THE Q-HOMOTOPY ANALYSIS METHOD (Q-HAM)

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ABSTRACT

In this paper, a more general method of homotopy analysis method (HAM) is introduced to solve non-linear differential equations, it is called (q-HAM). The interval of convergence of HAM, if exists, is increased when using q-HAM. The analysis shows that the series solution in the case of q-HAM is more likely to converge than that on HAM. The new method is applied to some nonlinear differential equations to illustrate the method of analysis.

Keywords: Homotopy Analysis Method (HAM), Partial Differential Equations

1 INTRODUCTION

The standard homotopy analysis method (HAM) is an analytic method that provides series solutions for nonlinear partial differential equations and has been firstly proposed by (Liao 1992). In recent years, this method has been successfully employed to solve many types of linear and non-linear differential equations in various fields of engineering and science (Liao 1995,1996,1997,2002,2003,2004,2009 ;Wang et al.2007;Bataineh et al.2009;Van Gorder and Vairavelu 2009). Different from all perturbation techniques (the techniques are strongly dependent upon small / large physical parameters (Nayfeh 1981,1985,2000;Lagerstrom 1988;Murdock 1991;Hinch 1991)) and non perturbation techniques, such as the Lyapunov artificial small parameter method (Lyapunov 1992) , the δ-expansion method (Karmishin et al. 1990; Awrejcewicz et al. 1998) , Adomians decomposition method (Rach 1984; Adomain and Adomain GE 1984,Adomain 1991) and so on which are formally independent of small/large physical parameters but they can not ensure the convergence of solution series, the homotopy analysis method has the following advantages:

(i) It is valid even if a given nonlinear problem does not contain any small/large parameters.
(ii) It can provide us with a convenient way to adjust and control the convergence region and rate of approximation series.
(iii) It can be employed to efficiently approximate a nonlinear problem by choosing different sets of base functions.
In recent years more and more researchers have been successfully applying (HAM) to various nonlinear problems in science and engineering, such as, the nonlinear water waves (Tao et al. 2007), groundwater flows (Song and Tao 2007), time-dependent Emden-Fowler type equations (Bataineh et al. 2007), solving high order nonlinear differential equations (Hany N Hassan and El-Tawil 2010), solving two points nonlinear boundary value problems (Hany N Hassan and El-Tawil 2010), time-fractional partial differential equations (O Abdulaaziz et al. 2008), Klein-Gordon equation (Alomari et al. 2008), Goursat problems (Syyed Ali Kazemipour and Ahmad Neyrameh 2010), one group neutron diffusion equation (Cavdar 2010), algebraic equations (Chin Fugn Yuen et al. 2010), integral and integro-differential equations (Hossein Zadeh et al. 2010), singular higher-order boundary value problems (A Sami Bataineh et al. 2011).

2 BASIC IDEA OF Q-HOMOTOPY ANALYSIS METHOD (Q-HAM)

Consider the following differential equation:

\[ N[u(x, t)] - f(x, t) = 0 \]  

(1)

where \( N \) is a nonlinear operator, \((x, t)\) denotes independent variables, \( f(x, t) \) is a known function and \( u(x, t) \) is an unknown function.

Let us construct the so-called zero-order deformation equation:

\[ (1 - nq)L[\Phi(x, t; q) - u_0(x, t)] = qhH(x, t)(N[\Phi(x, t; q)] - f(x, t)) \]  

(2)

Where \( n \geq 1 \), \( q \in [0, \frac{1}{n}] \) denotes the so-called embedded parameter, \( L \) is an auxiliary linear operator with the property \( L[f] = 0 \) when \( f = 0, h \neq 0 \) is an auxiliary parameter, \( H(x, t) \) denotes a non-zero auxiliary function.

It is obvious that when \( q = 0 \) and \( q = \frac{1}{n} \) equation (2) becomes:

\[ \Phi(x, t; 0) = u_0(x, t) \]  

,  

\[ \Phi(x, t; \frac{1}{n}) = u(x, t) \]  

(3)

respectively. Thus as \( q \) increases from 0 to \( \frac{1}{n} \), the solution \( \Phi(x, t; q) \) varies from the initial guess \( u_0(x, t) \) to the solution \( u(x, t) \).

Having the freedom to choose \( u_0(x, t), L, h, H(x, t) \), we can assume that all of them can be properly chosen so that the solution \( \Phi(x, t; q) \) of equation (2) exists for \( q \in [0, \frac{1}{n}] \).

Expanding \( \Phi(x, t; q) \) in Taylor series, one has:

\[ \Phi(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t)q^m \]  

(4)

Where:

\[ u_m(x, t) = \frac{1}{m!} \frac{\partial^m \Phi(x, t; q)}{\partial q^m} \bigg|_{q=0} \]  

(5)
Assume that $h, H(x, t), u_0(x, t), L$ are so properly chosen such that the series (4) converges at $q = \frac{1}{n}$ and:

$$u(x, t) = \emptyset \left( x, t; \frac{1}{n} \right) = u_0(x, t) + \sum_{m=1}^{+\infty} u_m(x, t) \left( \frac{1}{n} \right)^m \tag{6}$$

Defining the vector $u_r(x, t) = \{u_0(x, t), u_1(x, t), u_2(x, t), \ldots, u_r(x, t)\}$, differentiating equation (2) $m$ times with respect to $q$ and then setting $q = 0$ and finally dividing them by $m!$ we have the so-called $m^{th}$ order deformation equation:

$$L[u_m(x, t) - k_m u_{m-1}(x, t)] = hH(x, t)R_m(u_{m-1}(x, t)) \tag{7}$$

where:

$$R_m(u_{m-1}(x, t)) = \left. \frac{1}{(m-1)!!} \frac{\partial^{m-1}(N[\emptyset(x,t;q) - f(x,t)])}{\partial q^{m-1}} \right|_{q=0} \tag{8}$$

and:

$$k_m = \begin{cases} 0 & m \leq 1 \\ \frac{1}{n} & \text{otherwise} \end{cases} \tag{9}$$

It should be emphasized that $u_m(x, t)$ for $m \geq 1$ is governed by the linear equation (7) with linear boundary conditions that come from the original problem. Due to the existence of the factor $\left( \frac{1}{n} \right)^m$, more chances for convergence may occur or even much faster convergence can be obtained better than the standard HAM. It should be noted that the case of $n = 1$ in equation (2), standard HAM can be reached.

3 APPLICATIONS

3.1 The nonlinear homogeneous gas dynamics equation

Let

$$u_t + \frac{1}{2}(u^2)_x - u(1 - u) = 0, \quad 0 \leq x \leq 1, \quad t \geq 0 \tag{10}$$

with initial condition

$$u(x, 0) = e^{-x} \tag{11}$$

The exact solution of this problem is known to be

$$u(x, t) = e^{e^{-x}} \tag{12}$$

This problem was solved by HAM in (Hany Nasr Hassan 2010).

For $q$-HAM solution we choose the linear operator:

$$L[\emptyset(x, t; q)] = \frac{\partial \emptyset(x,t;q)}{\partial t} \tag{13}$$

with the property:

$$L[c_1] = 0$$
where \( c_1 \) is constant. Using initial approximation \( u_0(x, t) = e^{-x} \),

we define a nonlinear operator as

\[
N[\varnothing(x, t; q)] = \frac{\partial \varnothing(x, t; q)}{\partial t} + \frac{1}{2} \frac{\partial (\varnothing^2(x, t; q))}{\partial x} - \varnothing(x, t; q) + \varnothing^2(x, t; q)
\]

We construct the zero order deformation equation:

\[
(1 - nq)L[\varnothing(x, t; q) - u_0(x, t)] = qhH(x, t)N[\varnothing(x, t; q)].
\]

we can take \( H(x, t) = 1 \), and the \( m^{th} \) order deformation equation is:

\[
L[u_m(x, t) - k_m u_{m-1}(x, t)] = hR_m(u_{m-1}(x, t))
\]

with the initial conditions for \( m \geq 1 \)

\[
u_m(x, 0) = 0
\]

Where:

\[
k_m = \begin{cases} 0 & m \leq 1 \\ n & \text{otherwise} \end{cases}
\]

And:

\[
R_m(u_{m-1}(x, t)) = \frac{1}{(m-1)!!} \frac{\partial^{m-1}N[\varnothing(x, t; q)]}{\partial q^{m-1}} \bigg|_{q=0}
\]

\[
= \frac{\partial u_{m-1}(x, t)}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} \sum_{i=1}^{m-1} u_i(x, t) u_{m-1-i}(x, t) - u_{m-1}(x, t) + \\
\sum_{i=1}^{m-1} u_i(x, t) u_{m-1-i}(x, t)
\]

Now the solution of equation (10) for \( m \geq 1 \) becomes

\[
u_m(x, t) = k_m u_{m-1}(x, t) + h \int R_m(u_{m-1}(x, s)) ds + c_1
\]

where the constant of integration \( c_1 \) is determined by the initial conditions (15).

We can obtain components of the solution using q- HAM as follows:

\[
u_1(x, t) = -e^{-x}ht
\]

\[
u_2(x, t) = -e^{-x}ht + e^{-x}h^2(-t + \frac{t^2}{2})
\]

\[
u_3(x, t) = -\frac{1}{2} e^{-x}h^2(2ht + 2nt - 2ht^2 - nt^2 + \frac{ht^3}{3}) + n(-e^{-x}ht + e^{-x}h^2(-t + \frac{t^2}{2}))
\]

\( u_m(x, t) \), \( m = 4, 5, 6, ... \) can be calculated similarly. Then the series solution expression by q- HAM can be written in the form:

\[
u(x; n; h) \equiv U_M(x, t; n; h) = \sum_{i=0}^{M} u_i(x, t; n; h) \left( \frac{1}{n} \right)^i
\]

Equation (16) is an approximate solution to the problem (10) in terms of the convergence parameters $h$ and $n$. It can be noted that the fraction factor $\left(\frac{1}{n}\right)^m$ highly increase the convergence chances than that of HAM.

To find the valid region of $h$, the $h$-curves given by the 10th order q-HAM approximation at different values of $x, t$ and $n$ are drawn in figures (1,2,3 and 4). These figures show the interval of $h$ at which the value of $U_{10}(x,t;n)$ is constant at certain values of $x, t$ and $n$. We choose the horizontal line parallel to $x-axis (h)$ as a valid region which provides us with a simple way to adjust and control the convergence region of the series solution (16). From these figures, the valid intersection region of $h$ for the values of $x, t$ and $n$ in the curves becomes larger as $n$ increase as in the following Table (1):

<table>
<thead>
<tr>
<th>$n$</th>
<th>$h$ region</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-1.5 \leq h \leq -1.1$</td>
</tr>
<tr>
<td>1.5</td>
<td>$-2.4 \leq h \leq -1.5$</td>
</tr>
<tr>
<td>2</td>
<td>$-3.4 \leq h \leq -2.1$</td>
</tr>
<tr>
<td>3</td>
<td>$-4.8 \leq h \leq -3.1$</td>
</tr>
<tr>
<td>4</td>
<td>$-7 \leq h \leq -4.2$</td>
</tr>
<tr>
<td>5</td>
<td>$-8.1 \leq h \leq -5$</td>
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<tr>
<td>10</td>
<td>$-16 \leq h \leq -10$</td>
</tr>
<tr>
<td>20</td>
<td>$-29 \leq h \leq -22$</td>
</tr>
<tr>
<td>30</td>
<td>$-43 \leq h \leq -33$</td>
</tr>
<tr>
<td>40</td>
<td>$-59 \leq h \leq -44$</td>
</tr>
<tr>
<td>50</td>
<td>$-83 \leq h \leq -48$</td>
</tr>
<tr>
<td>100</td>
<td>$-172 \leq h \leq -95$</td>
</tr>
</tbody>
</table>

**Table (1): the increase of the convergence interval length with the increase of $n$**

Figures (5,6) show the comparison among $U_5, U_7$ and $U_{10}$ using different values of $n$ with the exact solution (12). Figures (7,8 and 9) show the comparison between $U_{10}$ of HAM and $U_{10}$ of q-HAM using different values of $n$ with the exact solution (12), which indicates that the speed of convergence for q-HAM with $n > 1$ is faster in comparison with $n = 1$.

The Absolute errors of the 10th order solutions q-HAM approximate at $x = 0$ using different values of $n > 1$ compared with 10th order solutions HAM approximate at $x = 0$ are calculated by the formula

$$\text{Absolute Error} = |u_{exact} - u_{approx}|$$ (17)

Figures (9,10) show that the series solution obtained by HAM is more accurate at $(0 < t \leq 4.5)$ but at larger $t$ the series solutions obtained by q-HAM at $n > 1$ converge faster than $n = 1$(HAM).
Figure (1) : $h$ - curve for the HAM (q-HAM; $n = 1$) approximation solution $U_{10}(x; t; 1)$ of problem (10) at different values of $x$ and $t$.

Figure (2) : $h$ - curve for the (q-HAM; $n = 1.5$) approximation solution $U_{10}(x; t; 1.5)$ of problem (10) at different values of $x$ and $t$. 
Figure (3): \( h \) - curve for the (q-HAM; \( n = 5 \)) approximation solution \( U_{10}(x; t; 5) \) of problem (10) at different values of \( x \) and \( t \).

Figure (4): \( h \) - curve for the (q-HAM; \( n = 10 \)) approximation solution \( U_{10}(x; t; 10) \) of problem (10) at different values of \( x \) and \( t \).
Figure (5): Comparison between $U_5, U_7, U_{10}$ of HAM ($q$-HAM; $n = 1$) and exact solution of (10) at $x = 0.5$ with $h = -1.2$, $0 < t \leq 15$

Figure (6): Comparison between $U_5, U_7, U_{10}$ of ($q$-HAM; $n = 5$) and exact solution of (10) at $x = 0.5$ with $h = -7.2$, $0 < t \leq 15$
Figure (7): Comparison between $U_{10}$ of HAM (q-HAM; $n = 1$) and (q-HAM ; $n = 1.5, 2, 3, 4, 5$) with exact solution of (10) at $x = 0.5$ with $(h = -1.2, -2, -2.7, -4.2, -5.6, -7.2)$, respectively, $0 < t \leq 15$

Figure (8): Comparison between $U_{10}$ of HAM (q-HAM (n = 1)) and q-HAM(n = 10, 20, 30, 40, 50, 100) with exact solution of (10) at $x = 0.5$ with $(h = -1.2, h = -12.5, h = -26, h = -40, h = -54, h = -73, h = -160), 0 < t \leq 20$
3.2 The Riccati equation

Let
\[ u' = u^2 - 2xu + x^2 + 1, \quad u(0) = 1 \]  

The exact solution is known to be
\[ u(x) = x + \frac{1}{1-x}, \quad |x| < 1 \]
For q-HAM solution we choose the linear operator:
\[ L[\phi(x; q)] = \frac{\partial \phi(x; q)}{\partial x} \]  
\[ (20) \]

With the property:
\[ L[c_1] = 0 \]

Where \( c_1 \) is constant. Using initial approximation
\[ u_0(x) = u(0) = 1 \]

We define a nonlinear operator as:
\[ N[\phi(x; q)] = \frac{\partial \phi(x; q)}{\partial x} - \phi^2(x; q) + 2x \phi(x; q) \]

We construct the zero order deformation equation
\[ (1 - nq)L[\phi(x; q) - u_0(x)] = qH(x)(N[\phi(x; q)] - f(x)) \]

Where \( f(x) = x^2 + 1 \). We can take \( H(x) = 1 \), and the \( m^{th} \) order deformation equation is:
\[ L[u_m(x) - k_m u_{m-1}(x)] = hR_m(u_{m-1}(x)) \]  
\[ (21) \]

With the initial conditions for \( m \geq 1 \):
\[ u_m(0) = 0 \]  
\[ (22) \]

Where:
\[ k_m = \begin{cases} 
0 & m \leq 1 \\
n & \text{otherwise} 
\end{cases} \]

and
\[ R_m(u_{m-1}(x)) = \frac{1}{(m-1)!} \frac{\partial^{m-1}(N[\phi(x; q)] - f(x))}{\partial q^{m-1}} \bigg|_{q=0} \]

\[ = \frac{d u_{m-1}(x)}{dx} - \sum_{i=0}^{m-1} u_i(x) u_{m-1-i}(x) + 2x u_{m-1}(x) - (1 - \frac{1}{n} k_m) (x^2 + 1) \]

Now the solution of equation (18) for \( m \geq 1 \) becomes:
\[ u_m(x) = k_m u_{m-1}(x) + h \int R_m(u_{m-1}(s)) ds + c_1 \]

Where the constant of integration \( c_1 \) is determined by the initial conditions (22).

We now obtain components of the solution using q-HAM as follows:
\[ u_1(x) = h(-2x + x^2 - \frac{x^3}{3}) \]
\[ u_2(x) = hn(-2x + x^2 - \frac{x^3}{3}) + h(-2hx + 3hx^2 - \frac{7hx^3}{3} - \frac{2hx^4}{3} - \frac{2hx^5}{15}) \]
\[ u_3(x) = h(-2h^2 x - 2hnx + 5h^2 x^2 + 3hnx^2 - 7h^2 x^3 - \frac{7}{3} hnx^3 + \frac{13h^2 x^4}{3} + \frac{2}{3} hnx^4 - \frac{9h^2 x^5}{5} + \frac{2}{45} hnx^5) + n(h(-2x + x^2 - \frac{x^3}{3}) + h(-2hx + 3hx^2 - \frac{7hx^3}{3} + \frac{2hx^4}{3} - \frac{2hx^5}{15})) \]

\[ u_m(x), \ (m = 4, 5, 6,...) \] can be calculated similarly. Then the series solution expression by q-HAM can be written in the form:

Equation (23) is a family of approximation solutions to the problem (18) in terms of the convergence parameters $h$ and $n$.

To find the valid region of $h$, the $h$-curves given by the 15th order $q$-HAM approximation at different values of $x$ and $n$ are drawn in figures (11,12). From these figures, the valid intersection region of $h$ for the values of $x$ and $n$ in the curves becomes larger as $n$ increases as in the following Table (2).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$h$ region</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-1.25 \leq h \leq -1.15$</td>
</tr>
<tr>
<td>2</td>
<td>$-2.7 \leq h \leq -2.5$</td>
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</tr>
<tr>
<td>4</td>
<td>$-5.3 \leq h \leq -4.9$</td>
</tr>
<tr>
<td>5</td>
<td>$-6.8 \leq h \leq -6.3$</td>
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<td>$-13.2 \leq h \leq -12.1$</td>
</tr>
<tr>
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<td>$-27 \leq h \leq -25$</td>
</tr>
<tr>
<td>30</td>
<td>$-41 \leq h \leq -36.8$</td>
</tr>
<tr>
<td>40</td>
<td>$-55.4 \leq h \leq -49$</td>
</tr>
<tr>
<td>50</td>
<td>$-68 \leq h \leq -60$</td>
</tr>
<tr>
<td>100</td>
<td>$-138 \leq h \leq -127$</td>
</tr>
</tbody>
</table>

Table (2): the increase of the convergence interval length with the increase of $n$

Figure (11): $h$ - curves for the HAM $(q$-HAM; $n = 1)$ approximation solution $U_{15}(x; 1)$ of problem (18) at different values of $x$. 

Figure (12): $h$ - curves for the (q-HAM; $n = 100$) approximation solution $U_{15}(x; 100)$ of problem (18) at different values of $x$.

Figures (13,14) show the comparison between $U_{15}$ of HAM and $U_{15}$ of q-HAM using different values of $n$ with the exact solution (19), which indicates that the speed of convergence for q-HAM with $n > 1$ is faster in comparison with $n = 1$.

Figure (13): Comparison between $U_{15}$ of HAM(q-HAM ($n = 1$)) and q-HAM($n = 2$) with exact solution of (18) with ($h = -1.2$, $h = -2.55$).
The absolute errors of the 15th order solutions q-HAM approximate using different values of n > 1 compared with 15th order solutions HAM approximate are shown in figures (15,16). These figures show that the series solution obtained by HAM is more accurate at (0 < x ≤ 0.55) but at larger x the series solution obtained by q-HAM at n > 1 converge faster than n = 1(HAM).

Figure (14): Comparison between U_{15} of HAM(q-HAM (n = 1)) and q-HAM(n = 100) with exact solution of (18) with (h = -1.2, h = -133).

Figure (15): The Absolute error of U_{15} of q-HAM (n = 1, n = 100) for problem (18) at 0 ≤ x ≤ 0.55 and using h = -1.2 and h = -133.
Figure (16): The Absolute error of $U_{15}$ of q-HAM ($n = 1, n = 100$) for problem (18) at $0.55 \leq x \leq 0.85$ and using $h = -1.2$ and $h = -133$

It should be noted that the absolute error is highly decreased when modifying the solution by taking more terms into consideration. Figures (17,18) illustrate this fact.

Figure 17: The Absolute error of $U_{10}$ ($n = 1, n = 100$) and $U_{30}(n = 100)$ of q-HAM at $0 \leq x \leq 0.5$ using $h = -1.35, h = -135$ and $h = -96.98$ respectively.
Figure 18: The Absolute error of $U_{10} (n = 1, n = 100)$ and $U_{30} (n = 100)$ of q-HAM at $0.5 \leq x \leq 0.9$ using $h = -1.35, h = -135$ and $h = -96.98$ respectively.

3.3 The logistic growth model

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) \quad (24)$$

where $r$ and $K$ are positive constants. Here $N = N(t)$ represents the population of the species at time $t$, and $K$ is the carrying capacity of the environment.

Let:
$$u(\tau) = \frac{N(t)}{K}, \quad \tau = rt$$

Then (1) becomes
$$\frac{du}{d\tau} = u(1 - u) \quad (25)$$

If $N(0) = N_0$ then $u(0) = \frac{N_0}{K}$, therefore, the analytical solution of (25) is easily obtained (such as Bernoulli method and separation of variables)
$$u(\tau) = \frac{1}{1 + (\frac{N_0}{K} - 1)e^{-\tau}} \quad (26)$$

For q- HAM solution we choose the linear operator:
$$L[\varphi(\tau; q)] = \frac{\partial \varphi(\tau; q)}{\partial \tau} \quad (27)$$

With the property:
$$L[c_1] = 0$$

Where $c_1$ is constant. Using initial approximation:
$$u_0(\tau) = \frac{N_0}{K} \quad (28)$$
We define a nonlinear operator as:
\[ N[\phi(\tau; q)] = \frac{\partial \phi(\tau; q)}{\partial \tau} - \phi(\tau; q) + \phi^2(\tau; q) \]

We construct the zeroth-order deformation equation:
\[(1 - nq)L[\phi(\tau; q) - u_0(\tau)] = qhH(\tau)N[\phi(\tau; q)] \]

We can take \( H(\tau) = 200 \), and the mth-order deformation equation is:
\[ L[u_m(\tau) - k_m u_{m-1}(\tau)] = h \ R_m(u_{m-1}(\tau)) \quad (29) \]

With the initial conditions for \( \geq 1 \):
\[ u_m(0) = 0 \quad (30) \]

Where:
\[ k_m = \begin{cases} 0 & m \leq 1 \\ n & \text{otherwise} \end{cases} \]

And:
\[ R_m(u_{m-1}(\tau)) = \left. \frac{1}{(m-1)!} \frac{\partial^{m-1}N[\phi(\tau; q)]}{\partial q^{m-1}} \right|_{q=0} = \frac{\partial u_{m-1}(\tau)}{\partial \tau} - u_{m-1}(\tau) + \sum_{i=0}^{m-1} u_i(\tau)u_{m-1-i}(\tau) \quad (31) \]

Now the solution of equation (29) for \( m \geq 1 \) becomes:
\[ u_m(\tau) = k_m u_{m-1}(\tau) + 200 \ h \int R_m(u_{m-1}(\tau)) d\tau + c_1 \]

Here the constant of integration \( c_1 \) is determined by the initial conditions (30).

For numerical results we take \( N_0 = 2 \ and \ K = 1 \), therefore (26) becomes:
\[ u(\tau) = \frac{2}{2-e^{-\tau}} \quad (32) \]

We now obtain components of the solution using q-HAM as follows:

\[ u_1(\tau) = 400h\tau \]
\[ u_2(\tau) = 400h\tau + 200h(400h\tau + 600h^2) \]
\[ u_3(\tau) = 200h(80000h^2\tau + 400h\tau + 240000h^2\tau^2 + 600h\tau^2 + \frac{520000h^2\tau^3}{3} + n(400h\tau + 200h(400h\tau + 600h^2\tau^2)) \]

\[ u_m(\tau) \ , (m = 4,5,6, ...) \] can be calculated similarly. Then the series solution expression by q-HAM can be written in the form:
\[ u(\tau; n; h) = U_M(\tau; n; h) = \sum_{i=0}^{M} u_i(\tau; n; h) \left( \frac{1}{n} \right)^i \quad (33) \]

Equation (33) is a family of approximation solutions to the problem (24) in terms of the convergence parameters \( h \ and \ n \).
To find the valid region of \( h \), the \( h \)-curves given by the 10th order q-HAM approximation at different values of \( \tau \), and \( n \) are drawn in figures (19, 20 and 21). This figures show the interval of \( h \) where the value of \( U_{10}(\tau; n) \) is constant at certain \( \tau \), and \( n \). We choose the horizontal line parallel to \( x - axis \) (\( h \)) as a valid region which provides us with a simple way to adjust and control the convergence region.

Figure (19): \( h \)-curves for the HAM (q-HAM; \( n = 1 \)) approximation solution \( U_{10}(\tau; 1) \) of problem (1) at different values of \( \tau \).

Figure (20): \( h \)-curves for the q-HAM (\( n = 10 \)) approximation solution \( U_{10}(\tau; 10) \) of problem (1) at different values of \( \tau \).
Figure (21): $h$ - curves for the $q$-HAM ($n = 100$) approximation solution $U_{10}(\tau; 100)$ of problem (1) at different values of $\tau$.

Figures (22,23 and 24) show the comparison between $U_{10}$ of HAM and $U_{10}$ of $q$-HAM using different values of $n$ with the solution (32)

Figure(22): Comparison between $U_{10}$ of HAM($q$-HAM ($n = 1$)) and $q$-HAM($n = 2$) and $u(\tau)$ for logistic growth model (1) with $h = -0.003, h = -0.005$ respectively.
Figure(23): Comparison between $U_{10}$ of HAM(q-HAM ($n = 1$)) and q-HAM($n = 10$) and $u(\tau)$ for logistic growth model (1) with ($h = -0.003, h = -0.019$) respectively.

The Absolute errors of the 10th order solutions q-HAM approximate using different values of $n > 1$ compared with 10th order solutions HAM approximate are calculated by the formula

$$\text{Absolute Error} = |u(\tau) - u_{\text{approx}}|.$$

Figures (25,26 and 27) show that the series solution obtained by HAM is divergent at ($1.3 \leq \tau \leq 3$) but, the series solutions obtained by q-HAM ($n > 1$) becomes more accurate as $n$ increases.
Figure (25): The Absolute error of $U_{10}$ of HAM ($q$-HAM ($n = 1$) and $q$-HAM ($n = 2$) for problem (1) using $h = -0.003$ and $h = -0.005$ with $0 \leq \tau \leq 3$.

Figure (26): The Absolute error of $U_{10}$ of HAM ($q$-HAM ($n = 1$) and $q$-HAM ($n = 10$) for problem (1) using $h = -0.003$ and $h = -0.019$ with $0 \leq \tau \leq 3$. 

4 CONCLUSION

A more general method (q-HAM) of the homotopy analysis method (HAM) was proposed. This method provides an approximate solution by taking different values of q. The results show that the convergence region of series solutions obtained by q-HAM is increasing as q is decreased. The q-HAM improves the performance of HAM and it was shown from the illustrative examples that the convergence of q-HAM is faster than the convergence of HAM.

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