propose equilibrium notions for incomplete information games involving players who perceive ambiguity about the types of others. Players have smooth ambiguity preferences (Klibanoff, Marinacci and Mukerji, 2005) and may be ambiguity averse. In the smooth ambiguity model it is possible to hold the agents’ information fixed while varying the agent’s ambiguity attitude from aversion to neutrality (i.e., expected utility). This facilitates a natural way to understand the effect of introducing ambiguity attitude into a strategic environment. Our focus is on extensive form games, specifically multi-stage games with observed actions, and on equilibrium concepts satisfying perfection (sequential optimality). We propose the notions of Perfect Equilibrium with Ambiguity (PEA) and Sequential Equilibrium with Ambiguity (SEA) for such games, investigate their properties and provide several examples applying them.

1 Introduction

Dynamic games are the subject of a large literature in economics, both theory and application, with diverse fields including models of firm competition, agency theory, auctions,
search, insurance and many others. Uncertainty is central to many of these models, leading to the important issue of preference and belief updating within equilibrium analysis. While there is consensus and there are strong justifications for Bayesian updating in the case of expected utility preferences, there is much less consensus about and, in fact, well-known difficulties with methods for updating more general preferences. One motivation for examining more general preferences has been the desire to model individuals who do not treat all uncertainty in the same way – specifically, who may make meaningful distinctions between probabilities about which they know more or less. This uncertainty about probabilities is most commonly referred to in the recent literature as ambiguity. Preferences that allow for decision makers to care about ambiguity have drawn increasing interest, motivated both by theoretical advances and by empirical verification of the relevance of such phenomenon in actual behavior. Exploring dynamic games under ambiguity sensitive preferences, in addition to bringing the possibility of new insights, can help evaluate the robustness or lack thereof of the standard conclusions.

Our overall goal is to present a model that would allow us to examine the effect of introducing ambiguity aversion in strategic settings, static and dynamic, involving incomplete information. To that end, we propose equilibrium notions for incomplete information games involving players who perceive ambiguity about the types of others. Players have smooth ambiguity preferences (Klibanoff, Marinacci and Mukerji, 2005) and may be ambiguity averse. In the smooth ambiguity model it is possible to hold the agents’ information fixed while varying the agent’s ambiguity attitude from aversion to neutrality (i.e., expected utility). This would facilitate a natural way to understand the effect of introducing ambiguity attitude into a strategic environment. Our focus will be on extensive form games, specifically multi-stage games with observed actions, and on equilibrium notions capturing perfection analogous to those in standard theories for ambiguity neutral players, such as Perfect Bayesian equilibrium (PBE) (e.g., Fudenberg and Tirole (1991a, b)) and Sequential equilibrium (Kreps and Wilson (1982)).

We first extend Bayesian Nash equilibrium to ex-ante and interim equilibrium concepts that allow aversion to ambiguity about the types of other players. When there is no type uncertainty, these definitions collapse to Nash equilibrium. Next, we refine our notion of equilibrium on the domain of multi-stage games with observed actions and incomplete information by imposing perfection in the form of a sequential optimality requirement – each player $i$’s strategy must be optimal at each stage given the strategies of the other players and $i$’s conditional beliefs about types. When there is no type uncertainty, sequential optimality reduces to subgame perfection. We then propose two refinements of sequential optimality that restrict any player $i$’s beliefs about types at stage games that could be reached only
immediately following a deviation of at least one other player $j \neq i$: Perfect Equilibrium with Ambiguity (PEA) and Sequential Equilibrium with Ambiguity (SEA). In addition to sequential optimality, PEA requires, under certain conditions, that any player $i$’s conditional beliefs about player $j \neq i$’s type remain the same as they were at the previous stage if player $j$ had no choice (or only one action) available at that stage. SEA replaces this additional requirement of PEA with a generalization of Kreps and Wilson’s (1982) consistency condition. Any SEA is shown to be a PEA. We show that these notions (sequential optimality, PEA and SEA) have a number of attractive properties: First, they exist for any multi-stage game with observed actions and incomplete information, and for any specification of players’ ambiguity aversion and initial beliefs. Second, we show that it is without loss of generality in identifying strategy profiles that are part of a sequential optimum/PEA/SEA to restrict attention to belief systems updated using a dynamically consistent generalization of Bayesian updating appropriate for smooth ambiguity preferences, called the smooth rule (Hanany and Klibanoff 2009). Third, we show that to verify that a candidate for a sequential optimum/PEA/SEA, consisting of a strategy profile and a belief system updated according to the smooth rule, is sequentially optimal, it is sufficient to check that there are no profitable one-stage deviations. Thus we show that a strategy profile is part of a sequential optimum if and only if there are no profitable one-stage deviations with respect to beliefs updated according to the smooth rule.

Building on ideas of Aumann (1974) and Bade (2011), we demonstrate that our equilibrium notions and framework can be used to model strategic ambiguity through strategies that are optimally chosen to be contingent on payoff irrelevant types about which there is ambiguity (e.g., possible draws from an Ellsberg urn). In that section, we also compare this aspect with both older literature on complete information games with ambiguity about others’ strategies and issues raised by Mouraviev, Riedel and Sass (2015).

We then provide several examples that apply our equilibrium notions. First, we present a game with a path that is played in an SEA given sufficient ambiguity aversion, but is never an ex-ante equilibrium (and thus also not sequentially optimal, a PEA, an SEA, a PBE or a sequential equilibrium) given ambiguity neutrality. This example is truly strategic in that it relies on one player recognizing the ambiguity aversion of others and changing play because of it. Second, we consider a game with a path that is played in an SEA given sufficient ambiguity aversion, but is never played in a PEA (or PBE) given ambiguity neutrality. This example illustrates how differential screening based on beliefs and ambiguity attitudes can give rise to strategic behavior ruled out under ambiguity neutrality. Third, we present an example of a Milgrom and Roberts (1982)-style limit pricing entry game with an SEA involving limit pricing and non-trivial smooth rule updating on the equilibrium path that
departs from Bayes’ rule. We provide conditions under which ambiguity aversion makes limit pricing more robust.

Finally, we discuss some alternative approaches to extensive form games with incomplete information and ambiguity and their relation to ours.

While there have been a very few papers investigating aspects of dynamic games with ambiguity aversion (e.g., Lo 1999, Eichberger and Kelsey 1999, 2004, Bose and Daripa 2009, Kellner and Le Quement 2013, 2015, Bose and Renou 2014, Mouraviev, Riedel and Sass 2015, Battigalli et al. 2015, Dominiak and Lee 2015), to the best of our knowledge this paper is the first to propose an equilibrium notion for dynamic games with incomplete information that requires sequential optimality while allowing for ambiguity averse preferences. The existing literature instead mostly either takes approaches related to optimality under consistent planning in the spirit of Strotz (1955-56) or takes a purely ex ante perspective. In Section 5, we formally define optimality under consistent planning and say more about how these approaches relate to ours.


2 Model

We consider games with incomplete information defined as follows:

**Definition 2.1** A (finite) game with incomplete information and (weakly) ambiguity aversion smooth ambiguity preferences \( G \) is a tuple \((N, (A_i)_{i \in N}, (\Theta_i)_{i \in N}, (\mu_i)_{i \in N}, (u_i, \phi_i)_{i \in N})\) where \( N \) is a finite set of players, \( A_i \) is the finite set of actions available to player \( i \), \( \Theta_i \) is the finite set of possible “type” for player \( i \), \( \mu_i \) is a probability over \( \Delta(\Theta) \) having finite support such that \( \sum_{\pi \in \Delta(\Theta)} \mu_i(\pi)\pi(\theta) > 0 \) for all \( i \in N \) and \( \theta \in \Theta \), where \( \Theta \subseteq \prod_{j \in N} \Theta_j \) and \( \Delta(\Theta) \) is the set of all probability measures over \( \Theta \), \( u_i : A \times \Theta \to \mathbb{R} \) is the utility payoff of player \( i \) given the action of each player \( A \equiv \prod_{j \in N} A_j \) and the type of each player, and \( \phi_i : u_i(A \times \Theta) \to \mathbb{R} \) is a continuously differentiable, concave and strictly increasing function.

Notice that we allow for games where players have different initial beliefs. Importantly, however, all of the definitions and formal results of this paper continue to hold if restricted to the class of games with a common \( \mu \) such that \( \mu_i = \mu \) for all players \( i \). Furthermore, none of our examples or the conclusions we draw from them will rely on differences in the \( \mu_i \).
To interpret $u_i$ in these definitions, one can think of this vNM utility function as coming from the composition of two more fundamental functions. The first function $c_i : A \times \Theta \to Z$ is a consequence function determined by the structure of the game – for each action and type profile, it specifies the consequence or prize or outcome $z \in Z$ received by player $i$. The second function is a vNM utility over consequences, $w_i : Z \to \mathbb{R}$. Assume that $Z$ is big enough so that $u_i(A \times \Theta)$ is interior in $w_i(Z)$. This will be convenient for some later optimality characterizations. Notice that $w_i$ is independent of $\theta$, so that even though it might appear from the expression (2.1) that $u_i = w_i \circ c_i$ is state-dependent, this does not mean that decision-theoretically we are in a state-dependent setting, since the dependence is only via the usual dependence of the consequence of an act on the state.

**Definition 2.2** A (mixed) strategy for player $i$ in a game $G$ is a function $\sigma_i : \Theta_i \to \Delta(A_i)$ specifying the distribution over $i$’s actions conditional on each possible type of player $i$. Denote the set of all such strategies for player $i$ by $\Sigma_i$. A strategy profile, $\sigma \equiv (\sigma_i)_{i \in N}$ is a strategy for each player.

**Definition 2.3** An ex ante equilibrium of a game $G$ is a strategy profile $\sigma^*$ such that, for each player $i \in N$ and each $\sigma_i \in \Sigma_i$, strategy profile $\sigma^*$ is preferred by player $i$ to $(\sigma_i; \sigma_{-i})$.

**Assumption 2.1** Given a game $G$, player $i$ prefers strategy profile $\bar{\sigma}$ to strategy profile $\hat{\sigma}$ if and only if $V_i(\bar{\sigma}) \geq V_i(\hat{\sigma})$ where

$$V_i(\sigma) \equiv \sum_{\pi \in \Delta(\Theta)} \phi_i \left( \sum_{\theta \in \Theta} \sum_{a \in A} u_i(a, \theta) \left( \prod_{j \in N} \sigma_j(\theta_j)(a_j) \right) \pi(\theta) \right) \mu_i(\pi). \quad (2.1)$$

The strategy set for player $i$ is the set of functions $\sigma_i$, which is non-empty, compact and convex (as is $\Delta(A_i)$). Since $\phi_i$ is continuous and concave, and the expression inside $\phi_i$ is linear (and so trivially concave), $V(\sigma_i; \sigma_{-i})$ is concave in $\sigma_i$ for all $\sigma_{-i}$. Thus for any $\mu_i$, an ex-ante equilibrium exists by Gilcksberg (1952).

When there is no type uncertainty, the definition collapses to Nash equilibrium. Thus, in a game with complete information, we have nothing new to say compared to the standard theory. Also, in the case where the $\phi_i$ are linear (expected utility) and $\mu_i = \mu$ for all players $i$, the definition reduces to the usual (ex-ante) Bayesian Nash Equilibrium definition.

**Definition 2.4** Let $\Theta_i, \tau_i \equiv \{ \theta \in \Theta \mid \theta_i = \tau_i \}$. An interim belief for player $i$ of type $\tau_i$ in a game $G$ is a finite support probability measure $\nu_i, \tau_i$ over $\Delta(\Theta)$ such that

$$\nu_i(\tau_i) \{ \pi \in \Delta(\Theta) \mid \pi(\Theta_i, \tau_i) > 0 \} = 1. \quad (2.2)$$
An interim belief profile, \( \nu \equiv (\nu_{i, \tau_i})_{i \in N, \tau_i \in \Theta_i} \), is an interim belief for each type of each player.

Condition (2.2) says that to be an interim belief, the probability measure \( \nu_{i, \tau_i} \) must respect a player’s knowledge of her own type.

**Definition 2.5** An interim equilibrium of a game \( G \) is a pair \( (\sigma^I, \nu^I) \) consisting of a strategy profile and interim belief profile such that, for each player \( i \in N \), each type \( \theta_i \in \Theta_i \) and each \( \sigma'_i \in \Sigma_i \), strategy profile \( \sigma^I \) is preferred given interim belief \( \nu_{i, \theta_i} \) to \( (\sigma'_i; \sigma_i^I) \).

**Assumption 2.2** Given a game \( G \) and given that her type is \( \tau_i \) and interim belief is \( \nu_{i, \tau_i} \), player \( i \) prefers strategy profile \( \tilde{\sigma} \) to strategy profile \( \tilde{\sigma} \) if and only if

\[
V_i(\tilde{\sigma} | \tau_i) \geq V_i(\tilde{\sigma} | \tau_i)
\]

where

\[
V_i(\sigma | \tau_i) \equiv \sum_{\{\pi \in \Delta(\Theta) | \pi(\Theta_i, \tau_i) > 0\}} \phi_i \left( \sum_{\theta \in \Theta_i, \tau_i} \sum_{a \in A} u_i(a, \theta) \left( \prod_{j \in N} \sigma_j(\theta_j)(a_j) \right) \frac{\pi(\theta)}{\pi(\Theta_i, \tau_i)} \right) \nu_{i, \tau_i}(\pi).
\]

(2.3)

Note that we leave the relationship between \( \mu_i \) and the \( \nu_{i, \tau_i} \) unrestricted for now. Similar strengthenings as above for ex ante equilibrium are possible also here. Additionally, one may want to require all players to use the same update rule to relate \( \mu_i \) and the \( \nu_{i, \tau_i} \).

We now move to the central domain of the paper, multi-stage games with observed actions and incomplete information (cf. Fudenberg and Tirole Chapter 8.2.3). It is on this domain that we will develop and apply our main equilibrium concept. While this class of games is broad enough to cover many applications to economics and elsewhere, it does embody some limitations. In such games, the only observation that a player may see while others do not is what is revealed to her at the start of the game by nature. There are no private observations as the game proceeds.

**Definition 2.6** A (finite) extensive-form multi-stage game with observed actions and incomplete information and (weakly) ambiguity averse smooth ambiguity preferences \( \Gamma \) is a tuple \( (N, T, (A^t_i)_{i \in N, t \in T}, (\Theta_i)_{i \in N}, (\mu_i)_{i \in N}, (u_i, \phi_i)_{i \in N}) \) where \( N \) is a finite set of players, \( T = \{0, 1, ..., T\} \) is the set of stages, \( A^t_i(\eta^t) \) gives the finite set of actions (possibly singleton) available to player \( i \) in stage \( t \) as a function of the partial history \( \eta^t \in H^t \) of action profiles up to (but not including) time \( t \), where the sets of partial histories are defined by \( H^0 = \{\emptyset\} \) and, for \( 1 \leq t \leq T + 1 \), \( H^t \equiv \{(\eta^{t-1}, a) | \eta^{t-1} \in H^{t-1}, a \in \prod_{j \in N} A_j^{t-1}(\eta^{t-1})\} \), \( \Theta_i \) is the finite set of possible “types” for player \( i \), \( \mu_i \) is a probability over \( \Delta(\Theta) \) having finite support such that

\[
\sum_{\pi \in \Delta(\Theta)} \mu_i(\pi) \pi(\theta) > 0 \text{ for all } i \in N \text{ and } \theta \in \Theta, \text{ where } \Theta \subseteq \prod_{j \in N} \Theta_j \text{ and } \Delta(\Theta) \text{ is the set}
\]
of all probability measures over \( \Theta \), \( u_i : H \times \Theta \to \mathbb{R} \) is the utility payoff of player \( i \) given the history of actions (\( H \equiv H^{T+1} \)) and the type of each player, and \( \phi_i : u_i(H \times \Theta) \to \mathbb{R} \) is a continuously differentiable, concave and strictly increasing function.

As was mentioned after defining normal form games, all of the definitions and formal results of this paper continue to hold if restricted to the class of games with a common \( \mu \) such that \( \mu_i = \mu \) for all players \( i \). Furthermore, none of our examples or the conclusions we draw from them will rely on differences in the \( \mu_i \).

Given a history \( h \in H \) and stage \( t \leq T + 1 \), \( h^t \in \prod_{j \in N} \prod_{s \leq t} A^j_s(h^s) \) is the partial history up to but not including \( t \) specified by \( h \). It is useful to define a strategy for player \( i \) as specifying the distribution over \( i \)'s actions conditional on each possible partial history and each possible type of player \( i \). Formally:

**Definition 2.7** A (behavior) strategy for player \( i \) in a game \( \Gamma \) is a function \( \sigma_i \) such that \( \sigma_i(h^t, \theta_i) \in \Delta(A^i_t(h^t)) \) for each type \( \theta_i \), history \( h \) and stage \( t \).

Given a strategy \( \sigma_i \) for player \( i \), the continuation strategy at stage \( t \) given partial history \( h^t, \sigma^h_i \), is the restriction of \( \sigma_i \) to the set of all partial histories starting with \( h^t \). Let \( \Sigma_i \) and \( \Sigma^h_i \) denote the set of all strategies for player \( i \) and the set of all continuation strategies for player \( i \) at stage \( t \) given partial history \( h^t \), respectively. A strategy profile, \( \sigma \equiv (\sigma_i)_{i \in N} \), is a strategy for each player. Similarly, the associated continuation strategy profile at stage \( t \) given partial history \( h^t \) is \( \sigma^h \equiv (\sigma^h_i)_{i \in N} \).

For a history \( h \), the action taken at stage \( s \) by player \( j \) is denoted by \( h_{s,j} \). Given a strategy profile \( \sigma \), type profile \( \theta \), history \( h \) and \( 0 \leq r < t \leq T + 1 \), the probability of reaching \( h^t \) starting from \( h^r \) is \( p_{\sigma,\theta}(h^t|h^r) \equiv \prod_{j \in N} \prod_{r \leq s < t} \sigma_j(h^s, \theta_j)(h_{s,j}) \). It will be useful in what follows to separate this probability into a part affected only by \( \sigma_i \) and \( \theta_i \) and a part affected only by \( \sigma_{-i} \) and \( \theta_{-i} \). These are \( p_{i,\sigma,\theta}(h^t|h^r) \equiv \prod_{r \leq s < t} \sigma_i(h^s, \theta_i)(h_{s,i}) \) and \( p_{-i,\sigma,\theta}(h^t|h^r) \equiv \prod_{j \neq i} \prod_{r \leq s < t} \sigma_j(h^s, \theta_j)(h_{s,j}) \) respectively, with \( p_{i,\sigma,\theta}(h^t|h^r)p_{-i,\sigma,\theta}(h^t|h^r) = p_{\sigma,\theta}(h^t|h^r) \). With this notation, we can now state formally the assumption that players ex-ante preferences over strategies are smooth ambiguity preferences (Klibanoff, Marinacci and Mukerji 2005) with the \( u_i, \phi_i \) and \( \mu_i \) as specified by the game.

**Assumption 2.3** Fix a game \( \Gamma \). Ex-ante (before own-types are known), each player \( i \) ranks strategy profiles \( \sigma \) according to

\[
V_i(\sigma) \equiv \sum_{\pi \in \Delta(\Theta)} \phi_i \left( \sum_{\hat{h} \in H} u_i(\hat{h}, \hat{\theta}) p_{\sigma,\hat{\theta}}(\hat{h}|h^0) p_{-i,\sigma,\hat{\theta}}(\hat{h}|h^0) \pi(\hat{\theta}) \right) \mu_i(\pi) \cdot (2.4)
\]
Using these preferences we define ex-ante equilibrium:

**Definition 2.8** Fix a game $\Gamma$. A strategy profile $\sigma^*$ is an ex-ante equilibrium if, for all players $i$,

$$V_i(\sigma^P) \geq V_i(\sigma'_i, \sigma^P_{-i})$$

for all $\sigma'_i \in \Sigma_i$.

We next turn to defining preferences beyond the ex-ante stage. Given a player $i$ of type $\tau_i$ a partial history $h^t$, and a strategy profile $\sigma$, consider the set of type profiles consistent with $\tau_i$ that make $h^t$ reachable without requiring a deviation from $\sigma$ by players other than $i$. It is possible that this set might be empty (i.e., $h^t$ can be reached only by some player(s) deviating from $\sigma_{-i}$). For this reason, consider the furthest point back from $h^t$ for which there is some $\theta_{-i}$ such that getting to $h^t$ from that point requires no deviation from $\sigma_{-i}$. Note that such a point always exists, as $h^t$ is always reachable from $h^t$ itself. We will be interested in the set of type profiles consistent with $\tau_i$ that make $h^t$ reachable from such a point without requiring a deviation from $\sigma$ by players other than $i$. Formally this set is the following:

**Notation 2.1** $\Theta_{i,\tau_i,h^t} \equiv \{ \theta \in \Theta \mid \theta_i = \tau_i \text{ and } p_{-i,\sigma,\theta}(h^t|h_{m_i}(h^t)) > 0 \}$, where

$$m_i(h^t) \equiv \min \bigcup_{\theta} \{ r \in \{0,\ldots,t\} \mid p_{-i,\sigma,\theta}(h^t|h^r) > 0 \} .$$

Using $m_i(h^t)$ we can make precise what it means for one partial history to be reachable from another:

**Definition 2.9** Given a strategy profile $\sigma$ and an interim belief system $\nu$, player $i$ views partial history $h^t$ with $t \geq 1$ as reachable from $h^s$ (where $0 \leq s \leq t$) if $m_i(h^t) \leq s$.

The following expresses a defining property for interim (second-order) beliefs of player $i$ of type $\tau_i$ given partial history $h^t$ and strategy profile $\sigma$: that they assign weight only to type distributions that assign positive probability to type profiles in $\Theta_{i,\tau_i,h^t}$.

**Definition 2.10** An interim belief for player $i$ of type $\tau_i$ in a game $\Gamma$ given partial history $h^t$ and strategy profile $\sigma$ is a finite support probability measure $\nu_{i,\tau_i,h^t}$ over $\Delta(\Theta)$ such that

$$\nu_{i,\tau_i,h^t} \left( \{ \pi \in \Delta(\Theta) \mid \pi(\Theta_{i,\tau_i,h^t}) > 0 \} \right) = 1. \quad (2.5)$$

Given a strategy profile $\sigma$, an interim belief system $\nu \equiv (\nu_{i,\tau_i,h^t})_{i \in N, \tau_i \in \Theta_i, h^t \in H^t}$ is an interim belief for each type of each player at each partial history. The associated interim belief profile given partial history $h^t$ is $\nu^{h^t} \equiv (\nu_{i,\tau_i,h^t})_{i \in N, \tau_i \in \Theta_i}$.
Fundamental to our equilibrium notion will be sequential optimality. It requires that each player plays optimally for each partial history and each own-type realization given the strategies of the others. This optimality is required even when the partial history or own-type realization is before-the-fact viewed as a null event according to the given strategy profile combined with the beliefs of the player. In order to describe optimality for $i$ given $\tau_i$ and $h^t$, we need to write $i$’s conditional preferences. These make use of interim beliefs.

**Assumption 2.4** Fix a game $\Gamma$ and an interim belief system $\nu$. Any player $i$ of type $\tau_i$ at partial history $h^t$ ranks strategy profiles $\sigma$ according to

$$V_{i,\tau_i,h^t}(\sigma; \nu) \equiv \sum_{\pi \in \Delta(\Theta) | \pi(\Theta_{i,\tau_i,h^t}) > 0} \phi_i \left( \sum_{\delta \in \Theta} \sum_{\hat{h} \in H | h^t = \hat{h}} u_i(\hat{h}, \hat{\theta}) p_{i,\sigma,\hat{\theta}}(\hat{h} | h^t) p_{-i,\sigma,\hat{\theta}}(\hat{h} | h^t) \pi_{\Theta_{i,\tau_i,h^t}}(\hat{\theta}) \right) \nu_{i,\tau_i,h^t}(\pi),$$

where

$$\pi_{\Theta_{i,\tau_i,h^t}}(\theta) = \frac{p_{-i,\sigma,\theta}(h^t | h^t_{m_i(h^t)}) \pi(\theta)}{\sum_{\hat{\theta} \in \Theta_{i,\tau_i,h^t}} p_{-i,\sigma,\hat{\theta}}(h^t | h^t_{m_i(h^t)}) \pi(\hat{\theta})} \text{ if } \theta \in \Theta_{i,\tau_i,h^t} \text{ and } 0 \text{ otherwise.} \quad (2.7)$$

Compared to the ex-ante preferences given in (2.4), the conditional preferences (2.6) differ only in that (1) the beliefs may have changed in light of $\tau_i$ and $h^t$ ($\nu_i$ is replaced by $\nu_{i,\tau_i,h^t}$ and $\pi$ by $\pi_{\Theta_{i,\tau_i,h^t}}$), and (2) the probabilities of reaching various histories according to the strategy profile are now calculated starting from $h^t$ rather than from the beginning of the game. Observe that, while $\pi_{\Theta_{i,\tau_i,h^t}}$ is calculated using Bayes’ formula (see the Remark below), there is no restriction placed at this point on the $\nu$ other than (2.5).

**Remark 2.1** To see that the conditioning formula for $\pi$ in (2.7) is the usual Bayes’ formula, note that, because $\theta_i = \tau_i$ for all $\theta \in \Theta_i_{i,\tau_i,h^t}$,

$$\frac{p_{-i,\sigma,\theta}(h^t | h^t_{m_i(h^t)}) \pi(\theta)}{\sum_{\hat{\theta} \in \Theta_{i,\tau_i,h^t}} p_{-i,\sigma,\hat{\theta}}(h^t | h^t_{m_i(h^t)}) \pi(\hat{\theta})} = \frac{p_{\sigma,\theta}(h^t | h^t_{m_i(h^t)}) \pi(\theta)}{\sum_{\hat{\theta} \in \Theta_{i,\tau_i,h^t}} p_{\sigma,\hat{\theta}}(h^t | h^t_{m_i(h^t)}) \pi(\hat{\theta})}$$

if $\theta \in \Theta_{i,\tau_i,h^t}$ and $p_{i,\sigma,\theta}(h^t | h^t_{m_i(h^t)}) > 0$.

Furthermore, as long as $m_i(h^t) < t$ (so that one may go back at least one stage from $h^t$ without a deviation by players other than $i$ and therefore $m_i(h^t) = m_i(h^{t-1})$), such
conditional probabilities are also related by the one-step-ahead Bayes’ formula

\[
\pi_{\theta_{i,\tau_i,h^t}}(\theta) = \frac{p_{i,\sigma_i,h^t|h^{t-1}}\pi_{\theta_{i,\tau_i,h^{t-1}}}(\theta)}{\sum_{\theta' \in \Theta_{i,\tau_i,h^t}} p_{i,\sigma_i,h^t|h^{t-1}}\pi_{\theta_{i,\tau_i,h^{t-1}}}(\theta)} \text{ if } \theta \in \Theta_{i,\tau_i,h^t} \text{ and } 0 \text{ otherwise. (2.8)}
\]

Using these preferences, we may now define sequential optimality:

**Definition 2.11** Fix a game \( \Gamma \). A pair \((\sigma^P, \nu^P)\) consisting of a strategy profile and interim belief system is sequentially optimal if, for all players \( i \), all types \( \tau_i \) and all partial histories \( h^t \),

\[
V_i(\sigma^P) \geq V_i(\sigma_i', \sigma_{-i}^P)
\]

and

\[
V_{i,\tau_i,h^t}(\sigma^P; \nu^P) \geq V_{i,\tau_i,h^t}((\sigma_i', \sigma_{-i}^P); \nu^P)
\]

for all \( \sigma_i' \in \Sigma_i \), where the \( V_i \) and \( V_{i,\tau_i,h^t} \) are as specified in (2.4) and (2.6).

Note that since \( V_{i,\tau_i,h^t}(\sigma; \nu) = V_{i,\tau_i,h^t}(\hat{\sigma}; \nu) \) if \( \sigma^{h^t} = \hat{\sigma}^{h^t} \) for type \( \tau_i \), requiring the inequalities for the \( V_{i,\tau_i,h^t} \) to hold as \( i \) changes only her continuation strategy given \( h^t \) and \( \tau_i \) would result in an equivalent definition.

While sequential optimality does not place restrictions on the interim belief system beyond (2.10), observe that \( V_{i,\tau_i,h^t} \) does entail the assumption that, even at stages immediately following a deviation by some player other than \( i \), player \( i \) continues to assume that the players \(-i\) will play according to \( \sigma_{-i} \) from the current stage onward.

Assuming a common \( \mu \), sequential optimality implies subgame perfection adapted to allow for smooth ambiguity preferences. In multistage games with observed actions, the only proper subgames occur at stages where all type uncertainty (if any) has been resolved. For any such proper subgame, (2.10) ensures that the continuation strategy profile derived from \( \sigma^P \) forms a Nash equilibrium of the subgame. For the overall game, (2.9) ensures \( \sigma^P \) is an ex-ante equilibrium, which, with common \( \mu \), is the natural extension of Nash equilibrium to allow for smooth ambiguity preferences.

Sequential optimality identifies a set of strategy profiles. Each such profile is sequentially optimal with respect to some interim belief system. Recall that we have placed little restriction on how a player \( i \)'s beliefs at different points in the game relate to one another and to the ex-ante beliefs \( \mu_i \). We now show (Theorem 2.1) that every such profile is sequentially optimal with respect to an interim belief system generated by one particular update rule. This update rule was proposed by Hanany and Klibanoff (2009) and is called the smooth rule. The smooth rule is defined as follows:
Definition 2.12 An interim belief system \( \nu \) satisfies the smooth rule using \( \sigma \) as the ex ante equilibrium if the following holds for each player \( i \) and type \( \tau_i \): First, for all \( \pi \in \Delta(\Theta) \) such that \( \pi(\Theta_{i,\tau_i,\theta}) > 0 \),

\[
\nu_{i,\tau_i,\theta}(\pi) \propto \frac{\phi_i' \left( \sum_{\theta \in \Theta} \sum_{h \in H} u_i(h, \hat{\theta}) p_{\sigma,\hat{\theta}}(h|\hat{\theta}) \pi(\hat{\theta}) \right)}{\phi_i' \left( \sum_{\theta \in \Theta} \sum_{h \in H} u_i(h, \hat{\theta}) p_{\sigma,\hat{\theta}}(h|\hat{\theta}) \pi_{\Theta_{i,\tau_i,\theta}}(\hat{\theta}) \right)} \pi(\Theta_{i,\tau_i,\theta}) \mu_i(\pi);
\]

Second, for each partial history \( h^t \) such that \( i \) views \( h^t \) as reachable from \( h^{t-1} \), for all \( \pi \in \Delta(\Theta) \) such that \( \pi(\Theta_{i,\tau_i,h^t}) > 0 \),

\[
\nu_{i,\tau_i,h^t}(\pi) \propto \frac{\phi_i' \left( \sum_{\theta \in \Theta} \sum_{\hat{h} \in H|h^{t-1} = h^t} u_i(h, \hat{\theta}) p_{\sigma,\hat{\theta}}(h|\hat{\theta}) \pi_{\Theta_{i,\tau_i,h^t-1}}(\hat{\theta}) \right)}{\phi_i' \left( \sum_{\theta \in \Theta} \sum_{\hat{h} \in H|h^{t-1} = h^t} u_i(h, \hat{\theta}) p_{\sigma,\hat{\theta}}(h|\hat{\theta}) \pi_{\Theta_{i,\tau_i,h^t}}(\hat{\theta}) \right)} \cdot \left( \sum_{\hat{\theta} \in \Theta_{i,\tau_i,h^t}} p_{-i,\sigma,\hat{\theta}}(h^t|h^{t-1}) \pi_{\Theta_{i,\tau_i,h^t-1}}(\hat{\theta}) \right) \nu_{i,\tau_i,h^{t-1}}(\pi).
\]

Note that under ambiguity neutrality (\( \phi_i \) linear, which is expected utility), \( \phi_i' \) is constant, and thus the \( \phi_i' \) terms appearing in the formula cancel and the smooth rule coincides with standard Bayesian updating. More generally, the \( \phi_i' \) ratio terms, which reflect changes in the motive to hedge against ambiguity (see Hanany and Klibanoff 2009 and Baliga, Hanany and Klibanoff 2013), are the only difference from Bayesian updating. These changes can be motivated via dynamic consistency. For ambiguity averse preferences, Bayesian updating does not ensure dynamic consistency. The smooth rule is dynamically consistent for all ambiguity averse smooth ambiguity preferences (Hanany and Klibanoff 2009).

We now show that, for the purposes of identifying sequentially optimal strategy profiles, restricting attention to beliefs updated according to the smooth rule is without loss of generality. Specifically, considering only interim belief systems satisfying the smooth rule yields the entire set of sequentially optimal strategy profiles. Later in the paper (Theorem 2.3), we show that with smooth rule updating, sequential optimality is equivalent to the absence of profitable one-stage deviations.

Theorem 2.1 Fix a game \( \Gamma \). Suppose \((\sigma^P,\nu^P)\) is sequentially optimal. Then, there exists an interim belief system \( \hat{\nu}^P \) satisfying the smooth rule using \( \sigma^P \) as the ex ante equilibrium
such that \((\sigma^P, \hat{\nu}^P)\) is sequentially optimal.

**Proof.** We show that \((\sigma^P, \hat{\nu}^P)\), where, for all \(i, \tau_i, h^t_i, \nu^P_{i, \tau_i, h^t_i} = \nu^P_{i, \tau_i, h^t_i}\) whenever \((t > 0, \text{and } m_i(h^t) = t)\) or \((t = 0, \text{and } \tau_i \text{ is } \mu_i\text{-null})\), and where, everywhere else, \(\hat{\nu}^P_{i, \tau_i, h^t_i}\) is derived via the smooth rule, is sequentially optimal. First, observe that \(\hat{\nu}^P\) does not enter into the function \(V_i\), so the fact that \((\sigma^P, \nu^P)\) is sequentially optimal directly implies that \(V_i(\sigma^P) \geq V_i(\sigma'_i, \sigma^P_{-i})\) for all \(\sigma'_i \in \Sigma_i\). Second, by construction, \(\hat{\nu}^P\) satisfies the smooth rule using \(\sigma^P\) as the ex ante equilibrium except, possibly, for \(i, \tau_i, h^t\) where \((t > 0, \text{and } m_i(h^t) = t)\) or \((t = 0, \text{and } \tau_i \text{ is } \mu_i\text{-null})\). However, from the definition of the smooth rule (Definition 2.12), observe that it is exactly for \(i, \tau_i, h^t\) where \((t > 0, \text{and } m_i(h^t) = t)\) or \((t = 0, \text{and } \tau_i \text{ is } \mu_i\text{-null})\) for which the smooth rule allows any interim beliefs. Thus \(\hat{\nu}^P\) satisfies the smooth rule using \(\sigma^P\) as the ex ante equilibrium. Finally, to see that \((\sigma^P, \hat{\nu}^P)\) satisfies \(V_i(\tau_i, h^t; (\sigma^P; \hat{\nu}^P)) \geq V_i(\tau_i, h^t; ((\sigma'_i, \sigma^P_{-i}); \nu^P))\) for all \(\sigma'_i \in \Sigma_i\), observe that (a) for \(i, \tau_i, h^t\) such that either \((h^t \neq \emptyset \text{ and } m_i(h^t) = t)\) or \((h^t = \emptyset \text{ and } \tau_i \text{ is } \mu_i\text{-null})\), it directly inherits this from \((\sigma^P, \nu^P)\) and (b) everywhere else, Lemma 5.2 shows that smooth rule updating ensures the required optimality.

Thus, sequential optimality plus requiring smooth rule updating identifies the same set of strategy profiles as sequential optimality alone. Note that the theorem would be false if we were to replace the smooth rule with Bayes’ rule – restricting attention to interim belief systems satisfying Bayesian updating generally rules out some (or all) sequentially optimal strategies. A Bayesian version of the theorem is true, however, if we restrict attention to expected utility preferences, for in that case the smooth rule and Bayes’ rule agree. Perfect Bayesian Equilibrium (PBE) imposes sequential optimality (defined using only expected utility preferences) and also that beliefs are related via Bayesian updating wherever possible (plus some auxiliary conditions, which we ignore for now). From our theorem, it follows that in the expected utility case, sequential optimality alone (i.e., without additionally requiring Bayesian updating) identifies the same set of strategy profiles as sequential optimality plus Bayesian updating. This Bayesian version of the result was first shown by Shimoji and Watson (1998) in the context of defining extensive form rationalizability.

To describe our proposed Perfect Equilibrium with Ambiguity (PEA) notion, we consider an auxiliary condition stated by some versions of PBE that relates to beliefs exactly at those points where sequential optimality has no implications for updating. These points are those a player \(i\) thinks are only reached immediately following deviation(s) of other player(s). Formally, these are partial histories \(h^t\) such that \(t > 0\) and \(m_i(h^t) = t\). The condition requires that, if players’ types are viewed as independent, there should be no updating of player \(i\)’s belief about player \(j\)’s type immediately following a partial history at which player \(j\) has no choice (i.e., only one action) available. This reflects the idea (see e.g., Fudenberg and Tirole 1991, JET, p. 241) that when players’ types are independent, only player \(j\) has
information to reveal about her own type and so $i$’s beliefs about player $j$’s type should not be affected by another player’s deviation. When $j$ has only one action, she has no means to reveal anything, and so, absent reasons related to hedging against ambiguity, player $i$ should not change her marginal on $j$’s type.

To formalize this in our setting we need to define $i$’s marginal on $j$’s type, as well as a condition ensuring that no change in ambiguity hedging concerns occurs in moving from $h_{t-1}^i$ to $h_t^i$.

**Definition 2.13** Given an interim belief system $\nu$, player $i$’s marginal on player $j$’s type at partial history $h_t^i$ is

$$\sum_{\pi \in \Delta(\Theta)} \pi_{\Theta_{i,\tau,ht}}\left(\{\theta_j\} \times \Theta_{-j}\right) \nu_{i,\tau,ht}(\pi).$$

**Definition 2.14** Given a strategy profile $\sigma$ and an interim belief system $\nu$, if player $i$ does not view a partial history $h_t^i$ with $t \geq 1$ as reachable from $h_{t-1}^i$, $i$ has no costly ambiguity exposure under $\sigma$ at $h_{t-1}^i$ and $h_t^i$ if

$$\phi_i^t\left(\sum_{\hat{h} \in H} \sum_{\hat{\theta} \in \Theta \hat{h} \in H|\hat{h}|=ht} u_i(\hat{h}, \hat{\theta})p_{\sigma,\hat{h}}(\hat{h}|h_{t-1}^i)\pi_{\Theta_{i,\tau,ht}}(\hat{\theta})\right)$$

is constant for all $\pi$ in the support of $\nu_{i,\tau,ht}$, and

$$\phi_i^t\left(\sum_{\hat{h} \in H} \sum_{\hat{h} \in H|\hat{h}|=ht} u_i(\hat{h}, \hat{\theta})p_{\sigma,\hat{h}}(\hat{h}|h_t^i)\pi_{\Theta_{i,\tau,ht}}(\hat{\theta})\right)$$

is constant for all $\pi$ in the support of $\nu_{i,\tau,ht}$.

Observe that there are essentially two ways that a player $i$ could have no costly ambiguity exposure under $\sigma$ at $h_{t-1}^i$ and $h_t^i$ – strategies might be such that $i$ is fully hedged against ambiguity from the points of view of the two partial histories (i.e., the conditional expected utility arguments of $\phi_i^t$ in the definition do not vary with $\pi$) or, where $i$ is exposed to fluctuations in these conditional expected utilities, $\phi_i^t$ is constant (i.e., $i$ is ambiguity neutral over some range) so the ambiguity exposure is not costly. We can now state our auxiliary condition:

**Definition 2.15** Fix a game $\Gamma$. A pair $(\sigma^P, \nu^P)$ consisting of a strategy profile and interim belief system naturally extends updating if, for all players $i, j \neq i$, all types $\tau_i$ and all partial histories $h_t^i$ with $t \geq 1$, if
(a) player $i$ does not view $h^t$ as reachable from $h^{t-1}$,
(b) $\sum_{\pi \in \Delta(\Theta)} \pi \omega^{\pi}_{i,\tau_i,h^{t-1}} \nu^P_{i,\tau_i,h^{t-1}}(\pi) \in \Delta(\Theta)$ is a product measure,
(c) player $j$ has no choice (i.e., only one action) available at $h^{t-1}$ and
(d) $i$ has no costly ambiguity exposure under $\sigma$ at $h^{t-1}$ and $h^t$;
then

$i$’s marginal on player $j$’s type at partial history $h^t$ must remain the same as it would be at $h^{t-1}$ if the smooth rule using $\sigma^P$ as the ex ante equilibrium were used to derive $\nu_{i,\tau_i,h^t}$ from $\nu^P_{i,\tau_i,h^m_i(h^{t-1})}$.

In the case where players are ambiguity neutral, Definition 2.15 is implied by Fudenberg and Tirole’s (1991b) PBE requirement that Bayes’ rule is used to update beliefs whenever possible (see Fudenberg and Tirole (1991b) condition (1) of Definition 3.1, p.242).

Adding this condition to sequential optimality leads to the following equilibrium definition.

**Definition 2.16** A perfect equilibrium with ambiguity (PEA) of a game $\Gamma$ is a pair $(\sigma^P, \nu^P)$ consisting of a strategy profile and interim belief system such that $(\sigma^P, \nu^P)$ is sequentially optimal and naturally extends updating.

Theorem 2.1 showed that interim belief systems using smooth rule updating generate all sequentially optimal strategy profiles. Similarly, they generate all PEA strategy profiles.

**Corollary 2.1** Fix a game $\Gamma$. Suppose $(\sigma^P, \nu^P)$ is a PEA of $\Gamma$. Then, there exists an interim belief system $\hat{\nu}^P$ satisfying the smooth rule using $\sigma^P$ as the ex ante equilibrium such that $(\sigma^P, \hat{\nu}^P)$ is a PEA of $\Gamma$.

**Proof.** We show that $(\sigma^P, \hat{\nu}^P)$, where, for all $i, \tau_i, \hat{\nu}^P_{i,\tau_i,h^t} = \nu^P_{i,\tau_i,h^t}$ whenever $(t > 0$ and $m_i(h^t) = t)$ or $(t = 0$ and $\tau_i$ is $\mu$-null), and where, everywhere else, $\hat{\nu}^P_{i,\tau_i,h^t}$ is derived via the smooth rule, is a PEA. By the proof of Theorem 2.1, $(\sigma^P, \hat{\nu}^P)$ is sequentially optimal. It remains to show that it naturally extends updating. This imposes restrictions on $i$’s beliefs only at partial histories $h^t$ where $i$ does not view $h^t$ as reachable from $h^{t-1}$. By construction of $\hat{\nu}^P$, at all such $h^t$, $\hat{\nu}^P_{i,\tau_i,h^t} = \nu^P_{i,\tau_i,h^t}$. Thus $(\sigma^P, \hat{\nu}^P)$ naturally extends updating because $(\sigma^P, \nu^P)$ does. □

Now that PEA has been defined, we show that every game $\Gamma$ has at least one PEA. Since the functions $\phi_i$ describing players’ ambiguity attitudes are part of the description of $\Gamma$, this result goes beyond the observation that a PEA would exist if players were ambiguity neutral, and ensures existence given any specified ambiguity aversion.

**Theorem 2.2** A PEA exists for any game $\Gamma$. 14
Proof. Fix a sequence $\varepsilon^k = (\varepsilon^k_{\theta_j, h^t})_{\theta_j \in \mathcal{N}, h^t \in \mathcal{H}^t}$ of strictly positive vectors of dimension $\left| \bigcup_{t \in T} H^t \right| \sum_{j \in N} |\Theta_j|$, converging in the sup-norm to 0 and such that $\varepsilon^k_{\theta_j, h^t} \leq \frac{1}{A^j_t(h^t)}$ for all types $\theta_j$ and all histories $h$ and all stages $t$. For any $k$, let $\Gamma^k$ be the restriction of the game $\Gamma$ defined such that the set of feasible strategy profiles is the set of all completely mixed $\sigma^k$ satisfying $\sigma^k_j(h^t, \theta_j)(a^t_j) \geq \varepsilon^k_{\theta_j, h^t}$ for all $j$, $\theta_j$, $h^t$ and actions $a^t_j \in A^j_t(h^t)$. Consider the agent normal form $G^k$ of the game $\Gamma^k$ (see e.g., Myerson, 1991, p.61). Since the payoff functions are concave and the set of strategies of each player in $G^k$ is non-empty, compact and convex, $G^k$ has an ex-ante equilibrium. Let $\hat{\sigma}^k$ be the strategy profile in the game $\Gamma^k$ corresponding to this equilibrium. By concavity of the payoff functions, $\hat{\sigma}^k$ is an ex-ante equilibrium of $\Gamma^k$. Let $\hat{\nu}^k$ be a belief system in $\Gamma^k$ that satisfies the smooth rule using $\hat{\sigma}^k$ as the ex ante equilibrium. Since the naturally extends updating condition restricts beliefs only off-path and $\hat{\sigma}^k$ is completely mixed, this condition is trivially satisfied by $(\hat{\sigma}^k, \hat{\nu}^k)$. Since the strategy of each player $(h^t, \theta_j)$ in $G^k$ according to $\hat{\sigma}^k$ is an ex-ante best response to $\hat{\sigma}^k$, and since all partial histories $h^t$ are on the equilibrium path, by Lemma 5.2 smooth rule updating ensures that $(\hat{\sigma}^k, \hat{\nu}^k)$ is sequentially optimal and is thus a PEA for $\Gamma^k$. By compactness of the set of strategy profiles, the sequence $\hat{\sigma}^k$ has a convergent sub-sequence, the limit of which is denoted by $\hat{\sigma}$. By continuity of the payoff functions, $\hat{\sigma}$ is an ex-ante equilibrium of $\Gamma$. Inspection of Definition 2.12 reveals that the beliefs generated by the smooth rule vary continuously in the ex-ante equilibrium $\sigma$, as $\sigma$ enters continuously in $p_{i, \sigma, \theta}^t(h^t | h^r)$ and $p_{-i, \sigma, \theta}^t(h^t | h^r)$ and only affects $m_i(h^t)$ when the weight on some action hits zero, in which case the smooth rule becomes less restrictive and so the same beliefs can be maintained in that case. By this continuity of the smooth rule in the ex-ante equilibrium $\hat{\sigma}^k$, the associated sub-sequence of $\hat{\nu}^k$ converges to a limit denoted by $\hat{\nu}$. Given any partial history $h^t$ and continuation strategy $\hat{\sigma}^t_j$ of player $j$ of type $\theta_j$ in $\Gamma$, let $\tilde{\sigma}^{k,h^t}_j$ be a feasible strategy in $\Gamma^k$ for this player that is closest (in the sup-norm) to $\hat{\sigma}^t_j$. Since for each $k$, $\tilde{\sigma}^{k,h^t}_j$ is weakly better than $\hat{\sigma}^{k,h^t}_j$ for player $j$ of type $\theta_j$ given belief $\hat{\nu}^k_{j, \theta_j, h^t}$, and since, along the sub-sequence, $\tilde{\sigma}^{k,h^t}_j$ converges to $\hat{\sigma}^t_j$ and $\hat{\nu}^k$ converges to $\hat{\nu}$, continuity of the payoff functions implies that $\hat{\sigma}^{h^t}_j$ is weakly better than $\tilde{\sigma}^{h^t}_j$ for this player given belief $\hat{\nu}_{j, \theta_j, h^t}$. Therefore $(\hat{\sigma}, \hat{\nu})$ satisfies sequential optimality. Observe that $(\hat{\sigma}, \hat{\nu})$ satisfies smooth rule consistency (since it is explicitly constructed as the limit of an appropriate sequence). Therefore Lemma 5.1 implies that it naturally extends updating. Therefore $(\hat{\sigma}, \hat{\nu})$ is a PEA of $\Gamma$. ■

To verify that a pair $(\sigma, \nu)$ where $\nu$ obeys smooth rule updating is sequentially optimal it is sufficient to check for profitable one-stage deviations. To show this, we formally define what it means to have no profitable one-stage deviations and what we mean by obeying the smooth rule in this context.
Definition 2.17 The pair \((\sigma, \nu)\) satisfies the no profitable one-stage deviation property if for each player \(i\) and each partial history \(h^t\), \(\sigma_i\) is optimal for \(i\) given interim belief \(\nu_{i,\tau_i, h^t}\) among all \(\sigma_i'\) satisfying \(\sigma'_i(\eta, \tau_i) = \sigma_i(\eta, \tau_i)\) for all \(\eta \in \bigcup_{t \in [T]} \mathcal{H}^t\) such that \(\eta \neq h^t\).

In order to describe conditions on updating under which the no profitable one-stage deviation property implies that \((\sigma, \nu)\) is a PEA, it is helpful to apply the smooth rule given a strategy profile \(\sigma\), even when \(\sigma\) is not necessarily an ex ante equilibrium. We say that an interim belief system \(\nu\) satisfies extended smooth rule updating using \(\sigma\) as the strategy profile if \(\nu\) satisfies the conditions in Definition 2.12 with respect to \(\sigma\).

It will also be useful for the proof of the next result to refer to a player’s “local ambiguity neutral measure” after some partial history in the game. Given \((\sigma, \nu)\), for any player \(i\) and type \(\tau_i\), let \(q^{\sigma,i}(h, \theta)\) denote \(i\)'s ex-ante \(\sigma\)-local measure, defined for each \(\theta \in \Theta\) and \(h \in \mathcal{H}\) by

\[
q^{\sigma,i}(h, \theta) \equiv \sum_{\pi \in \Delta(\Theta \times \mathcal{H})} \phi_i' \left( \sum_{\hat{h} \in H} \sum_{\theta' \in \Theta} u_i(\hat{h}, \theta') p_{\sigma, \hat{h}}(\hat{h} | h^0) \pi(\theta') \right) \cdot p_{-i, \sigma \theta}(h | h^0) \pi(\theta) \mu_i(\pi);
\]

additionally, for any partial history \(\eta \in \bigcup_{t \in [T]} \mathcal{H}^t\), let \(q^{(\sigma, \nu),i, \tau_i, \eta}(h, \theta)\) denote \(i\)'s \((\sigma, \nu)\)-local measure given \(\tau_i\) and \(\eta\), defined for each \(\theta \in \Theta\) and \(h \in \mathcal{H}\) with \(\theta_i = \tau_i\) and \(h^t = \eta\) and by

\[
q^{(\sigma, \nu),i, \tau_i, \eta}(h, \theta) \equiv \sum_{\pi \in \Delta(\Theta' \times \mathcal{H}) | \pi(\Theta_i, \tau_i, \eta) > 0} \phi_i' \left( \sum_{\hat{h} \in \mathcal{H}|h^t = \eta} \sum_{\theta' \in \Theta} u_i(\hat{h}, \theta') p_{\sigma, \hat{h}}(\hat{h} | h^0) \pi_{\Theta_i, \tau_i, \eta}(\theta') \right) \cdot p_{-i, \sigma \theta}(h | h^0) \pi_{\Theta_i, \tau_i, \eta}(\theta) \nu_i, \tau_i, \eta(\pi).
\]

Theorem 2.3 If \((\sigma, \nu)\) satisfies the no profitable one-stage deviation property and \(\nu\) satisfies extended smooth rule updating using \(\sigma\) as the strategy profile then \((\sigma, \nu)\) is sequentially optimal. If, in addition, \((\sigma, \nu)\) naturally extends updating, then \((\sigma, \nu)\) is a PEA.

Proof. Suppose that \((\sigma, \nu)\) satisfies the no profitable one-stage deviation property and naturally extends updating and \(\nu\) satisfies extended smooth rule updating using \(\sigma\) as the strategy profile. First, for each player \(i\), the no profitable one-stage deviation property implies conditional optimality of \(\sigma_i\) according to \(\nu_{i,\tau_i, \eta^T}\) for all \(\eta^T \in \mathcal{H}^T\). Next we proceed by induction on the stage \(t\). Fix any \(t\) such that \(0 < t < T\), and suppose that, for each player \(i\), \(\sigma_i\) is conditionally optimal according to \(\nu_{i,\tau_i, \eta^t}\) for all \(\eta^t \in \mathcal{H}^t\). We claim that, for each player \(i\), \(\sigma_i\) is conditionally optimal according to \(\nu_{i,\tau_i, \eta^{t-1}}\) for all \(\eta^{t-1} \in \mathcal{H}^{t-1}\). The argument for this is as follows: Fix \(\eta^{t-1} \in \mathcal{H}^{t-1}\) and fix a player \(i\). Consider any strategy \(\sigma'_i\) for player
i. For any \( h \in \mathcal{H} \) such that \( h_{t-1} = \eta_{t-1} \) and \( i \) views \( h_t \) as reachable from \( \eta_{t-1} \), the conditional optimality of \( \sigma_i \) according to \( \nu_{i,\tau_i,h_t} \) implies (see HK 2009, Lemma A.1)

\[
\sum_{\theta \in \Theta} \sum_{\theta_i = \tau_i, h \in H | h_t = h^t} u_t(h, \theta) p_{i,\sigma,\theta}(\bar{h} | h^t) q(\sigma,\nu, i, \tau_i, h^t | h, \bar{\theta}) \geq \sum_{\theta \in \Theta} \sum_{\theta_i = \tau_i, h \in H | h_t = h^t} u_t(h, \theta) p_{i,\sigma,\theta,-i}(\bar{h} | h^t) q(\sigma,\nu, i, \tau_i, h^t | h, \bar{\theta}).
\]

(2.13)

Since \( i \)'s preferences satisfy extended smooth rule updating using \( \sigma \), for all such \( h, q(\sigma,\nu, i, \tau_i, h^t | h, \bar{\theta}) \propto q(\sigma,\nu, i, \tau_i, \eta_{t-1} | h, \bar{\theta}) \) for all \( \bar{\theta} \in \Theta \) and \( h \in H \) with \( \bar{\theta}_i = \tau_i \) and \( h^t = h^t \). Thus, after substituting for \( q(\sigma,\nu, i, \tau_i, h^t | h, \bar{\theta}) \), cancelling the constant of proportionality and multiplying by \( p_{i,\sigma,\theta,-i}(\bar{h} | \eta_{t-1}) \) (which is identical for all \( \bar{\theta} \) with \( \bar{\theta}_i = \tau_i \), as it depends only on \( \tau_i \), and for all \( h \) with \( h^t = h^t \)), (2.13) becomes

\[
\sum_{\theta \in \Theta} \sum_{\theta_i = \tau_i, h \in H | h_t = h^t} u_t(h, \theta) p_{i,\sigma,\theta}(\bar{h} | h^t) q(\sigma,\nu, i, \tau_i, \eta_{t-1} | h, \bar{\theta}) \geq \sum_{\theta \in \Theta} \sum_{\theta_i = \tau_i, h \in H | h_t = h^t} u_t(h, \theta) p_{i,\sigma,\theta,-i}(\bar{h} | h^t) q(\sigma,\nu, i, \tau_i, \eta_{t-1} | h, \bar{\theta}) = \sum_{\theta \in \Theta} \sum_{\theta_i = \tau_i, h \in H | h_t = h^t} u_t(h, \theta) p_{i,\sigma,\theta,-i}(\bar{h} | h^t) q(\sigma,\nu, i, \tau_i, \eta_{t-1} | h, \bar{\theta}).
\]

Summing for all such \( h \), yields:

\[
\sum_{\theta \in \Theta} \sum_{\theta_i = \tau_i, h \in H | h_t = h^t} u_t(h, \theta) p_{i,\sigma,\theta}(\bar{h} | h^t) q(\sigma,\nu, i, \tau_i, \eta_{t-1} | h, \bar{\theta}) \geq \sum_{\theta \in \Theta} \sum_{\theta_i = \tau_i, h \in H | h_t = h^t} u_t(h, \theta) p_{i,\sigma,\theta,-i}(\bar{h} | h^t) q(\sigma,\nu, i, \tau_i, \eta_{t-1} | h, \bar{\theta})
\]

where the second and third equalities follow since, for all \( \bar{\theta} \in \Theta \mid \bar{\theta}_i = \tau_i \) and all \( h \in H \mid \eta_{t-1} = \eta_{t-1} \) with \( m_i(h^t) > t - 1 \), \( q(\sigma,\nu, i, \tau_i, \eta_{t-1} | h, \bar{\theta}) = 0 \) because \( p_{-i,\sigma,\theta}(\bar{h} | \eta_{t-1}) = 0 \).

Since \((\sigma, \nu)\) satisfies the no profitable one-stage deviation property, the conditional optimality
of \( \sigma_i \) according to \( \nu_{i,\tau_i,\eta^t-1} \) among all strategies deviating only at \( t-1 \) implies

\[
\sum_{\hat{\theta} \in \Theta \mid \hat{\theta}_i = \tau_i} \sum_{\bar{h} \in H \mid H^{t-1} = \eta^t-1} u_i(\bar{h}, \hat{\theta}) p_{i,\sigma_i,\hat{\theta}}(\bar{h} | \eta^{t-1}) q^{(\sigma,\nu), i, \tau_i, \eta^t}(\bar{h}, \hat{\theta}) \geq \sum_{\hat{\theta} \in \Theta \mid \hat{\theta}_i = \tau_i} \sum_{\bar{h} \in H \mid H^{t-1} = \eta^t-1} u_i(\bar{h}, \hat{\theta}) p_{i,\sigma'_i,\hat{\theta}}(\bar{h} | \eta^{t-1}) p_{i,\sigma,\hat{\theta}}(\bar{h} | \eta^t) q^{(\sigma,\nu), i, \tau_i, \eta^t}(\bar{h}, \hat{\theta}).
\]  

(2.15)

Combining (2.14) and (2.15) implies

\[
\sum_{\hat{\theta} \in \Theta \mid \hat{\theta}_i = \tau_i} \sum_{\bar{h} \in H \mid H^{t-1} = \eta^t-1} u_i(\bar{h}, \hat{\theta}) p_{i,\sigma_i,\hat{\theta}}(\bar{h} | \eta^{t-1}) q^{(\sigma,\nu), i, \tau_i, \eta^t}(\bar{h}, \hat{\theta}) \geq \sum_{\hat{\theta} \in \Theta \mid \hat{\theta}_i = \tau_i} \sum_{\bar{h} \in H \mid H^{t-1} = \eta^t-1} u_i(\bar{h}, \hat{\theta}) p_{i,\sigma'_i,\hat{\theta}}(\bar{h} | \eta^{t-1}) q^{(\sigma,\nu), i, \tau_i, \eta^t}(\bar{h}, \hat{\theta}).
\]  

(2.16)

Since (2.16) holds for any \( \sigma'_i, \sigma_i \) is conditionally optimal according to \( \nu_{i,\tau_i,\eta^t-1} \). Since this conclusion holds for any \( \eta^t-1 \in H^{t-1} \), the induction step is completed. It follows that \( (\sigma, \nu) \) satisfies the second set of inequalities in the definition of sequentially optimal.

To show it also satisfies the first set of inequalities, note that for any \( \tau_i \in \Theta_i \), the conditional optimality of \( \sigma_i \) according to \( \nu_{i,\tau_i,\hat{\theta}} \) implies

\[
\sum_{\hat{\theta} \in \Theta \mid \hat{\theta}_i = \tau_i} \sum_{\bar{h} \in H} u_i(\bar{h}, \hat{\theta}) p_{i,\sigma_i,\hat{\theta}}(\bar{h} | \eta^0) q^{(\sigma,\nu), i, \tau_i, \hat{\theta}}(\bar{h}, \hat{\theta}) \geq \sum_{\hat{\theta} \in \Theta \mid \hat{\theta}_i = \tau_i} \sum_{\bar{h} \in H} u_i(\bar{h}, \hat{\theta}) p_{i,\sigma'_i,\hat{\theta}}(\bar{h} | \eta^0) q^{(\sigma,\nu), i, \tau_i, \hat{\theta}}(\bar{h}, \hat{\theta}).
\]  

(2.17)

Since \( \nu_i \)’s preferences satisfy extended smooth rule updating using \( \sigma \), for all \( \tau_i, q^{(\sigma,\nu), i, \tau_i, \hat{\theta}}(\bar{h}, \hat{\theta}) \propto q^{|\sigma,i}(\bar{h}, \hat{\theta}) \) for all \( \hat{\theta} \in \Theta \) and \( \bar{h} \in H \). Substituting back into (2.17), cancelling the constant of proportionality and summing for all \( \tau_i \), yields:

\[
\sum_{\hat{\theta} \in \Theta} \sum_{\bar{h} \in H} u_i(\bar{h}, \hat{\theta}) p_{i,\sigma_i,\hat{\theta}}(\bar{h} | \eta^0) q^{|\sigma,i}(\bar{h}, \hat{\theta}) \geq \sum_{\hat{\theta} \in \Theta} \sum_{\bar{h} \in H} u_i(\bar{h}, \hat{\theta}) p_{i,\sigma'_i,\hat{\theta}}(\bar{h} | \eta^0) q^{|\sigma,i}(\bar{h}, \hat{\theta}).
\]  

(2.18)

Since (2.18) holds for any \( \sigma'_i, (\sigma, \nu) \) also satisfies the first set of inequalities in the definition of sequentially optimal, thus it is sequentially optimal. Therefore \( (\sigma, \nu) \) is a PEA. ■

The above has shown that, fixing the strategies of \( i \)’s opponents, at any stage in the game where \( i \) has a non-trivial strategy choice, \( i \) may identify an optimal strategy for the remainder of the game using a “folding back” algorithm. It works as follows – for each partial history
leading to a final stage where \( i \) may move, \( i \) calculates an optimal mixture over the actions she has available at that stage given the partial history, the strategies of the other players and \( i \)'s beliefs about types. Then, holding these fixed, \( i \) repeats this process for partial histories one stage earlier in the game, and so on. The only thing that (in common with the standard approach under ambiguity neutrality) cannot be calculated via folding back are the beliefs about types at each partial history. These may be determined by updating according to the smooth rule (which reduces to Bayes' rule under ambiguity neutrality). Recall that smooth rule updating is without loss of generality for the purposes of identifying sequentially optimal strategies or PEA.

2.1 Sequential Equilibrium with Ambiguity

In this section, we consider a stronger auxiliary condition than naturally extending updating. This condition is an extension of Kreps and Wilson’s (1982) consistency to accommodate ambiguity aversion by replacing Bayes’ rule in their definition with the (extended) smooth rule. Recall that if we simply limited attention to Bayesian updating then sequentially optimal strategies might fail to exist because Bayesian updating is dynamically inconsistent under ambiguity aversion. Also recall that this is a true extension because Bayes’ rule and the smooth rule coincide under ambiguity neutrality.

**Definition 2.18** Fix a game \( \Gamma \). A pair \((\sigma^S, \nu^S)\) consisting of a strategy profile and interim belief system satisfies smooth rule consistency if there exists a sequence of completely mixed strategy profiles \(\{\sigma^k\}_{k=1}^{\infty}\), with \(\lim_{k \to \infty} \sigma^k = \sigma^S\), such that \(\nu^S = \lim_{k \to \infty} \nu^k\), where \(\nu^k\) is determined by extended smooth rule updating using \(\sigma^k\) as the strategy profile.

**Definition 2.19** A sequential equilibrium with ambiguity (SEA) of a game \(\Gamma\) is a pair \((\sigma^S, \nu^S)\) consisting of a strategy profile and interim belief system such that \((\sigma^S, \nu^S)\) is sequentially optimal and satisfies smooth rule consistency.

The following result tells us that smooth rule consistency implies naturally extending updating, and thus any SEA is also a PEA.

**Theorem 2.4** Fix a SEA, \((\sigma^S, \nu^S)\), of a game \(\Gamma\). \((\sigma^S, \nu^S)\) is also a PEA of \(\Gamma\).

**Proof.** That \((\sigma^S, \nu^S)\) satisfies sequential optimality follows directly from the definition of a SEA. By Lemma 5.1, and the fact that \((\sigma^S, \nu^S)\) satisfies smooth rule consistency, \((\sigma^S, \nu^S)\) naturally extends updating and is thus a PEA of \(\Gamma\). ■

Despite being a stronger equilibrium notion, existence is maintained:
Theorem 2.5  A SEA exists for any game $\Gamma$.

Proof. For any game $\Gamma$, the argument in the proof of Theorem 2.2 constructs a SEA of $\Gamma$. ■

Finally, we show that smooth rule consistency implies extended smooth rule updating in the limit, and use this plus Theorem 2.3 to conclude that replacing sequential optimality in the definition of SEA by the no profitable one-stage deviation property would not change the set of equilibria.

Lemma 2.1 If $(\sigma, \nu)$ satisfies smooth rule consistency then $\nu$ satisfies extended smooth rule updating using $\sigma$ as the strategy profile.

Proof. By smooth rule consistency and upper semi-continuity of the extended smooth rule in the strategy profile at $i, \tau_i, h^t$ such that $i$ does not view $h^t$ as reachable from $h^{t-1}$ in the limit and continuity of the extended smooth rule in the strategy profile everywhere else, $\nu$ satisfies extended smooth rule updating using $\sigma$ as the strategy profile. (The upper semi-continuity comes from the fact that at unreachable partial histories the extended smooth rule is less restrictive than at reachable partial histories.) ■

Corollary 2.2 $(\sigma, \nu)$ satisfies the no profitable one-stage deviation property and smooth rule consistency if and only if $(\sigma, \nu)$ is an SEA.

Proof. It is enough to show that the no profitable one-stage deviation property and smooth rule consistency imply $(\sigma, \nu)$ is sequentially optimal. This follows directly from Lemma 2.1 and Theorem 2.3. ■

3 Modeling Strategic Ambiguity

At first glance, since the only source of ambiguity in our framework is ambiguity about players’ types, one might think that this is too restrictive to address strategic ambiguity (i.e., ambiguity about the strategies of other players). We show that, in fact, ambiguous strategies may be modeled within our framework. The approach builds on that introduced by Bade (2011) in normal form games who in turn built on Aumann (1974). The basic idea is as follows: in the framework of this paper, players may perceive ambiguity about types. Thus, a type-contingent strategy of a single player may be viewed as ambiguous since the ambiguity about that player’s type will translate through to ambiguity about that player’s actions. However, since players’ payoff functions may also depend on the types, this method of generating strategic ambiguity might in general be confounded with the desire to
make actions type contingent due to this payoff dependence. To allow for “pure” strategic ambiguity, we can impose some structure on the type space so that some aspects of a player’s type are assumed not to affect payoffs, i.e., players’ payoff functions are constant with respect to those aspects of the types. In such a game, if a player prefers, in equilibrium, to make his strategy responsive to the realization of such “action” types, the only reason for this can be a desire to affect the ambiguity that other players’ perceive about his strategy. One may think of a mixed strategy as choosing a strategy contingent on the outcome of a roulette wheel or other randomizing device. Here, instead of a roulette wheel, there is an “Ellsberg urn” and the player may make his strategy contingent on the draw from the urn(s). Observe that sequential optimality ensures that whenever a player chooses to condition her strategy on these payoff-irrelevant but ambiguous aspects of types, she also necessarily wants to carry out that conditioning even after the type is realized. Thus our approach to strategic ambiguity is able to satisfy this important implementation issue that was raised in the context of Ellsberg strategies and normal form games by Riedel and Sass (2013).

Formally, consider placing the following structure on the finite set of types for each player $i$: $\Theta_i \equiv \Theta_i^{\mu} \times \Theta_i^{A}$ (with generic element $\theta_i \equiv (\theta_i^{\mu}, \theta_i^{A})$) where $\Theta_i^{A} \equiv \prod_{t \in T} \prod_{\eta^t \in H^t} \Theta_{i,\eta^t}$ and $|\Theta_{i,\eta^t}| = |A_i^{\eta_t}|$ and the utility payoff function for each player $i$ depends only on the $\Theta_i^{\mu} \equiv \prod_{j \in N} \Theta_j^{\mu}$ component of the type profile, i.e., for each history $h \in H$, $u_i(h, \theta) = u_i(h, \hat{\theta})$ for all $\theta, \hat{\theta} \in \Theta$ such that $\theta_i^{\mu} = \hat{\theta}_i^{\mu}$ for all players $j$.

We call a strategy for a player an Ellsberg strategy if it makes play depend on the (payoff-irrelevant) $\Theta_i^{A}$ component of $i$’s type.

**Definition 3.1** A strategy for player $i$, $\sigma_i$, is an Ellsberg strategy if $\sigma_i(\eta^t, \theta_i) \neq \sigma_i(\eta^t, \hat{\theta}_i)$ for some $\theta_i, \hat{\theta}_i \in \Theta_i$ such that $\theta_i^{\mu} = \hat{\theta}_i^{\mu}$, partial history $\eta^t \in H^t$, and $t \in \{1, \ldots, T + 1\}$.

Notice that if $\mu$ makes $\Theta_i^{A}$ ambiguous then an Ellsberg strategy allows player $i$ to create ambiguity about his strategy. Because ignoring $\theta_i^{A}$ is always an option, if, in a sequential optimum/PEA/SEA, a player uses an Ellsberg strategy it must be the case that she views choosing to create this strategic ambiguity (and follow through on it) as a best response. This is a key difference with the older literature on complete information games with ambiguity about others’ strategies (e.g., Dow and Welang (1994), Lo (1996, 1999), Klibanoff (1996), Eichberger and Kelsey (2000), Marinacci (2000), Mukerji and Shin (2002)). In that literature, while each player is assumed to best respond to the ambiguity she has about the others’ strategies, the strategies in the support of that ambiguity are not all required to be part of others’ best responses. A notable exception is Lo (1996, 1999), which does require this.
best response property. Even in Lo (1996, 1999) however, there is no choice on the part of a player to create (or not) ambiguity about her play.

Here we make some observations on the relation to Kuhn’s theorem issues raised by Moraviev, Riedel and Sass (2015) in the context of their notion of Ellsberg strategies: They define Ellsberg behavior strategies and Ellsberg mixed strategies and show that in games of perfect recall they are not generally equivalent. In this regard it is useful to observe that (1) our notion of Ellsberg strategies involves conditioning $i$’s choice of traditional (as opposed to Ellsberg) behavior strategy on the realization of $\Theta_i^A$; (2) if we instead allowed $i$ to choose a traditional mixed strategy conditional on the realization of $\Theta_i^A$, then since Kuhn’s Theorem says that any such traditional mixed strategy can be replicated by a traditional behavioral strategy and vice-versa, this could not make any difference. How does this relate to Moraviev, Riedel and Sass (2015)? What they show may not have sufficient richness to replicate the conditioning of traditional mixed strategies on an ambiguous urn are behavioral Ellsberg strategies that allow only, for each node, conditioning of the mixture over actions at that node on an ambiguous urn for that node. Since all conditioning of strategies is through the type space in our framework, the same issue does not arise for us. In this sense, our Ellsberg strategies correspond more closely with what Moraviev, Riedel and Sass (2015) call mixed Ellsberg strategies. Also notice that, within the context of multistage games with observed actions, our approach to Ellsberg strategies does not allow for a player to view her own future play as ambiguous once the ex ante stage has passed. Such a feature is present in the behavioral Ellsberg strategies of Moraviev, Riedel and Sass (2015).

4 Examples

In this section, we present a number of examples designed to illustrate different aspects of our equilibrium concepts and compare with standard concepts that limit attention to ambiguity neutral players. All of our examples look at games where there is a common belief $\mu$ such that $\mu_i = \mu$ for all players $i$. Thus the behavior in our examples is never driven by differences in ex-ante beliefs.

4.1 Ambiguity Aversion Generates New Strategic Behavior in Equilibrium

4.1.1 Example 1: Opting in and different hedges

We present a 3-player game, with incomplete information about player 1, in which a path of play can occur as part of an SEA when players 2 and 3 are sufficiently ambiguity averse, but
never occurs as part of even an ex-ante equilibrium if we modify the game by making players 2 and 3 ambiguity neutral (expected utility). Furthermore, under the SEA we construct, player 1 achieves a higher expected payoff than under any ex-ante equilibrium of the game with ambiguity neutral players, and even outside the convex hull of such ex-ante equilibrium payoffs. The game is depicted in Figure 4.1.

There are three players: 1, 2 and 3. First, nature determines whether player 1 is of type I or type II and 1 observes her own type. Players 2 and 3 have only one type, so there is complete information about them. The payoff triples in Figure 4.1 describe vNM utility payoffs given players’ actions and players’ types (i.e., \((u_1, u_2, u_3)\) means that player \(i\) receives \(u_i\)). Players 2 and 3 have ambiguity about player 1’s type and have smooth ambiguity preferences with an associated \(\phi_2 = \phi_3 = \phi\) and \(\mu_2 = \mu_3 = \mu\). Player 1 also has smooth ambiguity preferences, but nothing in what follows depends on either \(\phi_1\) or \(\mu_1\). Player 1’s first and only move in the game is to choose between action \(P(\text{lay})\) which leads to players 2 and 3 playing a simultaneous move game in which their payoffs depend on 1’s type, and action \(Q(\text{uit})\), which ends the game (equivalently think of it leading to a stage where all

Figure 4.1: SEA Example
players have only one action).\footnote{Note that to eliminate any possible effects of varying players’ risk aversion, think of the payoffs being generated using lotteries over two “physical” outcomes, the better of which has utility $u$ normalized to $5/2$ and the worse of which has $u$ normalized to $0$. So, for example, the payoff 1 can be thought of as generated by the lottery giving the better outcome with probability $2/5$ and the worse outcome with probability $3/5$.}

**Theorem 4.1** Suppose players 2 and 3 are ambiguity neutral and have a common belief $\mu$. There is no ex-ante equilibrium such that player 1 plays $P$ with positive probability.

**Proof.** Observe that player 1 is willing ex-ante to play $P$ with positive probability if and only if, after the play of $P$, $(U, R)$ will be played with probability at least $\frac{1}{2}$. Suppose there is an ex-ante equilibrium, $\sigma$, in which $P$ is played with positive probability. Let $p_I$ and $p_{II}$ denote the probabilities according to $\sigma$ that types I and II, respectively, of player 1 play $P$. Then player 2 is finds it optimal to play $U$ with positive probability if and only if

$$p_I \sum_{\pi \in \Delta(\Theta)} \mu(\pi)(\pi(I)) + p_{II} \sum_{\pi \in \Delta(\Theta)} \mu(\pi)(1 - \pi(I)) \geq \frac{5}{2} p_{II} \sum_{\pi \in \Delta(\Theta)} \mu(\pi)(1 - \pi(I))$$

which is equivalent to

$$p_I \sum_{\pi \in \Delta(\Theta)} \mu(\pi)(\pi(I)) \geq \frac{3}{2} p_{II} \sum_{\pi \in \Delta(\Theta)} \mu(\pi)(1 - \pi(I)). \quad (4.1)$$

Similarly, player 3 finds it optimal to play $R$ with positive probability if and only if

$$p_I \sum_{\pi \in \Delta(\Theta)} \mu(\pi)(\pi(I)) + p_{II} \sum_{\pi \in \Delta(\Theta)} \mu(\pi)(1 - \pi(I)) \geq \frac{5}{2} p_I \sum_{\pi \in \Delta(\Theta)} \mu(\pi)(\pi(I))$$

which is equivalent to

$$p_I \sum_{\pi \in \Delta(\Theta)} \mu(\pi)(\pi(I)) \leq \frac{2}{3} p_{II} \sum_{\pi \in \Delta(\Theta)} \mu(\pi)(1 - \pi(I)). \quad (4.2)$$

Since (4.1) and (4.2) cannot both be satisfied when $p_I + p_{II} > 0$ (i.e., $P$ is played with positive probability), $\sigma$ must specify that the history $(P, U, R)$ is never realized. This implies that player 1 has an ex-ante profitable deviation to the strategy of always playing $Q$, contradicting the assumption that $\sigma$ is an ex-ante equilibrium. □

Since $\sigma$ being part of a sequentially optimal $(\sigma, \nu)$ implies $\sigma$ is an ex-ante equilibrium, Theorem 4.1 immediately implies that none of the stronger concepts such as PEA, PBE or Sequential Equilibrium can admit the play of $P$ with positive probability under ambiguity.
neutrality. The next result shows that the situation changes dramatically under sufficient ambiguity aversion.

**Theorem 4.2** There exist $\phi$ and $\mu$ such that in an SEA both types of player 1 play $P$ with probability 1, and $(U, R)$ is played with probability greater than $\frac{1}{2}$.

**Proof.** Let $\mu$ put probability $\frac{1}{2}$ on $\pi_0$ and $\frac{1}{2}$ on $\pi_1$, where $\pi_0(I) = 1$ and $\pi_1(I) = 0$.\(^2\) Let $\phi(x) \equiv -e^{-x}$. Let $\sigma$ be a strategy profile specifying that both types of player 1 play $P$ with probability 1, player 2 plays $U$ with probability $\lambda^*$ if given the move and player 3 plays $R$ with probability $\lambda^*$ if given the move, where $\lambda^* = 1 - \frac{2}{5} \ln(3/2)$. Notice that according to $\sigma$, the history $(P, U, R)$ occurs with probability $(1 - \frac{2}{5} \ln(3/2))^2 > \frac{7}{10}$. Observe that player 1 strictly prefers ex-ante to play $P$ with probability 1 for both types if and only if, after the play of $P$, $(U, R)$ will be played with probability greater than $\frac{1}{2}$. The same is true for each type of player 1 after her type is realized as well. Player 2 ex-ante chooses the probability, $\lambda \in [0, 1]$, with which to play $U$ if given the move to maximize

$$\frac{1}{2} e^{-\lambda} - \frac{1}{2} e^{-(\lambda + \frac{5}{2}(1-\lambda))}.$$ 

One can verify that the maximum is reached at $\lambda = \lambda^*$. Similarly, player 3 ex-ante chooses the probability, $\lambda \in [0, 1]$, with which to play $R$ if given the move to maximize

$$\frac{1}{2} e^{-(\lambda + \frac{5}{2}(1-\lambda))} - \frac{1}{2} e^{-\lambda}$$

which is again maximized at $\lambda = \lambda^*$.

Now consider the following sequence of completely mixed strategies with limit $\sigma$: $\sigma^k$ has each type of player 1 play $P$ with probability $1 - \frac{1}{2k}$, and leaves the strategies otherwise the same as in $\sigma$. The associated interim belief system, $\nu^k$, is derived from $\sigma^k$ according to the extended smooth rule and we define $\nu$ as $\lim_{k \to \infty} \nu^k$. Recall that only player 1 has more than one possible type. Thus $\nu_{1, l, \eta}^k(\pi_0) = 1$ and $\nu_{1, l, \eta}^k(\pi_1) = 1$ for all partial histories $\eta$. Furthermore, $\nu_{2, 2, \eta}^k(\pi_0) = \frac{1}{2} \frac{\phi'(\frac{1-\pi}{2})(\lambda^*)}{\phi'(\frac{1-\lambda^*}{2})} = \frac{5}{2} \frac{\phi'(\frac{1-\pi}{2})(\lambda^*)}{\phi'(\frac{1-\lambda^*}{2})}$ for all partial histories $\eta$. Also, $\nu_{2, 2, \eta}^k(\pi_0) = \frac{1}{2} \frac{\phi'(\frac{1-\pi}{2})(\lambda^*)}{\phi'(\frac{1-\lambda^*}{2})}$ for all partial histories $\eta$. Furthermore, $\nu_{3, 2, \eta}^k(\pi_0) = \frac{1}{2} \frac{\phi'(\frac{1-\pi}{2})(\lambda^*)}{\phi'(\frac{1-\lambda^*}{2})} + \frac{1}{2} \frac{\phi'(\frac{1-\pi}{2})(\lambda^*)}{\phi'(\frac{1-\lambda^*}{2})}$. $\nu_{3, 2, \eta}^k(\pi_0) = \frac{1}{2} \frac{\phi'(\frac{1-\pi}{2})(\lambda^*)}{\phi'(\frac{1-\lambda^*}{2})}$ for all partial histories $\eta$. Furthermore, $\nu_{3, 2, \eta}^k(\pi_0) = \frac{1}{2} \frac{\phi'(\frac{1-\pi}{2})(\lambda^*)}{\phi'(\frac{1-\lambda^*}{2})} + \frac{1}{2} \frac{\phi'(\frac{1-\pi}{2})(\lambda^*)}{\phi'(\frac{1-\lambda^*}{2})}$.

\(^2\) The degeneracy of the $\pi$ in the support of $\mu$ is not necessary for the argument to go through – it merely shortens some calculations and reduces the ambiguity aversion required.

\(^3\) Any more concave $\phi$ will also work, as will any $\phi$ more concave than $-e^{-ax}$ for $\alpha = \frac{-4\ln(2/3)}{5(2-\sqrt{2})} \approx 0.554$. 

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\[ \frac{\frac{1}{2}(1-\frac{1}{2^k}) \phi'(\frac{(1-\frac{1}{2^k})(1-\lambda^*) + \frac{1}{2}(1-\lambda^*)}{\phi'(\lambda^* + \frac{1}{2}(1-\lambda^*))})}{\frac{1}{2}(1-\frac{1}{2^k}) \phi'(\frac{(1-\frac{1}{2^k})(1-\lambda^*) + \frac{1}{2}(1-\lambda^*)}{\phi'(\lambda^* + \frac{1}{2}(1-\lambda^*))}) + \frac{1}{2}(1-\frac{1}{2^k}) \phi'(\frac{(1-\frac{1}{2^k})(1-\lambda^*)}{\phi'(\lambda^*)})} = \frac{1}{1 + e^{-\frac{1}{4}(\phi(\lambda^*) + \sqrt{\frac{1}{4}(\phi(\lambda^*)^2 - \phi'(\lambda^*)^2)})}}. \]

Since \( \lim_{k \to \infty} \nu_{2}^{k,\theta_2,\nu}(\pi_0) = \lim_{k \to \infty} \nu_{2}^{k,\theta_2,\nu}(\pi_0) = \lim_{k \to \infty} \nu_{3}^{k,\theta_3,\nu}(\pi_0) = \lim_{k \to \infty} \nu_{3}^{k,\theta_3,\nu}(\pi_0) = \frac{1}{3} \), \( \sigma \) remains optimal for players 2 and 3 following the play of \( P \) given \( \nu \). (The beliefs at other partial histories can also be calculated using the smooth rule formula, but they will not matter for sequential optimality because the relevant player has no moves left.) Thus, \( (\sigma, \nu) \) is sequentially optimal and, by construction, satisfies smooth rule consistency. It is therefore an SEA.

If one modifies the example in Figure 4.1 by reversing the 0’s and 2’s in player 1’s payoffs then, by similar reasoning as in the proof of Theorem 4.2, it should be possible to construct an SEA where \( Q \) is played with probability 1 by both types of player 1. There should not be such an SEA under ambiguity neutrality (and thus no such sequential equilibrium), though there would be such a PEA.

4.1.2 Example 2: Hedging plus Screening on beliefs

We now present an example of a 3-player game, with incomplete information about player 3, in which a path of play can occur as part of an SEA when player 2 is sufficiently ambiguity averse, but never occurs as part of a PEA (nor a PBE) when player 2 has expected utility preferences (and is thus ambiguity neutral). Furthermore, under the SEA we construct, player 3 achieves a higher expected payoff than under any PEA with player 2 having expected utility preferences. The example is constructed so that if player 2 is sufficiently ambiguity averse then 3 changes his strategy to allow an action by 2 that is favorable to 3. The role of player 1 is to effectively “screen” player 2 and prevent the part of the game that has the play path in question from being reached when 2 puts sufficiently high weight on player 3 being of a particular type (type II). This screening, by design, catches player 2 for a smaller range of parameters when 2 is more ambiguity averse. When 2 is ambiguity neutral, the screening works for a large enough range of parameters that the part of the game in question is reached only when player 2 does not have incentive to carry out the action favorable to player 3, thus 3 opts out of this portion of the game. The example also serves to help the reader understand the PEA and SEA definitions and shows that they can be quite easy to apply. The game is depicted in Figure 4.2.

There are three players: 1, 2 and 3. First, nature determines whether player 3 is of type I or type II and 3 observes his own type. Players 1 and 2 have only one type, so there is complete information about them. The payoff triples in Figure 4.2 describe vNM utility payoffs given players’ actions and players’ types (i.e., \((u_1, u_2, u_3)\) means that player \(i\) receives \(u_i\)). Player 2 has ambiguity about player 3’s type and has smooth ambiguity preferences.
Figure 4.2: 3-Player Game
with an associated $\phi_2$ and $\mu_2$. Players 1 and 3 also have smooth ambiguity preferences, but nothing in what follows depends on either $\phi_j$ or $\mu_j$ for $j = 1, 3$. Player 1’s first and only move in the game is to choose between action $T(\text{wo})$ which gives the move to player 2 and action $(\text{th})R(\text{ee})$, which gives the move to player 3. If $T$, then 2 makes a single move that ends the game, by choosing between $F(\text{ixed})$ and $B(\text{et})$ (i.e., player 2 effectively chooses between a fixed payoff and betting that player 3 is of type II). If $R$, then player 3’s move is a choice between $C(\text{ontinue})$ which leads to player 2 being given the move, and $S(\text{top})$ which ends the game. If $C$, then player 2 has a choice between $G(\text{amble})$ and $H(\text{edge})$ after which the game ends.\footnote{Note that to eliminate any possible effects of varying 2’s risk attitude, think of the payoffs of player 2 being generated using lotteries over two “physical” outcomes, the better of which has utility $u_2$ normalized to 6 and the worse of which has $u_2$ normalized to 0. So, for example, the payoff 2 can be thought of as generated by the lottery giving the better outcome with probability 1/3 and the worse with probability 2/3.}

In any sequential optimum where only type I of player 3 plays $C$ with positive probability, (2.6) requires player 2, following $C$, to put weight only on type I. Thus, 2 would then always play $G$ if given the move in any such sequential optimum. Notice that if 2 plays $G$, player 3 is always better off playing $S$ than $C$. Therefore no sequential optimum can have only type I of player 3 play $C$ with positive probability. Similarly, no sequential optimum can have only type II of player 3 play $C$ with positive probability as 2 would play $H$ and type I would gain from deviating to $C$. Observe that for any pure strategy sequential optimum, player 3 plays $(C, C)$ if and only if player 2 plays $H$. Thus $(C, C)$, $H$ is part of a sequential optimum if and only if Player 2 is behaving optimally by playing $H$ (with with sufficiently high probability) from the point of view of both stage 1 and stage 2. From (2.6), the only difference in the point of view of these stages can be the beliefs and the event the $\pi$ are conditioned on. For extreme beliefs such as putting all weight on player 3 being type II with probability 1, $H$ is indeed optimal. For other beliefs, such as, putting all weight on player 3 being type I with probability 1, $H$ is not optimal. Are there beliefs supporting it as part of an SEA?

Our first result shows that $C$ may be played on the equilibrium path as part of an SEA.

**Theorem 4.3** There exist $\phi_2$ and $\mu_2$ such that $C$ is played on the equilibrium path with probability 1 as part of an SEA.

**Proof.** Since type uncertainty is only about player 3 and there are only two possible types, represent probabilities over the types by the probability, $p$, of Type I. Think of player 2 being subjectively uncertain whether nature used $\hat{p} \geq 1/2$ or $\hat{p} = 1 - \hat{p}$ to determine player 3’s type. Specifically, let $\mu_2(\hat{p}) = \mu_2(\hat{p}) = 1/2$. Also assume $\phi_2$ is smooth and strictly concave.
We now show that \((R, F, (C, C), \lambda^*H + (1 - \lambda^*)G)\) is an SEA given \(\hat{\mu} = \frac{3}{4}\) and \(\phi_2(x) = -e^{-2x}\) and defining \(\lambda^*\) by
\[
\lambda^* = \arg\max_{\lambda \in [0, 1]} \frac{1}{2} \phi_2(2\lambda + 6(1 - \lambda)\hat{\mu}) + \frac{1}{2} \phi_2(2\lambda + 6(1 - \lambda)\hat{\mu}).
\] (4.3)

First, we show that the strategy profile is an ex-ante equilibrium of the game. Player 1 is best responding (for any specification of \(\phi_1\) and \(\mu_1\)) because he gets a payoff of 1 on path and would get less than 1 by deviating since 2 plays \(F\) following \(T\). Player 2 is best responding on-path by the definition of \(\lambda^*\). Player 3 has \((C, C)\) as a best response (for any specification of \(\phi_3\) and \(\mu_3\)) if and only if \(\lambda^* \geq 1/2\). First-order conditions for an interior \(\lambda^*\) are given by
\[
(1 - 3\hat{\mu})\phi_2'(2\lambda^* + 6(1 - \lambda^*)\hat{\mu}) + (3\hat{\mu} - 2)\phi_2'(2\lambda^* + 6(1 - \lambda^*)(1 - \hat{\mu})) = 0.
\] (4.4)

Notice that the left-hand side of (4.4) is always negative for \(\lambda^* = 1\). By concavity therefore, \(\lambda^* < 1\). Observe that \(\lambda^* \geq 1/2\) if and only if the left-hand side of (4.4) is non-negative at \(\lambda^* = 1/2\), i.e.,
\[
(1 - 3\hat{\mu})\phi_2'(1 + 3\hat{\mu}) + (3\hat{\mu} - 2)\phi_2'(1 + 3(1 - \hat{\mu})) \geq 0.
\] (4.5)

Observe that this is satisfied for our choice of \(\hat{\mu}\) and \(\phi_2\). In particular \(\lambda^* = 1 - \frac{\ln(5)}{6}\). Thus 3 is best responding and the strategy profile is an ex-ante equilibrium.

Before specifying interim beliefs observe that, since there is no type uncertainty about players 1 or 2, player 3’s beliefs are trivial. Also, since player 1’s payoffs do not depend on 3’s type or actions, 1’s best response to the strategies of the others is independent of his beliefs. Thus, the only important beliefs to specify are those of player 2. We construct such beliefs to satisfy smooth rule consistency. Consider the following sequence of strategy profiles with limit \((R, F, (C, C), \lambda^*H + (1 - \lambda^*)G)\):
\[
\sigma^k \equiv \left( \frac{k}{k + 1} R + \frac{1}{k + 1} T, \frac{k}{k + 1} F + \frac{1}{k + 1} B, \left( \frac{k}{k + 1} C + \frac{1}{k + 1} S, \frac{k}{k + 1} C + \frac{1}{k + 1} S \right), \lambda^*H + (1 - \lambda^*)G \right).
\]

Applying the extended smooth rule using \(\sigma^k\) (in particular, applying the formula in (5.1)),
\[
\nu^k_{2,(R,C)}(p) \propto \frac{\phi_1' \left( p\left( \frac{k}{k + 1} (\frac{k}{k + 1} 2 + \frac{1}{k + 1} 0) + \frac{k}{k + 1} (\frac{1}{k + 1} 0 + \frac{k}{k + 1} (2\lambda^* + 6(1 - \lambda^*)))) \right) \right)}{\phi_2' \left( p(2\lambda^* + 6(1 - \lambda^*)) + (1 - p)(2\lambda^* + 0(1 - \lambda^*)) \right)} \left( \frac{k}{k + 1} \right)^2 \frac{1}{2}.
\]
Thus,
\[
\nu^k_{2,(R,C)}\left(\frac{3}{4}\right) = \frac{(3 + \lambda^*)(-2 - 4k + k^2(-9 + 5\lambda^*))}{2(-30 + 8k(-3 + \lambda^*) + 14\lambda^* + k^2(-27 + 6\lambda^* + 5(\lambda^*)^2))}
\]
and so
\[
\nu_{2,(R,C)}\left(\frac{3}{4}\right) = \lim_{k \to \infty} \nu^k_{2,(R,C)}\left(\frac{3}{4}\right) = \frac{1}{2} = \lim_{k \to \infty} \nu^k_{2,(R,C)}\left(\frac{1}{4}\right) = \nu_{2,(R,C)}\left(\frac{1}{4}\right).
\]
Furthermore,
\[
\nu^k_{2,T}(p) \propto
\phi_t\left(\frac{p\left(\frac{1}{k+1}(\frac{k}{k+1}2 + \frac{1}{k+1}0) + \frac{k}{k+1}(\frac{1}{k+1}0 + \frac{k}{k+1}(2\lambda^* + 6(1 - \lambda^*)))\right)}{\phi_t\left(p\left(\frac{1}{k+1}2 + \frac{1}{k+1}0\right) + (1 - p)\left(\frac{1}{k+1}2 + \frac{1}{k+1}4\right)\right)}\right) \frac{1}{k+1} \frac{2}{2}
\]
Thus,
\[
\nu^k_{2,T}\left(\frac{3}{4}\right) = \frac{(3 + 2k)(-2 - 4k + k^2(-9 + 5\lambda^*))}{2(-6 - 16k + k^2(-23 + 7\lambda^*) + 4k^3(-3 + \lambda^*))}
\]
and so
\[
\nu_{2,T}\left(\frac{3}{4}\right) = \lim_{k \to \infty} \nu^k_{2,T}\left(\frac{3}{4}\right) = \frac{9 - 5\lambda^*}{12 - 4\lambda^*} = \frac{24 + 5\ln(5)}{48 + 4\ln(5)} \approx 0.59.
\]
We now verify that 2 is playing optimally with respect to these interim beliefs. By the definition of \(\lambda^*\) (via (4.3)), since \(\nu_{2,(R,C)}\left(\frac{3}{4}\right) = \frac{1}{2}\), player 2 is indeed best responding after 1 plays \(R\) and 3 plays \(C\). If player 2 is given the move after 1 plays \(T\), 2 playing \(F\) if a best response if and only if
\[
1 \in \arg \max_{\gamma \in [0,1]} \nu_{2,T}\left(\frac{3}{4}\right)\phi_2(2\gamma + 4(1 - \gamma)\frac{1}{4}) + (1 - \nu_{2,T}\left(\frac{3}{4}\right))\phi_2(2\gamma + 4(1 - \gamma)\frac{3}{4}).
\]
This may be verified for \(\nu_{2,T}\left(\frac{3}{4}\right) = \frac{24 + 5\ln(5)}{48 + 4\ln(5)}\) and the \(\phi_2\) assumed.

Putting everything together, \((R, F, (C, C), \lambda^*H + (1 - \lambda^*)G)\) is sequentially optimal with respect to beliefs satisfying smooth rule consistency. Therefore all the conditions for an SEA are satisfied. ■

Note that when player 2 is ambiguity neutral, i.e., \(\phi\) linear, \(((C, C), H)\) is not part of any PEA or SEA because \(\lambda^* = 0\) in (4.3). One might wonder how sensitive this finding is to the assumption in the proof of the above result that player 2’s reduced measure on the type of player 3 is \((1/2, 1/2)\) (i.e., player 2 views the types of player 3 symmetrically). We now show that the finding can be made fully robust to 2’s initial measure.

**Theorem 4.4** Regardless of the beliefs of any player, if player 2 is ambiguity neutral (\(\phi_2\) affine), then no PEA results in \(C\) being played on the equilibrium path with positive proba-
Proof. Suppose player 2 is ambiguity neutral (without loss of generality, take $\phi_2$ to be the identity). Let $\gamma$ be player 2’s initial reduced probability that 3 is of type I. For $C$ to be played on the equilibrium path, player 1 must play $R$ with positive probability, which can be a best response if and only if player 1’s expected payoff following $T$ is less than or equal to 1, the sure payoff after $R$. This is possible if and only if 2’s strategy plays $F$ with probability at least $\frac{5}{6}$ following $T$. If $T$ is played with positive probability in equilibrium, then 2 playing $F$ with probability at least $\frac{5}{6}$ following $T$ is optimal for 2 if and only if $\gamma \geq 1/2$. In the explanation before Theorem 4.3, we showed that no PEA can have only type I of player 3 play $C$ with positive probability on the equilibrium path. Suppose type II of player 3 plays $C$ with positive probability on the path. Optimal for 3 implies this can be true only if 2 plays $H$ with probability weakly higher than $G$. But then type I of player 3 finds it strictly optimal to play $C$ with probability 1. Note however that in this case 2 strictly prefers $G$ over $H$, making $C$ strictly worse than $S$ for both types of player 3. It follows that playing $C$ with positive probability on the equilibrium path cannot satisfy condition (2.9) of sequential optimality (and thus PEA) where $T$ is played with positive probability.

It remains to consider the case where 1 plays $R$ with probability 1. Then $T$ is now an off equilibrium path action and thus condition (2.9) places no restrictions on 2’s play following $T$. However, the naturally extends updating condition requires that 2’s updated beliefs following $T$ must continue to place weight $\gamma$ on 3 being of type I because 3 has no move at that stage, 2’s marginal over 3’s type is a product measure, and there is no costly ambiguity exposure (since $\phi'$ is constant) for 2. From sequential optimality, it then follows that 2’s best response to $T$ is $B$ whenever $\gamma < 1/2$, which contradicts the optimality of 1 playing $R$. Now suppose $\gamma \geq 1/2$. The same argument as used above after establishing that $\gamma \geq 1/2$ shows that $C$ cannot be played with positive probability on the equilibrium path. In sum, when player 2 is ambiguity neutral, in any PEA if $\gamma < 1/2$ then 1 plays $T$ and 3 never is given the move, while if $\gamma \geq 1/2$ then 3 never plays $C$ if given the move. 

One important finding, then, is that, under ambiguity neutrality, no PEA can ever result in play of $C$, while, when there is enough ambiguity aversion there are SEA involving the play of $C$ with probability 1. Note that if we were looking only for profiles satisfying sequential optimality then $(R, F, (C, C), H)$ could be the equilibrium strategies under ambiguity neutrality if 2’s initial reduced probability that 3 is of type II were sufficiently high. This is compatible with 2’s play of $F$ if given the move by 1 by specifying (off-path) beliefs for 2 following $T$ that place sufficient weight on type I. Such off-path beliefs are unrestricted by sequential optimality, but are not compatible with PEA or SEA.
4.2 Example 3: Limit Pricing under Ambiguity

In this section we present an example of a game with a PEA involving non-trivial updating on the equilibrium path that departs from Bayes rule. The example is based on the Milgrom and Roberts (1982) limit pricing entry model. In this example, an incumbent has private information concerning his production costs. The incumbent sets a price, an entrant observes the price and decides whether or not to enter, in which case he pays a fixed cost $K > 0$. Then the private information is revealed and last stage of game played, either by both firms competing in a Cournot duopoly or by only the incumbent acting as a monopolist. Suppose there are three possible costs for the incumbent $(H, M, L)$.

We show a PEA where in the 1st period, types $M$ and $L$ pool at the monopoly quantity for $L$, and type $H$ plays the monopoly quantity for $H$. Then the entrant, with known cost, finds it optimal to enter for sure after observing the monopoly quantity for $H$ and not to enter after the monopoly quantity for $L$ (this strategy may be extended to include the possibility of observing any non-negative quantity by having the entrant enter for sure if the quantity observed is strictly below the monopoly quantity for $L$ and stay out for sure if it is weakly higher than the monopoly quantity for $L$). We will see that after observing the monopoly quantity for $L$ (on-path), under ambiguity aversion sequential optimality will require that the entrant’s updating departs from Bayes rule. Sequential optimality also ensures that strategies in the duopoly game form a Nash equilibrium of that one-period complete information game (there are ex-ante equilibria violating sequential optimality that involve the incumbent deterring all entry by threatening to flood the market if entry occurs).

We assume that the inverse market demand is given by $P(Q) = a - bQ$, $a, b > 0$. Given the incumbent’s cost $c_I$ and quantity $q_I$ and entrant cost $c_E$ and quantity $q_E$, the complete information Cournot reaction functions are given by

$$q_E(q_I) = \arg \max (P(q_E + q_I) - c_E)q_E = \frac{a - c_E}{2b} - \frac{q_I}{2}$$

and

$$q_I(q_E) = \arg \max (P(q_E + q_I) - c_I)q_I = \frac{a - c_I}{2b} - \frac{q_E}{2}.$$ 

This yields equilibrium values:

$$q_I = \frac{a + c_E - 2c_I}{3b}, q_E = \frac{a + c_I - 2c_E}{3b}.$$ 

The use of three costs is necessary to have non-trivial updating on the equilibrium path under pure strategy limit pricing. With only two possible costs pure limit pricing strategies involve full pooling.
and corresponding profits:

\[ b\left(\frac{a+c_E-2c_I}{3b}\right)^2, b\left(\frac{a+c_I-2c_E}{3b}\right)^2. \]

Similarly, if there is only one firm in the market, with cost \( c_I \) and quantity \( q_I \), the monopoly quantity is defined by

\[ \arg \max_{q_I} (P(q_I) - c_I)q_I = \frac{a - c_I}{2b}. \]

Thus monopoly profits are

\[ b\left(\frac{a - c_I}{2b}\right)^2. \]

We check whether the strategies described above satisfy condition (2.9) of sequential optimality. First, we check optimality from the incumbent’s point of view.

1. Check that type H does not prefer to pool with M,L at the monopoly quantity for L thus deterring entry. Profits for H in the conjectured equilibrium are

\[ b\left(\frac{a+c_E-2c_H}{3b}\right)^2 + b\left(\frac{a+c_E-2c_H}{3b}\right)^2. \]

Profits if it instead pools with M,L at monopoly quantity for L and deters entry are

\[ \frac{a-c_L}{2b}(a - \frac{a-c_L}{2} - c_H) + b\left(\frac{a-c_H}{2b}\right)^2. \]

For H to be better off not pooling, must have

\[ b\left(\frac{a+c_E-2c_H}{3b}\right)^2 \geq \frac{a-c_L}{2b}(a - \frac{a-c_L}{2} - c_H). \]

This is equivalent to

\[ \left(\frac{a+c_E-2c_H}{3}\right)^2 \geq \frac{a-c_L}{2}(a - \frac{a-c_L}{2} - c_H) \quad (4.6) \]

2. Check that type M does not prefer to produce the monopoly quantity for M and fail to deter entry. Profits for M in the conjectured equilibrium are

\[ \frac{a-c_L}{2b}(a - \frac{a-c_L}{2} - c_M) + b\left(\frac{a-c_M}{2b}\right)^2. \]

If it instead produced at the monopoly quantity for M and fails to deter entry, profits are

\[ b\left(\frac{a-c_M}{2b}\right)^2 + b\left(\frac{a+c_E-2c_M}{3b}\right)^2. \]

For M to be better off pooling with L, must have

\[ \frac{a-c_L}{2b}(a - \frac{a-c_L}{2} - c_M) \geq b\left(\frac{a+c_E-2c_M}{3b}\right)^2. \]

This is equivalent to

\[ \frac{a-c_L}{2}(a - \frac{a-c_L}{2} - c_M) \geq \left(\frac{a+c_E-2c_M}{3}\right)^2. \quad (4.7) \]

3. Obvious that strategy for L is optimal.
Now for the entry decision of the entrant. We assume that the entrant views each of the three types \((L, M, H)\) as non-null events ex-ante. Functionally, this means \(\sum \mu(\pi(t)) > 0\) for \(t \in \{L, M, H\}\). Denote \(v_t = b(a + c\xi - 2c\xi) - K\). As a best-response to the incumbent’s strategy, ex-ante the entrant wants to maximize

\[
\sum \mu(\pi) \phi[\lambda_L(\pi(L)v_L + \pi(M)v_M) + \lambda_H(\pi(H)v_H)]
\]  

with respect to \(\lambda_H, \lambda_L \in [0, 1]\), where \(\lambda_H\) and \(\lambda_L\) are the mixed-strategy probabilities of entering contingent on seeing the monopoly quantity for \(H\) and the monopoly quantity for \(L\), respectively, and \(K \geq 0\) is the fixed cost of entry. We need it to be the case that this is maximized at \(\lambda_H = 1\) and \(\lambda_L = 0\). Notice, by monotonicity, some maximum involves \(\lambda_H = 1\) if and only if

\[
v_H \geq 0
\]

and the strict version of this is equivalent to \(\lambda_H = 1\) being part of every maximum. Notice that this simply says that entering against a known high cost incumbent is profitable. Assuming this is satisfied, so that \(\lambda_H = 1\) is optimal, then \(\lambda_L = 0\) is optimal if and only if the derivative of (4.8) with respect to \(\lambda_L\) evaluated at \(\lambda_L = 0\) and \(\lambda_H = 1\) is non-positive:

\[
\sum \mu(\pi)(\pi(L)v_L + \pi(M)v_M)\phi'(\pi(H)v_H) \leq 0.
\]  

Notice that, since \(\phi' > 0\), a necessary condition for (4.10) is \(v_L < 0\) (i.e., entering against a known low cost incumbent is not profitable). To sum up, the equilibrium strategies we described will satisfy condition (2.9) of sequential optimality if and only if the four inequalities (4.6), (4.7), (4.9) and (4.10) are satisfied.

Now, since Theorem 2.1 says imposing the smooth rule is without loss of generality, we impose the smooth rule and examine sequential optimality at later stages of the game. Notice that \(v_L < 0\) plus (4.9), means that, in checking optimality off-path, it will be easy to find beliefs to support the desired entry/no-entry threshold strategy because point masses on high cost and low cost respectively will then generate entry and no entry respectively following off-path quantities in the first period.

Thus, we only need to worry about checking on-path optimality for this candidate equilibrium. Furthermore, if the monopoly quantity for \(H\) is observed in the first period, then sequential optimality is immediate because the updated beliefs put all weight on type \(H\). It remains to focus on the path where the monopoly quantity for \(L\) is observed in the first period. What does smooth rule updating look like here? Let \(\mu_{L,eq}\) denote the update of \(\mu\)
after observing the monopoly quantity for \(L\) and assuming the above equilibrium strategy of the entrant is ex-ante optimal. Sequential optimality requires that not entering remain optimal at this interim stage. This is equivalent to the following:

\[
\sum_{\pi: \pi(H) < 1} \mu_{L,eq}(\pi) \frac{1}{1 - \pi(H)} (\pi(L)v_L + \pi(M)v_M)\phi'(0) \leq 0
\]

According to the smooth rule,

\[
\mu_{L,eq}(\pi) \propto \mu(\pi)(1 - \pi(H))\frac{\phi'(\pi(H)v_H)}{\phi'(0)}
\]

for all \(\pi\) such that \(\pi(H) < 1\) and \(\mu_{L,eq}(\pi) = 0\) otherwise. Substituting yields that not entering remaining optimal is equivalent to (4.10).

We collect here the conditions on the parameters assumed explicitly or implicitly already: \(a, b > 0, K \geq 0, c_H > c_M > c_L \geq 0, c_E \geq 0\) plus the following restrictions coming from the condition that the monopoly and duopoly quantities are non-negative: \(a \geq c_H, a + c_E - 2c_H \geq 0\) and \(a + c_L - 2c_E \geq 0\).

The following theorem provides sufficient conditions for the existence of PEA of the form described above.

**Theorem 4.5** Fix any \(a, b > 0, K \geq 0, c_H > c_M > c_L \geq 0, c_E \geq 0\) satisfying \(a \geq c_H, a + c_E - 2c_H \geq 0, a + c_L - 2c_E \geq 0\), (4.6), (4.7) and the strict version of (4.9). For any \(\mu \in \Delta(\Delta(\{H, M, L\}))\) with finite support such that \(\mu(\{\pi | \pi(L)v_L + \pi(M)v_M < 0\}) > 0\) or \(\mu(\{\pi | \pi(L)v_L + \pi(M)v_M = 0\}) = 1\) and such that, for all \(\pi, \pi'\) in the support of \(\mu\), \(\frac{\pi(L)}{\pi(M)} \geq \frac{\pi'(L)}{\pi'(M)}\) if and only if \(\pi(H) \leq \pi'(H)\), (a special case of this last condition is for all \(\pi, \pi'\) in the support of \(\mu\), \(\pi(M) = \pi'(M)\) – i.e., \(M\) is unambiguous) there exists a strictly increasing and twice continuously differentiable concave function \(\phi\) such that (4.10) is satisfied (and continues to be so for all \(\hat{\phi}\) at least as concave as \(\phi\) (thus for all more ambiguity averse entrants, too)).

**Proof.** Assume the conditions of the theorem. We show that (4.10) is satisfied for concave enough \(\phi\). If \(\mu(\{\pi | \pi(L)v_L + \pi(M)v_M \leq 0\}) = 1\) then (4.10) is trivially satisfied for any \(\phi\). Suppose \(\mu(\{\pi | \pi(L)v_L + \pi(M)v_M < 0\}) > 0\) and \(\mu(\{\pi | \pi(L)v_L + \pi(M)v_M > 0\}) > 0\). Let \(\Pi^- \equiv \{\pi | \pi(L)v_L + \pi(M)v_M < 0\}, \Pi^+ \equiv \{\pi | \pi(L)v_L + \pi(M)v_M > 0\}, N \equiv \sum_{\pi \in \Pi^-} \mu(\pi)(\pi(L)v_L + \pi(M)v_M), \text{ and } P \equiv \sum_{\pi \in \Pi^+} \mu(\pi)(\pi(L)v_L + \pi(M)v_M). \text{ Let } \pi^- \in \arg \max_{\pi \in \Pi^-} \pi(H) \text{ and }
\( \pi^+ \in \arg \min_{\pi \in \Pi^+} \pi(H) \). The left-hand side of (4.10) can be bounded from above as follows:

\[
\sum_{\pi \in \Pi^-} \mu(\pi)(\pi(L)v_L + \pi(M)v_M)\phi'(\pi(H)v_H) + \sum_{\pi \in \Pi^+} \mu(\pi)(\pi(L)v_L + \pi(M)v_M)\phi'(\pi(H)v_H) \\
\leq \sum_{\pi \in \Pi^-} \mu(\pi)(\pi(L)v_L + \pi(M)v_M)\phi'(\pi^-(H)v_H) + \sum_{\pi \in \Pi^+} \mu(\pi)(\pi(L)v_L + \pi(M)v_M)\phi'(\pi^+(H)v_H) \\
= N\phi'\left(\pi^-(H)v_H\right) + P\phi'\left(\pi^+(H)v_H\right).
\]

Consider \( \phi(x) = -e^{-\alpha x}, \alpha > 0 \). The upper bound above becomes

\[\alpha Ne^{-\alpha \pi^-(H)v_H} + \alpha Pe^{-\alpha \pi^+(H)v_H}.\]

for \( \alpha > 0 \). We show that this upper bound is non-positive for sufficiently large \( \alpha \), implying (4.10). The upper bound is non-positive if and only if \( Pe^{-\alpha \pi^+(H)v_H} \leq -Ne^{-\alpha \pi^-(H)v_H} \) if and only if \( e^{\alpha (\pi^-(H) - \pi^+(H))v_H} \leq -\frac{N}{P} \) if and only if \( \alpha (\pi^-(H) - \pi^+(H))v_H \leq \ln(-\frac{N}{P}) \). Since \( \pi^-(L)v_L + \pi^-(M)v_M < 0 < \pi^+(L)v_L + \pi^+(M)v_M \) and \( c_L < c_M \), we have \( v_L < 0 < v_M \). Thus, \( \frac{\pi^-(L)}{\pi^-(M)} > \frac{v_M}{v_L} > \frac{\pi^+(L)}{\pi^+(M)} \). By our assumption on the support of \( \mu \) and Lemma 4.1, this implies \( \pi^-(H) - \pi^+(H) < 0 \). Therefore, \( \alpha (\pi^-(H) - \pi^+(H))v_H \leq \ln(-\frac{N}{P}) \) if and only if \( \alpha \geq \frac{\ln(-\frac{N}{P})}{(\pi^-(H) - \pi^+(H))v_H} \).

To complete the proof, fix \( \alpha \) satisfying this inequality and consider \( \phi \) such that \( \phi(x) = h(-e^{-\alpha x}) \) for all \( x \) with \( h \) concave and strictly increasing on \((-\infty, 0)\). We show that (4.10) holds. Observe that \( \phi'(x) = h'(-e^{-\alpha x})\alpha e^{-\alpha x} \). Since \( \pi^-(H) - \pi^+(H) < 0 \) and \( v_H > 0 \), we have

\[-e^{-\alpha \pi^-(H)v_H} \leq -e^{-\alpha \pi^+(H)v_H}\]

and, by concavity of \( h \),

\[h'(-e^{-\alpha \pi^-(H)v_H}) \geq h'(-e^{-\alpha \pi^+(H)v_H}).\]

Therefore the upper bound derived above satisfies

\[
N\phi'\left(\pi^-(H)v_H\right) + P\phi'\left(\pi^+(H)v_H\right) \\
= \alpha Ne^{-\alpha \pi^-(H)v_H}h'(-e^{-\alpha \pi^-(H)v_H}) + \alpha Pe^{-\alpha \pi^+(H)v_H}h'(-e^{-\alpha \pi^+(H)v_H}) \\
\leq (\alpha Ne^{-\alpha \pi^-(H)v_H} + \alpha Pe^{-\alpha \pi^+(H)v_H})h'(-e^{-\alpha \pi^-(H)v_H}) \leq 0
\]

by the first part of the proof and the assumption on \( \alpha \). This implies (4.10).

**Lemma 4.1** If the support of \( \mu \) can be ordered in the likelihood-ratio ordering, then, for any \( \pi, \pi' \in \text{supp} \mu \), \( \frac{\pi(L)}{\pi(M)} \geq \frac{\pi'(L)}{\pi'(M)} \) if and only if \( \pi(H) \leq \pi'(H) \).

**Proof.** Suppose the support of \( \mu \) can be so ordered. Then \( \frac{\pi(L)}{\pi(M)} \geq \frac{\pi'(L)}{\pi'(M)} \) if and only if
\[ \pi(M) \leq \pi'(M) \] if and only if \[ \frac{\pi(H)}{\pi(M)} \leq \frac{\pi'(H)}{\pi'(M)}. \] Thus, \[ \frac{\pi(L)}{\pi(M)} \geq \frac{\pi'(L)}{\pi'(M)} \] if and only if \[ \frac{\pi'(L)}{\pi'(M)} \leq \frac{\pi'(H)}{\pi(H)}. \] But, \[ \frac{\pi'(L)}{\pi'(M)} \leq \frac{\pi'(M)}{\pi(M)} \leq \frac{\pi(H)}{\pi'(H)} \implies \pi'(H) \geq \pi(H) \] since all three ratios cannot be less than 1 without violating the total probability summing to 1. If \( \pi'(H) \geq \pi(H) \), then want to show that \( \pi' \) is above \( \pi \) in the likelihood-ratio ordering. Suppose not, then \( \pi \) strictly above \( \pi' \) in the likelihood-ratio ordering, i.e. \[ \frac{\pi(L)}{\pi(H)} \leq \frac{\pi(M)}{\pi'(H)} \leq \frac{\pi'(H)}{\pi'(M)} \] with at least one inequality strict. But this is impossible when \( \frac{\pi'(H)}{\pi'(M)} \leq 1 \), a contradiction. ■

One observation following from the above result is that for any \( \mu \in \Delta(\Delta\{H,M,L\}) \) such that \( \mu(\{\pi | \pi(L) v_L + \pi(M) v_M < 0\}) > 0 \) and (4.10) is violated when \( \phi \) is linear, there exists a strictly increasing and twice continuously differentiable concave function \( \phi \) such that (4.10) is satisfied. In this way, ambiguity aversion leads to an expansion in the set of \( \mu \) that can support such a semi-pooling equilibrium. The numerical example below shows that this expansion can be strict. Similarly, increasing ambiguity aversion increases the set of \( \mu \) that can support such a semi-pooling equilibrium. We also conjecture that in the limit as ambiguity aversion becomes infinite, the set of such \( \mu \) approaches the set of all \( \mu \) satisfying \( \mu(\{\pi | v_L(\pi) < 0\}) > 0 \) or \( \mu(\{\pi | v_L(\pi) = 0\}) = 1 \).

An example of parameters that satisfy the four inequalities so that the limit pricing strategies are part of a PEA are the following: \( \mu \) puts equal weight on type distributions \( \pi_0 = (1/6, 1/3, 1/2) \) and \( \pi_1 = (1/2, 1/3, 1/6) \), where the vector notation gives the probabilities of \( L, M, H \) respectively, \( \phi(x) = -e^{-\alpha x} \), with \( \alpha > \frac{189}{63} \log\left(\frac{39}{23}\right) \approx 1.53546 \), \( a = 2, b = \frac{7}{128}, c_H = \frac{3}{2}, c_M = \frac{11}{8}, c_L = 1, c_E = \frac{5}{4} \) and \( K = 1 \). With these parameters, \( v_L = -\frac{31}{63}, v_M = \frac{35}{63} \) and \( v_H = \frac{65}{63} \). Applying Bayesian updating after observing the monopoly price for \( L \) gives \( \mu_{L,eq}(\pi_0) = \frac{3}{8} \). Applying the smooth rule, the updated beliefs after observing the monopoly price for \( L \) are \( \mu_{L,eq}(\pi_0) = \frac{3e^{-\alpha(c_H + \lambda_M q)}}{3e^{-\alpha(c_H + \lambda_M q)} + 5e^{-\alpha(c_M + \lambda_M q)}} = \frac{1}{1 + \frac{5}{3}e^{\alpha(c_M - c_H)q}} < \frac{3}{8} \). For example, when \( \alpha = 2 \), \( \mu_{L,eq}(\pi_0) = \frac{1}{1 + \frac{5}{3}e^{\alpha(c_M - c_H)q}} \approx 0.232 \). If the entrant had used Bayesian updating in this example then these limit pricing strategies would not have been sequentially optimal. Specifically, after observing the monopoly quantity for \( L \) the entrant would wish to deviate by entering.

**Conjecture 1** The PEA strategies in this example are also SEA strategies.

**Proof.** [To be completed...] Idea for proof that this PEA is also a SEA: Consider a fully mixed strategy profile in the first period (assuming a finite support for the incumbent), which is identical to the second period PEA strategy profile (the argument below can be replicated under full mixing in both periods without important changes), where \( \gamma_{t,q} \) is the probability that type \( t \) of the incumbent chooses monopoly quantity \( q \), and \( \lambda_q \) is the probability that the entrant enters after observing monopoly quantity \( q \). Under this strategy profile and smooth
rule updating, the entrant’s beliefs after observing \( q \) satisfy

\[
\mu_q(\pi) \propto \frac{\phi'(\sum_q \lambda_q \sum_t \gamma_{t,q}(t) v_t)}{\phi'(\lambda_q \sum_t \pi_q(t) v_t)} \left( \frac{\sum_t \gamma_{t,q}(t)}{\mu(\pi)} \right)
\]

for all \( \pi \) in the support of the ex-ante belief \( \mu \), where \( \pi_q(t) \equiv \frac{\gamma_{t,q}(t)}{\sum_t \gamma_{t,q}(t)} \). Consider the sequence of strategy profiles such that \( \gamma_{t,q}^k \equiv \frac{\beta_{t,q}^k}{\sum_q \beta_{t,q}^k} \) for \( k = 1, 2, \ldots \), where \( \beta_{t,q}^k \) is defined by

<table>
<thead>
<tr>
<th>( t )</th>
<th>( q = q_H )</th>
<th>( q = q_L )</th>
<th>( q_H \neq q &lt; q_L )</th>
<th>( q &gt; q_L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L )</td>
<td>( k^3 )</td>
<td>( k^3 )</td>
<td>( k )</td>
<td>( k^2 )</td>
</tr>
<tr>
<td>( M )</td>
<td>( k )</td>
<td>( k )</td>
<td>( k )</td>
<td>( k )</td>
</tr>
<tr>
<td>( H )</td>
<td>( k^3 )</td>
<td>( k )</td>
<td>( k^2 )</td>
<td>( k )</td>
</tr>
</tbody>
</table>

and \( \lambda_q^k \) for \( k = 1, 2, \ldots \) converges to 1 as \( k \to \infty \) when \( q < q_L \) and converges to 0 otherwise. Note that this sequence converges to the PEA strategy profile. Note also that

\[
\lim_{k \to \infty} \pi_q^k(t) = \lim_{k \to \infty} \frac{\sum_t \gamma_{t,q}^k(t) \pi(t)}{\sum_t \gamma_{t,q}^k(t)} = \frac{\sum_t \gamma_{t,q}(t) \pi(t)}{\sum_t \gamma_{t,q}(t) \pi(t)},
\]

where \( I \) is the indicator function. Thus the corresponding beliefs for the entrant converge to the PEA beliefs, and in particular assign point masses on high cost and low cost respectively as required to generate entry and no entry respectively following off-path quantities in the first period. Thus we have smooth rule consistency and SEA. \( \square \)

## 5 Alternative Approaches

In order to compare with some of the existing literature investigating dynamic games with ambiguity, the following condition, describing a consistent planning requirement in the spirit of Strotz (1955-56), is useful.

**Definition 5.1** Fix a game \( \Gamma \) and a pair \((\sigma^P, \nu^P)\) consisting of a strategy profile and interim belief system. Specify \( V_i \) and \( V_{i,\tau_i, h_i} \) as in (2.4) and (2.6). For each player \( i \), type \( \tau_i \) and
history $h$, let

$$CP_{i,T+1} \equiv \Sigma_i.$$  

Then, inductively, for $0 \leq t \leq T$, let

$$CP_{i,\tau_i,h^t} \equiv \arg\max_{\hat{\delta}_i \in \hat{\tau}_i \in \Theta_i \cap CP_{i,\tau_i,h^t+1}} V_i,\tau_i,h^t((\hat{\delta}_i, \sigma^{P}_{-i}); \nu^P).$$

Finally, let

$$CP_i \equiv \arg\max_{\hat{\delta}_i \in \hat{\tau}_i \in \Theta_i \cap CP_{i,\tau_i,0}} V_i(\hat{\delta}_i, \sigma^{P}_i).$$

$(\sigma^P, \nu^P)$ is optimal under consistent planning if, for all players $i$,

$$\sigma^P_i \in CP_i.$$

Equivalently, $(\sigma^P, \nu^P)$ is such that for all players $i$, all types $\tau_i$ and all partial histories $h^t$,

$$V_i(\sigma^P) \geq V_i(\hat{\delta}_i, \sigma^{P}_{-i}) \text{ for all } \hat{\delta}_i \in \cap CP_{i,\tau_i,0}$$

and

$$V_i,\tau_i,h^t((\hat{\delta}_i, \sigma^{P}_{-i}); \nu^P) \geq V_i,\tau_i,h^t(\sigma^P; \nu^P) \text{ for all } \hat{\delta}_i \in \cap CP_{i,\tau_i,h^t+1}.$$  

If $(\sigma^P, \nu^P)$ is sequentially optimal then it is also optimal under consistent planning. However, if $(\sigma^P, \nu^P)$ is optimal under consistent planning it may fail to be sequentially optimal (even when limiting attention to ambiguity neutrality). For such a failure to occur, there must be a failure of dynamic consistency at some point – the optimal strategy from player $i$’s point of view at some earlier stage has a continuation that fails to be optimal from the viewpoint of some later reachable stage. This is what makes the extra constraints imposed in the optimization inequalities under consistent planning bind.

Sequential optimality is a feature of PEA and SEA. The above shows that sequential optimality is not generally satisfied by the consistent planning approach taken in much existing literature using extensive-form games of incomplete information with ambiguity. Sequential optimality is a relatively uncontroversial part of the main equilibrium concepts for extensive-form games with incomplete information under ambiguity neutrality, such as Perfect Bayesian Equilibrium and Sequential Equilibrium. Thus, it is both important and interesting to explore sequential optimality in the context of games with ambiguity.
Furthermore, when updating is according to the smooth rule, \((\sigma^P, \nu^P)\) optimal under consistent planning implies \((\sigma^P, \nu^P)\) is sequentially optimal, making the two equivalent under smooth rule updating. This follows from Theorem (2.3) and the fact that consistent planning implies no profitable one-stage deviations. Thus, under smooth rule updating, sequential optimality, optimality under consistent planning and no profitable one-stage deviations are equivalent. These observations are generalizations of the fact that updating according to Bayes’ rule makes all three concepts equivalent for expected utility preferences.

**Appendix**

Rather than the “one-step ahead” formulation of smooth rule updating from one partial history from an immediately prior one, one could alternatively (and equivalently) write the smooth rule as updating \(\nu_{i,\tau_i,h^t}\) to \(\nu_{i,\tau_i,h^t}\) all at once:

For each partial history \(h^t\), for all \(\pi \in \Delta(\Theta)\) such that \(\pi(\Theta_{i,\tau_i,h^t}) > 0\),

\[
\nu_{i,\tau_i,h^t}(\pi) \propto \frac{\phi_i' \left( \sum_{\hat{\theta} \in \Theta} \sum_{h \in H} \pi(\Theta_{i,\tau_i,h^t}=h^t_{m_i(h^t)}) u_i(\hat{h}, \hat{\theta}) p_{\sigma,\hat{\theta}}(\hat{h}|h^t_{m_i(h^t)}) \pi_{\Theta_{i,\tau_i,h^t}}(\hat{\theta}) \right)}{\phi_i' \left( \sum_{\hat{\theta} \in \Theta} \sum_{h \in H} \pi(\Theta_{i,\tau_i,h^t}=h^t) u_i(\hat{h}, \hat{\theta}) p_{\sigma,\hat{\theta}}(\hat{h}|h^t) \pi_{\Theta_{i,\tau_i,h^t}}(\hat{\theta}) \right)} \cdot \left( \sum_{\hat{\theta} \in \Theta_{i,\tau_i,h^t}} p_{-i,\sigma,\hat{\theta}}(h^t_{m_i(h^t)}) \pi_{\Theta_{i,\tau_i,h^t}}(\hat{\theta}) \right) \nu_{i,\tau_i,h^t}(\pi).
\]

**Lemma 5.1** Any \((\hat{\sigma}, \hat{\nu})\) satisfying smooth rule consistency also naturally extends updating.

**Proof.** Fix any \((\sigma, \nu)\) such that \(\nu\) satisfies extended smooth rule updating using \(\sigma\) as the strategy profile. Consider players \(i, j \neq i\), type \(\tau_i\) and partial histories \(h^t\) that player \(i\) views as reachable from \(h^{t-1}\) and for which conditions (b)-(d) in the definition of naturally extends updating are satisfied. Since player \(i\) has no costly ambiguity exposure under \(\sigma\) at \(h^{t-1}\) and \(h^t\), the smooth rule updating formula (2.11) simplifies to

\[
\nu_{i,\tau_i,h^t}(\pi) = A \cdot \left( \sum_{\hat{\theta} \in \Theta_{i,\tau_i,h^t}} p_{-i,\sigma,\hat{\theta}}(h^t|h^{t-1}) \pi_{\Theta_{i,\tau_i,h^t-1}}(\hat{\theta}) \right) \nu_{i,\tau_i,h^{t-1}}(\pi)
\]

for \(\pi\) such that \(\pi(\Theta_{i,\tau_i,h^t}) > 0\), where \(A\) is the normalization factor ensuring the left-hand side sums (over \(\pi\)) to 1.
Since player $j$ has only one action at $h^{t-1}$, $p_{-i,\sigma}(h^t|h^{t-1}) = \prod_{k \neq i, j} \sigma_k(h^{t-1}, \hat{\theta}_k)(h_{t-1,k})$. Thus, for each $\theta_j$,

$$
\pi_{\Theta_i,\tau_i,h^t}(\{\theta_j\} \times \Theta_j)\nu_{i,\tau_i,h^t}(\pi) = A\pi_{\Theta_i,\tau_i,h^t}(\{\theta_j\} \times \Theta_j)\left(\sum_{\hat{\theta} \in \Theta_i,\tau_i,h^t} \left(\prod_{k \neq i, j} \sigma_k(h^{t-1}, \hat{\theta}_k)(h_{t-1,k})\right)\pi_{\Theta_i,\tau_i,h^t-1}(\hat{\theta})\right)\nu_{i,\tau_i,h^{t-1}}(\pi).
$$

Summing over the $\pi$ yields,

$$
\sum_{\pi \text{ s.t. } \nu_{i,\tau_i,h^{t-1}}(\pi) > 0 \text{ and } \pi(\Theta_i,\tau_i,h^t) > 0} \pi_{\Theta_i,\tau_i,h^t}(\{\theta_j\} \times \Theta_j)\nu_{i,\tau_i,h^t}(\pi)
$$

$$
= A\sum_{\pi \text{ s.t. } \nu_{i,\tau_i,h^{t-1}}(\pi) > 0 \text{ and } \pi(\Theta_i,\tau_i,h^t) > 0} \pi_{\Theta_i,\tau_i,h^t}(\{\theta_j\} \times \Theta_j)\nu_{i,\tau_i,h^{t-1}}(\pi)
$$

$$
\cdot \left(\sum_{\hat{\theta} \in \Theta_i,\tau_i,h^t} \left(\prod_{k \neq i, j} \sigma_k(h^{t-1}, \hat{\theta}_k)(h_{t-1,k})\right)\pi_{\Theta_i,\tau_i,h^{t-1}}(\hat{\theta})\right).
$$

By (2.5) applied to $\nu_{i,\tau_i,h^{t-1}}$, we can replace the left hand side of (5.2) with

$$
\sum_{\pi \in \Delta(\Theta)} \pi_{\Theta_i,\tau_i,h^t}(\{\theta_j\} \times \Theta_j)\nu_{i,\tau_i,h^t}(\pi).
$$

By definition of $\pi_{\Theta_i,\tau_i,h^t}$,

$$
\pi_{\Theta_i,\tau_i,h^t}(\{\theta_j\} \times \Theta_j) = \frac{\sum_{\hat{\theta} \in \Theta_i,\tau_i,h^t \cap (\{\theta_j\} \times \Theta_j)} \pi_{\Theta_i,\tau_i,h^t-1}(\hat{\theta})\left(\prod_{k \neq i, j} \sigma_k(h^{t-1}, \hat{\theta}_k)(h_{t-1,k})\right)}{\sum_{\hat{\theta} \in \Theta_i,\tau_i,h^t} \pi_{\Theta_i,\tau_i,h^t-1}(\hat{\theta})\left(\prod_{k \neq i, j} \sigma_k(h^{t-1}, \hat{\theta}_k)(h_{t-1,k})\right)}.
$$
Thus, (5.2) becomes,

$$\sum_{\pi \in \Delta(\Theta)} \pi \Theta_{t_{i_{j}}, h_{t_{j}}} (\{ \theta_{j} \} \times \Theta_{-j}) \nu_{i_{j}, h_{t_{j}}} (\pi)$$

$$= A \sum_{\hat{\theta} \in \Theta_{t_{i_{j}}, h_{t_{j}} \cap (\{ \theta_{j} \} \times \Theta_{-j})} \left( \prod_{k \neq i_{j}}{\sigma_{k} \left( h_{t_{j} - 1}, \hat{\theta}_{k} \right) \left( h_{t_{j} - 1, k} \right)} \right)$$

$$\cdot \sum_{\pi \text{ s.t. } \nu_{i_{j}, h_{t_{j} - 1}} (\pi) > 0 \text{ and } \pi(\Theta_{t_{i_{j}}, h_{t_{j}}}) > 0} \pi \Theta_{t_{i_{j}}, h_{t_{j} - 1}} (\hat{\theta}) \nu_{i_{j}, h_{t_{j} - 1}} (\pi).$$

By (2.5) applied to $\nu_{i_{j}, h_{t_{j} - 1}}$, since $\hat{\theta} \in \Theta_{t_{i_{j}}, h_{t_{j}} \cap (\{ \theta_{j} \} \times \Theta_{-j})}$ (so that if $\pi(\Theta_{t_{i_{j}}, h_{t_{j}}}) = 0$ then $\pi \Theta_{t_{i_{j}}, h_{t_{j} - 1}} (\hat{\theta}) = 0$),

$$\sum_{\pi \text{ s.t. } \nu_{i_{j}, h_{t_{j} - 1}} (\pi) > 0 \text{ and } \pi(\Theta_{t_{i_{j}}, h_{t_{j}}}) > 0} \pi \Theta_{t_{i_{j}}, h_{t_{j} - 1}} (\hat{\theta}) \nu_{i_{j}, h_{t_{j} - 1}} (\pi) = \sum_{\pi \in \Delta(\Theta)} \pi \Theta_{t_{i_{j}}, h_{t_{j} - 1}} (\hat{\theta}) \nu_{i_{j}, h_{t_{j} - 1}} (\pi).$$

Since $\nu_{i_{j}, h_{t_{j} - 1}}$ satisfies the condition that the reduced measure $\sum_{\pi \in \Delta(\Theta)} \pi \Theta_{t_{i_{j}}, h_{t_{j} - 1}} (\hat{\theta}) \nu_{i_{j}, h_{t_{j} - 1}} (\pi) \in \Delta (\Theta)$ is a product measure, for $\hat{\theta} \in \Theta_{t_{i_{j}}, h_{t_{j}} \cap (\{ \theta_{j} \} \times \Theta_{-j})}$

$$\sum_{\pi \in \Delta(\Theta)} \pi \Theta_{t_{i_{j}}, h_{t_{j} - 1}} (\hat{\theta}) \nu_{i_{j}, h_{t_{j} - 1}} (\pi)$$

$$= \prod_{k \in N} \sum_{\pi \in \Delta(\Theta)} \pi \Theta_{t_{i_{j}}, h_{t_{j} - 1}} (\{ \hat{\theta}_{k} \} \times \Theta_{-k}) \nu_{i_{j}, h_{t_{j} - 1}} (\pi)$$

$$= \sum_{\pi \in \Delta(\Theta)} \pi \Theta_{t_{i_{j}}, h_{t_{j} - 1}} (\{ \hat{\theta}_{j} \} \times \Theta_{-j}) \nu_{i_{j}, h_{t_{j} - 1}} (\pi) \left( \prod_{k \neq i_{j}}{\sum_{\pi \in \Delta(\Theta)} \pi \Theta_{t_{i_{j}}, h_{t_{j} - 1}} (\{ \hat{\theta}_{k} \} \times \Theta_{-k}) \nu_{i_{j}, h_{t_{j} - 1}} (\pi)} \right)$$

$$= \sum_{\pi \in \Delta(\Theta)} \pi \Theta_{t_{i_{j}}, h_{t_{j} - 1}} (\{ \hat{\theta}_{j} \} \times \Theta_{-j}) \nu_{i_{j}, h_{t_{j} - 1}} (\pi) \left( \prod_{k \neq i_{j}}{\sum_{\pi \in \Delta(\Theta)} \pi \Theta_{t_{i_{j}}, h_{t_{j} - 1}} (\{ \hat{\theta}_{k} \} \times \Theta_{-k}) \nu_{i_{j}, h_{t_{j} - 1}} (\pi)} \right).$$
Substituting into (5.4) yields,

\[
\sum_{\pi \in \Delta(\theta)} \pi_{\theta_i, \tau_i, h^t} (\{\theta_j\} \times \Theta_{-j}) \nu_{i, \tau_i, h^t}(\pi) \tag{5.5}
\]

\[
= A \sum_{\hat{\theta} \in \Theta_{i, \tau_i, h^t \cap \{\theta_j\} \times \Theta_{-j}}} \left( \prod_{k \neq i, j} \sigma_k \left( h^{t-1}, \hat{\theta}_k \right) (h_{t-1,k}) \right) \cdot \left( \prod_{k \neq i, j \pi \in \Delta(\theta)} \pi_{\theta_i, \tau_i, h^{t-1}} (\{\theta_k\} \times \Theta_{-k}) \nu_{i, \tau_i, h^{t-1}}(\pi) \right) \left( \sum_{\pi \in \Delta(\theta)} \pi_{\theta_i, \tau_i, h^{t-1}} (\{\theta_j\} \times \Theta_{-j}) \nu_{i, \tau_i, h^{t-1}}(\pi) \right)
\]

\[
= B \left( \sum_{\pi \in \Delta(\theta)} \pi_{\theta_i, \tau_i, h^{t-1}} (\{\theta_j\} \times \Theta_{-j}) \nu_{i, \tau_i, h^{t-1}}(\pi) \right)
\]

where

\[
B = A \sum_{\hat{\theta} \in \Theta_{i, \tau_i, h^t \cap \{\theta_j\} \times \Theta_{-j}}} \left( \prod_{k \neq i, j} \left( \sigma_k \left( h^{t-1}, \hat{\theta}_k \right) (h_{t-1,k}) \right) \sum_{\pi \in \Delta(\theta)} \pi_{\theta_i, \tau_i, h^{t-1}} (\{\theta_k\} \times \Theta_{-k}) \nu_{i, \tau_i, h^{t-1}}(\pi) \right)
\]

showing that \(i\)’s marginal on player \(j\)’s type at partial history \(h^t\) remains the same as at \(h^{t-1}\).

Since all the operations in the above argument are continuous in \(\sigma\), it is also true that to ensure that \(i\)’s marginal on player \(j\)’s type at partial history \(h^t\) remains approximately the same as at \(h^{t-1}\) it is sufficient to know that conditions (b) and (d) in the definition of naturally extends updating (Definition 2.15) hold to within a given approximation (condition (c) always holds exactly, as it is part of the structure of the game).

By smooth rule consistency of \((\tilde{\sigma}, \tilde{\nu})\), there exists a sequence of completely mixed strategy profiles \(\{\sigma^k\}_{k=1}^\infty\), with \(\lim_{k \to \infty} \sigma^k = \tilde{\sigma}\), such that \(\tilde{\nu} = \lim_{k \to \infty} \nu^k\), where \(\nu^k\) is determined by extended smooth rule updating using \(\sigma^k\) as the strategy profile. Fix any \(i, \tau_i, h^t\) such that conditions (a)-(d) in the definition of naturally extends updating are satisfied for \((\tilde{\sigma}, \tilde{\nu})\). Then, by the argument above applied to \((\sigma^k, \nu^k)\) sufficiently far along the sequence, \((\tilde{\sigma}, \tilde{\nu})\) naturally extends updating.

**Lemma 5.2** Fix a game \(\Gamma\) and \((\sigma, \nu)\) such that \(\sigma\) is an ex ante equilibrium. For \(i, \tau_i, h^t\) such that neither \((t \neq 0 \text{ and } m_i(h^t) = t)\) nor \((t = 0 \text{ and } \tau_i \text{ is } \mu_i\text{-null})\), if \(\nu_{i, \tau_i, h^t}\) is derived from \(\nu_{i, \tau_i, h^t, i(h^t)}\) (or, if \(t = 0\), from \(\mu_i\)) via the smooth rule using \(\sigma\) as the ex ante equilibrium and \(\sigma_i\) is optimal for player \(i\) given \(\tau_i, h^{m_i(h^t)}\) (or, if \(t = 0\), given ex ante optimality) and
\(\sigma_i\), then \(\sigma_i\) is optimal for player \(i\) given \(\tau_i, h^t\) and \(\sigma_{-i}\): for all \(\sigma'_i \in \Sigma_i\),

\[
V_{i,\tau_i, h^t}(\sigma; \nu) \geq V_{i,\tau_i, h^t}(\sigma'_i, \sigma_{-i}; \nu).
\]

**Proof.** Consider first the case where \(t \neq 0\) and \(m_i(h^t) \neq t\). By assumption, \(\sigma_i\) is optimal given \(\tau_i, h^{m_i(h^t)}\) and \(\sigma_{-i}\). This is equivalent (see HK 2009, Lemma A.1) to the condition that \(\sigma_i\) solves

\[
\max_{\sigma'_i \in \Sigma_i} \sum_{\tilde{\theta} \in \Theta} \sum_{\tilde{h} \in H} \sum_{\tilde{h} = \tilde{h}^{m_i(h^t)} = h^{m_i(h^t)}} u_i(\tilde{h}, \tilde{\theta}) p_i(\sigma'_i, \sigma_{-i}, \tilde{\theta})(\tilde{h} | h^{m_i(h^t)}) q(\sigma, \nu, i, \tau_i, h^{m_i(h^t)})(\tilde{h}, \tilde{\theta}), \tag{5.6}
\]

where \(q(\sigma, \nu, i, \tau_i, h^{m_i(h^t)})\) is \(i\)'s \((\sigma, \nu)\)-local measure given \(\tau_i\) and \(h^{m_i(h^t)}\) (defined in (2.12)). Notice that the objective function in (5.6) can be equivalently written as

\[
\sum_{\tilde{\theta} \in \Theta} \sum_{\tilde{h} \in H} \sum_{\tilde{h} = \tilde{h}^{m_i(h^t)} = h^{m_i(h^t)}} u_i(\tilde{h}, \tilde{\theta}) p_i(\sigma'_i, \sigma_{-i}, \tilde{\theta})(\tilde{h} | h^{m_i(h^t)}) q(\sigma, \nu, i, \tau_i, h^{m_i(h^t)})(\tilde{h}, \tilde{\theta})
\]

\[
+ p_i(\sigma'_i, \sigma_{-i}, \tilde{\theta})(h^t | h^{m_i(h^t)}) \sum_{\tilde{\theta} \in \Theta} \sum_{\tilde{h} \in H, \tilde{h} = h^t} u_i(\tilde{h}, \tilde{\theta}) p_i(\sigma'_i, \sigma_{-i}, \tilde{\theta})(\tilde{h} | h^t) q(\sigma, \nu, i, \tau_i, h^{m_i(h^t)})(\tilde{h}, \tilde{\theta}).
\]

The advantage of doing so is making clear that only the term

\[
\sum_{\tilde{\theta} \in \Theta} \sum_{\tilde{h} \in H} u_i(\tilde{h}, \tilde{\theta}) p_i(\sigma'_i, \sigma_{-i}, \tilde{\theta})(\tilde{h} | h^t) q(\sigma, \nu, i, \tau_i, h^{m_i(h^t)})(\tilde{h}, \tilde{\theta})
\]

is affected by \(\sigma'_i(\eta, \theta_i)\) for partial histories \(\eta\) that include \(h^t\) and only those same components of \(\sigma'_i\) affect

\[
\sum_{\tilde{\theta} \in \Theta} \sum_{\tilde{h} \in H, \tilde{h} = h^t} u_i(\tilde{h}, \tilde{\theta}) p_i(\sigma'_i, \sigma_{-i}, \tilde{\theta})(\tilde{h} | h^t) q(\sigma, \nu, i, \tau_i, h^{m_i(h^t)})(\tilde{h}, \tilde{\theta}).
\]

Therefore (5.6) implies that \(\sigma_i\) also solves

\[
\max_{\sigma'_i \in \Sigma_i} \sum_{\tilde{\theta} \in \Theta} \sum_{\tilde{h} \in H} u_i(\tilde{h}, \tilde{\theta}) p_i(\sigma'_i, \sigma_{-i}, \tilde{\theta})(\tilde{h} | h^t) q(\sigma, \nu, i, \tau_i, h^{m_i(h^t)})(\tilde{h}, \tilde{\theta}).
\]

We want to show that \(\sigma_i\) is optimal given \(\tau_i, h^t\) and \(\sigma_{-i}\). This is equivalent to the condition that \(\sigma_i\) solves

\[
\max_{\sigma'_i \in \Sigma_i} \sum_{\tilde{\theta} \in \Theta} \sum_{\tilde{h} \in H} u_i(\tilde{h}, \tilde{\theta}) p_i(\sigma'_i, \sigma_{-i}, \tilde{\theta})(\tilde{h} | h^t) q(\sigma, \nu, i, \tau_i, h^t)(\tilde{h}, \tilde{\theta}).
\]
where \( q^{(\sigma,\nu),i,\tau_i,h^t} \) is the gradient of the interim indifference curve given \( \tau_i \) and \( h^t \) at \( \sigma \), which, for each \( \bar{\theta} \in \Theta \) and \( \bar{h} \in H \) with \( \bar{\theta} = \tau_i \) and \( \bar{h} = h^t \), is given by

\[
q^{(\sigma,\nu),i,\tau_i,h^t}(\bar{h}, \bar{\theta}) = \sum_{\pi \in \Delta(\Theta) | \pi(\Theta_{i,\tau_i,h^t}) > 0} \phi_i^I \left( \sum_{\hat{\theta} \in \Theta} \sum_{h \in H | h = h^t} u_i(\hat{h}, \hat{\theta}) p_{\sigma,\hat{\theta}}(\hat{h}|h^t) \pi_{\Theta_{i,\tau_i,h^t}}(\hat{\theta}) \right) \cdot p_{-i,\sigma,\hat{\theta}}(\hat{h}|h^t) \pi_{\Theta_{i,\tau_i,h^t}}(\hat{\theta}) \nu_{i,\tau_i,h^t}(\pi).
\]

Thus it is sufficient to show that

\[
q^{(\sigma,\nu),i,\tau_i,h^t}(\bar{h}, \bar{\theta}) \propto q^{(\sigma,\nu),i,\tau_i,h mi(h^t)}(\bar{h}, \bar{\theta})
\]

for each \( \bar{\theta} \in \Theta \) and \( \bar{h} \in H \) with \( \bar{\theta} = \tau_i \) and \( \bar{h} = h^t \). This follows by using the local measure definition (2.12), applying the smooth rule to substitute for \( \nu_{i,\tau_i,h^t}(\pi) \) for all \( \pi \in \Delta(\Theta) \) such that \( \pi(\Theta_{i,\tau_i,h^t}) > 0 \) (as \( \nu_{i,\tau_i,h^t}(\pi) = 0 \) for other \( \pi \)) and then using the definitions of \( \pi_{\Theta_{i,\tau_i,h^t}}(\hat{\theta}) \) and \( \pi_{\Theta_{i,\tau_i,h mi(h^t)}} \) and cancelling terms.

The case where \( t = 0 \) and \( \tau_i \) is not \( \mu_i \)-null is similar.

References


