Robust Economic Model Predictive Control  
with Linear Average Constraints

Florian A. Bayer and Frank Allgöwer

Abstract—In this paper, we extend the idea of a tube-based economic MPC framework for linear systems by taking linear average constraints into account. Using a specifically defined integral stage cost, we can explicitly consider the influence of the disturbance. The satisfaction of the average constraints is guaranteed despite the disturbances acting on the system, and we show results on optimal steady-state operation. Moreover, we make use of two algorithmic approaches from tube-based MPC which provide differences in the closed-loop behavior and in the average performance.

I. INTRODUCTION

Model Predictive Control (MPC) is a modern control concept which has attracted a lot of attention in theory as well as in practice. Not only has a thorough theory been developed (see e.g. [1]), but MPC has also found its way to many applications (cf. [2]) where the success of MPC is based on two major advantages: (i) it can deal with hard input and state constraints and (ii) a performance criterion can explicitly be taken into account.

Stabilization of a given setpoint or tracking of a predefined reference trajectory is in the focus of most literature available on MPC (see e.g. [3] and the references therein). This can, for example, be achieved by employing a stage cost which is positive definite with respect to the desired setpoint. However, stabilization of a pre-defined setpoint might not always be the most desired behavior, and thus, being driven by problems from the process industry, the idea came up to associate the stage cost with a favored economic relation. Such economic relations could, for example, allow for minimization of the energy consumption or maximization of an output. The loss of a direct relation to a specific steady-state – in particular the stage cost does not need to be positive definite about a specific steady-state – is the major difference to stabilizing (or tracking) MPC. Because of the economic relation, this framework has been presented by the name economic MPC (see e.g. [4], [5], [6],[7]).

When real applications are taken into account, most of them are affected by either disturbances or uncertainties. Because of this, a wide number of literature deals with robust (stabilizing) MPC (see e.g. [8], [9]), where the general goal is to provide feasibility and stability despite the disturbances and uncertainties. Most of these approaches in robust MPC aim to determine an invariant set either for the disturbed system or for the difference between the disturbed system and an artificially introduced undisturbed – or nominal – system. In linear MPC, approaches of the second kind are usually referred to as “tube MPC” since the disturbed system is kept in a “tube” around the nominal system. Even though considering disturbances and uncertainties is of practical relevance, only few publications can be found on economic MPC for disturbed systems. In [10], based on a formulation that is related to tracking MPC, a stability result for robust economic MPC is presented. A scenario based approach is studied in [11], also imposing probabilistic constraints. In [12], the uncertainty is considered within the constraints and robustness of steady-state optimality is investigated.

The contribution of this paper is the consideration of linear average constraints for linear systems in robust economic MPC. This is an extension of the results in [13] and [14]. We employ the robust economic MPC approach based on averaging over invariant error sets, which was shown to be advantageous compared to applying tube-based stabilizing MPC in an economic framework. By using the well-known idea of set tightening – in tube-based MPC this is used to guarantee feasibility – an appropriate tightening of the linear average constraints is introduced which allows to provide feasibility of the average constraints for the disturbed system. Moreover, when the nominal initial state at the next MPC iteration is determined by the nominal system dynamics, we can derive a statement for steady-state optimality despite the disturbances. This statement is based on the concept of dissipativity. On the other hand, if the nominal initial state is left as an optimization variable at each iteration, the average constraint is still satisfied but no statement on steady-state optimality can be given. However, simulations impose that using this additional degree of freedom can increase the performance significantly.

The remainder of this paper is organized as follows. In Section II, the problem is described and the concept of robust economic MPC based on averaging over invariant sets is summarized. The idea of applying linear average constraints for linear systems within this robust economic MPC concept is presented in Section III. A numerical example is provided in Section IV and the paper is concluded in Section V.

Notation: By \( \mathbb{I}_{\geq 0} \), we denote the set of non-negative integers and by \( I_{[a,b]} \) the set of all integers in the interval \( [a,b] \in \mathbb{R} \). For the sets \( X, Y \subseteq \mathbb{R}^n \), the Minkowski set addition is defined by \( X + Y := \{x + y : x \in X, y \in Y\} \); the Pontryagin set difference is defined by \( X \ominus Y := \{z : z + y \in X, \forall y \in Y\} \). Following the definition in [4], the set...
of asymptotic averages of a bounded signal \( v : I_{\geq 0} \to \mathbb{R}^{n_v} \) is defined by
\[
Av[v] := \{ \bar{v} \in \mathbb{R}^{n_v} : \exists t_n \to \infty : \lim_{n \to \infty} \sum_{k=0}^{t_n} v(k) \over t_n + 1 = \bar{v} \},
\]
where \( t_n \) is a sampling sequence. The average is always nonempty but need not be a singleton.

II. PROBLEM SETUP AND THE CONCEPT OF ROBUST ECONOMIC MPC

In this section, we recapitulate the approach for robust economic MPC for linear systems. Note that in [13] and [14], this approach is presented for general nonlinear systems. Here, we consider linear time-invariant disturbance-affected discrete-time systems of the form
\[
x(t+1) = Ax(t) + Bu(t) + w(t), \quad x(0) = x_0,
\]
with \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \) and \( (A,B) \) stabilizable. Moreover, \( x(t) \in X \subseteq \mathbb{R}^n \) is the state system at time \( t \in \mathbb{I}_{\geq 0} \) and \( u(t) \in U \subseteq \mathbb{R}^m \) is the input to the system. The disturbance \( w(t) \) is assumed to be unknown but bounded satisfying \( w(t) \in \mathcal{W} \subset \mathbb{R}^n \) for all \( t \in \mathbb{I}_{\geq 0} \). The disturbance set \( \mathcal{W} \) is a compact and convex set containing the origin. Moreover, we assume pointwise in time constraints on the states and inputs which might be coupled, that is, \( (x(t), u(t)) \in \mathcal{Z} \subseteq X \times U \) for all \( t \in \mathbb{I}_{\geq 0} \), where \( \mathcal{Z} \) is compact.

Since we want to deal in the following with an economic cost (rather than with stabilization of a given setpoint or tracking of a given reference trajectory) our overall goal is to find a feasible input sequence that minimizes the asymptotic average performance of the system
\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \ell(x(t), u(t)),
\]
where \( \ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) is the continuous stage cost. As pointed out in [4], one of the key features of economic MPC is to have a stage cost \( \ell \) which is not positive definite around the desired setpoint \( (x_s, u_s) \). This means that in economic MPC, the stage cost does not need to satisfy
\[
0 = \ell(x_s, u_s) \leq \ell(x, u)
\]
for all \( (x, u) \in \mathcal{Z} \), but the stage cost can be arbitrary.

As the system is disturbance affected, we cannot determine the real system state exactly a priori. However, as proposed in [13] and [14], we can make use of set based control concepts in order to determine bounds on the state of the real system.

In many set based approaches for robust MPC for linear systems, the goal is to keep the distance between the real (disturbed) system (1) and its associated nominal system
\[
z(t+1) = Az(t) + Bv(t), \quad z(0) = z_0
\]
bounded, where \( z \) is the state and \( v \) is the input of the nominal system, respectively. This distance
\[
e(t) = x(t) - z(t)
\]
is usually referred to as error. In order to determine bounds for the error, one needs to find an appropriate feedback
\[
u(t) = v(t) + K(x(t) - z(t)) =: \varphi(v(t), x(t), z(t))
\]
with which the error system results in
\[
e(t+1) = (A + BK)e(t) + w(t).
\]
Finding such bounds can be formalized using concepts from set based control.

Definition 1: ([11]) A set \( \Omega \subseteq \mathbb{R}^n \) is robustly positive invariant (RPI) for the system \( x^+ = g(x, w) \), where \( g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) and \( w \in \mathcal{W} \) if for every \( x \in \Omega \) it holds that \( g(x, w) \in \Omega \) for all \( w \in \mathcal{W} \).

In the literature, many set theoretic approaches on determining an invariant set for the error dynamics (5) have been presented for linear systems (see e.g. [8], [15]). As is the standard idea in tube MPC, the system to be considered within the optimization is the nominal system (2), while the feedback (4) is applied to the real system. By means of this feedback, we can guarantee that the “real” state \( x \) is always within the compact RPI set \( \Omega \) around the nominal state \( z \). Even though the control is derived for the nominal system, we still want the real system to satisfy its “original” constraints \( (x, u) \in \mathcal{Z} \). Thus, we have to introduce an appropriately tightened set \( \mathcal{Z} = \mathcal{Z} \cap (\Omega \times K\Omega) \). This guarantees that if \( (z, v) \in \mathcal{Z} \), then \( (x, u) \in \mathcal{Z} \). The projection of \( \mathcal{Z} \) on \( X \) is denoted by \( \overline{X} \).

The results presented in the remainder of this section are based on [13] and [14] and are supposed to provide a brief summary of the conceptual idea. When an economic stage cost is taken into account and when disturbances are acting on a system, it might not be the best to consider only the performance achieved by the nominal system, but to consider the performance of the real system by the knowledge that we have about it. As the exact state of the real system cannot be predicted during the optimization, a different approach has been presented in [13] which is based on the following idea: The real state is only known to lie in the RPI set around the nominal state, and thus, we have to consider all states within the RPI set representing all possible states of the real system. This can be interpreted as an averaging over all possible states of the real system. In order to account for all possible real states, the integrated cost is introduced by
\[
\ell^\text{int}(z, v) = \int_{v \in z(\{v\}) \in \Omega} \ell(x, \varphi(v, x, z)) dx.
\]
By means of this integrated cost, we can introduce the optimization problem solved at each iteration by
\[
\min_{\nu(t)} \sum_{k=t}^{N+t-1} \ell^\text{int}(z(k|t), v(k|t)) \quad \text{s.t. } \begin{cases} z(k+1|t) = Az(k|t) + Bu(k|t), \\ (z(k|t), v(k|t)) \in \mathcal{Z} \quad \forall k \in \mathbb{I}_{[t,N+t-1]}, \\ z(N+t|t) = z_s, \\ z(t|t) = z(t). \end{cases}
\]
denoted by \( v^*(t) = \{v^*(t|t), \ldots, v^*(t+N-1|t)\} \). The terminal predicted state \( z(N+t|t) \) is required to be equal to the robust optimal steady-state \((z_N,v_N)\) which is defined for a disturbance-affected system \((1)\) and a cost \(\ell(x,u)\) by

\[
(z_N,v_N) = \arg\min_{(z,v)\in Z,N} \int_{x\in\mathbb{X}} \ell(x,\varphi(v,x,z))dx.
\]

This terminal constraint is needed in order to provide recursive feasibility of the MPC algorithm. For ease of presentation we restrict this paper to terminal equality constraints. In [14], also terminal set constraints are taken into account.

The robust economic MPC algorithm is given as follows.

**Algorithm 1 Robust Economic MPC Algorithm [13]**

given: initial state \( x(0) \)

for \( t = 0, 1, 2, \ldots \) do

solve \((P_{R\text{EMPC}})\)

apply \( u(t) = \varphi(v^*(t|t),x(t),z(t)) \) to system \((1)\)

apply \( v(t) = v^*(t|t) \) to the nominal system \((2)\)

end for

Note that Algorithm 1 provides a trajectory for the nominal system \((2)\), i.e., the nominal closed-loop system is given by

\[
z(t+1) = Az(t) + Bv^*(t|t), \quad z(0) = z_0.
\]

Furthermore, the real closed-loop system is given by

\[
x(t+1) = Ax(t) + B\varphi(v^*(t|t),x(t),z(t)) + w(t), \quad x(0) = x_0.
\]

**Remark 1:** Up to now, the initial state of the nominal system \( z_0 \) is not determined. In fact, a certain degree of freedom is given for the nominal initial state. The only condition that needs to be satisfied is that \( x_0 \in \{z_0\} \oplus \Omega \). In [14], a more thorough discussion is given on this topic.

In [13] and [14], recursive feasibility for the optimization problem \((P_{\text{R\text{EMPC}}})\) can be guaranteed by using Algorithm 1. Moreover, it can be shown that the asymptotic average performance averaged over all possible disturbances is no worse than that of the robust optimal steady-state, i.e.,

\[
\limsup_{T \to \infty} \frac{\sum_{k=0}^{T-1} \ell^m(z(k),v^*(k|k))}{T} \geq \ell^m(z_N,v_N)
\]

(see [13, Theorem 4]). Asymptotic stability can be proven for the composite system \((7)\) and \((6)\) under some specific assumptions, cf. [13, Theorem 9]. Furthermore, under some weaker assumptions one can show the system to be robustly optimally operated at steady-state. This means that compared to the average integral cost for any trajectory of the nominal system, staying at the steady-state provides the best average integral cost (see [14, Theorem 2]).

**Remark 2:** Note that there might exist some special disturbance sequences, for example \( w(t) \equiv 0 \), for which a better performance could be achieved when using some tube-based MPC approach with an optimization based on the original (non-integrated) stage cost \(\ell\). However, since the disturbance is unknown a priori, we propose to incorporate the influence of the disturbance when determining the – in an economic setting – best closed-loop trajectory. This is done through the integration over the RPI set which includes all possible states of the real system. Other possibilities to incorporate the influence of the disturbance could be considered as well, for example, one could use the maximum of \(\ell\) over \(\{z\} \oplus \Omega\) which would lead to a worst case analysis.

**Remark 3:** In Algorithm 1, we fixed the nominal initial state \( z(t) \) to be determined through the nominal closed loop \((6)\). In the following section, we will compare two approaches taking average constraints into account, one where the nominal initial state is fixed by the closed-loop behavior and one where the nominal initial state is an optimization variable.

III. LINEAR AVERAGE CONSTRAINTS IN THE FRAMEWORK OF ROBUST ECONOMIC MPC

When considering stabilizing MPC, the average behavior as well as the average performance are determined by the setpoint to be stabilized or by the trajectory to be tracked. However, when economic MPC approaches are taken into account, and especially when stability is not the key issue, it might be necessary to impose constraints on the system that need only be satisfied on average, meaning they must not be considered as pointwise in time constraints, but can be violated as long as on average this violation is compensated.

In the case of undisturbed economic MPC (see e.g. [4], [7]), average constraints can be considered by means of an auxiliary output \( y = h(x,u) \), where \( h : \mathbb{X} \to \mathbb{R}^p \) is a continuous map. By means of this output function the average constraints are given by

\[
Av[y] \subseteq \mathbb{Y},
\]

where \( \mathbb{Y} \subseteq \mathbb{R}^p \) is a closed set. According to the definition in [7], we assume the average constraint set \( \mathbb{Y} \) to be a polyhedron of the form \( \mathbb{Y} := \{y \in \mathbb{R}^p : A_c y \leq b_c\} \), for some \( A_c \in \mathbb{R}^{n_c \times p} \) and some \( b_c \in \mathbb{R}^{n_c} \).

If disturbance-affected systems are considered, the same problem mentioned before reappears, namely we cannot determine the exact trajectory of the real system over the prediction horizon a priori. Still, we want to guarantee \((8)\). However, as is the case when determining the asymptotic average performance, some knowledge is given about the real system by means of the RPI set. We will make use of this in the framework of linear average constraints.

A. Linear Average Constraints for Linear Systems

In the following, we consider a linear auxiliary output

\[
y(t) = h(x(t),u(t)) = h_x x(t) + h_u u(t) + h_v,
\]

where \( h_x \in \mathbb{R}^{p \times n} \), \( h_u \in \mathbb{R}^{p \times m} \) and \( h_v \in \mathbb{R}^p \), for which we want to guarantee that \( Av[y] \subseteq \mathbb{Y} \). For many applications, it might be of particular interest to constrain the average behavior of the states \( Av[x] \subseteq \bar{X} \) and/or inputs \( Av[u] \subseteq \bar{U} \) (see e.g. [16]). The considered linear auxiliary output \((9)\) is a generalization of these average constraints.

In the undisturbed case, satisfaction of the average constraints can be guaranteed by introducing an additional
constraint \[ \sum_{k=t}^{N+t-1} h(x(k|t), u(k|t)) \in \mathbb{Y}_t \] (10)
within the optimization problem, where \( \mathbb{Y}_{t+1} = \mathbb{Y}_t \oplus \mathbb{Y} \oplus \{-h(x(t|t), u(t|t))\} \) with \( \mathbb{Y}_0 = NH_z \Omega \oplus \mathbb{Y}_{00} \), and by guaranteeing the nestedness condition \( h(x_s, u_s) \in \mathbb{Y} \) for the steady-state \((x_s, u_s)\) of the real system (see e.g. [4], [7]). The set \( \mathbb{Y}_{00} \subset \mathbb{R}^p \) can be any arbitrary compact set containing the desired setpoint. This constraint must be satisfied for each iteration of the MPC algorithm. Note that this constraint considers the predictions of the real system state and dynamics.

Going back to our - disturbed - case, we would still want to satisfy (10), but due to the disturbances the predictions of \( x \) and \( u \) cannot be determined exactly. Following the idea for satisfying the pointwise in time constraints when considering disturbances, we use tightening in order to find an appropriate average constraint for the nominal system. Using the error (3) and the error feedback, we can rewrite (10) leading to
\[ \sum_{k=t}^{N+t-1} h(z(k|t), v(k|t)) \in \mathbb{Y}_t \] (11)
for all possible disturbance sequences. Applying Algorithm 1 with the optimization problem \((P_{\text{LinAvg}})\), we end up with the error feedback \( u^*(t) = v^*(t|t) + K(x(t) - z(t)) \), and thus, with the linear closed-loop real system
\[ x(t+1) = Ax(t) + Bu^*(t) + w(t), \quad x(0) = x_0, \] (12a)
\[ y(t) = h(x(t), u^*(t)), \] (12b)
for which we can state the following result.

**Theorem 1:** Let the optimization problem \((P_{\text{LinAvg}})\) be feasible at time \( t = 0 \). Then it is feasible for all \( t \in \mathbb{I}_{\geq 0} \) and it holds for the closed-loop system (12) that
\[ Av[\epsilon_{\text{int}}(z, v)] \subseteq (-\infty, \epsilon_{\text{int}}(\bar{z}_s, \bar{v}_s)], \]
\[ (x(t), u(t)) \in \mathbb{Z}, \quad \forall t \in \mathbb{I}_{\geq 0}, \]
\[ Av[y] \subseteq \mathbb{Y}. \]

Note that the nominal closed loop is given by \( z(t+1) = Az(t) + Bu^*(t|t) \). The proof for Theorem 1 follows along the lines of the proof for the nominal case in [4].

**Proof:** Recursive feasibility of the pointwise in time constraints follows from the appropriate tightening \((z, v) \in \mathbb{Y}_{t+1}\). The proof for asymptotic performance follows directly from [13, Theorem 4]. For proving the average constraint, we first consider feasibility of the average sum (11) for the nominal trajectory. Using the usual candidate solution at time \( t + 1 \), \( \bar{v}(t+1) = \{v^*(t+1|t), v^*(t+2|t), \ldots, v^*(N-t-1|t), \bar{v}_s\} \), and the associated state trajectory \( \bar{z} \), one can derive
\[ \sum_{k=t}^{N+l-1} h(\bar{z}(k), \bar{v}(k)) \in \mathbb{Y}_t \oplus \mathbb{Y} \oplus \{-h(z(t|t), v^*(t|t))\} = \mathbb{Y}_{t+1}. \]

This shows to recursive feasibility of (11). Concerning the average constraints for the real system affected by disturbances, it follows from the tightening that
\[ \sum_{k=t}^{N+l-1} h(x_w(k), u_w(k)) \in \sum_{k=t}^{N+l-1} h(z(k|t), v(k|t)) \]
\[ \subseteq \mathbb{Y}_t \oplus \mathbb{Y}_{00} \oplus \mathbb{Y}_{00} \]
\[ y(t) = y_0 + (t + N) \mathbb{Y} \oplus \left\{ -\sum_{k=0}^{t-1} y(k) \right\}. \]

Following the proof in [4] and by letting \( t \) grow to infinity along any subsequence \( t_n \) such that a limit exists for \( \sum_{k=0}^{t_n} y(k)/(t_n + 1) \), it holds that
\[ \lim_{n \to \infty} \frac{1}{t_n + 1} \sum_{k=0}^{t_n} y(k) \in \mathbb{Y}, \]
and hence, we have shown that $A_v[y] \subseteq \mathbb{Y}$.

As mentioned above, the concept of robust optimal operation at steady-state is introduced in [13]. In the following, we want to investigate the conditions under which a system is robustly optimally operated at steady-state despite the linear average constraints (9).

Definition 2: System (1) is said to be robustly optimally operated at steady-state on averagely constrained solutions with respect to the given stage cost $\ell$, the constraints $(z, u) \in \mathbb{Z}$, and the average constraints if for any feasible nominal input sequence $\nu(\cdot)$ and its associated nominal state sequence $z(\cdot)$ according to (2) that satisfy $A_v[h(z, \nu)] \subseteq \mathbb{Y}$, it holds that

$A_v[\ell^\text{int}(z, \nu)] \subseteq [\ell^\text{int}(\bar{z}, \bar{\nu}), \infty)$,

where $(\bar{z}, \bar{\nu})$ is the robust optimal steady-state.

Note that this definition is a combination of [4, Definition 6.2] (for the nominal case with average constraints) and [14, Definition 3] (for the uncertain case without average constraints). Using the polyhedral definition of $\mathbb{Y}$, we can write the tightened average constraints $\mathbb{Y}$ by $\mathbb{Y} = \{y \in \mathbb{R}^n : A_y y \leq \bar{b}_y\}$. [5]

Definition 3 ([4]): The nominal system (2) is dissipative with respect to the supply rate $s : \mathbb{Z} \rightarrow \mathbb{R}$ if there exists a continuous storage function $\lambda : \mathbb{X} \rightarrow \mathbb{R}$ such that $\lambda(f(z, v, 0)) - \lambda(z) \leq s(z, v)$ for all $(z, v) \in \mathbb{Z}$.

By means of this definition, we can derive a statement on robust steady-state optimality under average constraints.

Theorem 2: Consider system (1) and let $\Omega$ be an RPI set for the associated linear error dynamics $e(t + 1) = (A + BK)e(t) + w(t)$. If there exists a multiplier $\lambda \in [0, \infty)_{\mathbb{R}}$ such that the nominal system is dissipative with respect to the supply rate

$s(z, v) = \ell^\text{int}(z, \nu) - \ell^\text{int}(\bar{z}, \bar{\nu}) + \bar{X}^T[A_x h(z, v) - \bar{b}_c]$, 

then system (1) is robustly optimally operated at steady-state on averagely constrained solutions.

Proof: The proof follows directly from the proof of Proposition 6.4 in [4] and is based on the feasibility of the tightened average constraints. \[ \blacksquare \]

B. Nominal Initial State as Optimization Variable

Up to now, the initial state of the nominal system $z(t|0)$ was given at each iteration through the closed-loop solution of the nominal system. However, as is presented in [9] for stabilizing tube MPC, the nominal initial state can be taken into account as an optimization variable at each iteration. It is pointed out in [1] that this has two advantages compared to the approach with a fixed initial state: (i) an additional optimization variable can improve the performance and (ii) the applied control only depends on the current real state and not on the real and nominal system.

Taking the nominal initial state as an optimization variable, we can formulate the following optimization problem to be solved at each iteration.

$$\min_{z(t), u(t)} \sum_{k=t}^{N+1-1} \ell^\text{int}(z(k|t), v(k|t))$$

s.t. $z(k+1) = A_z(z|k) + Bv(k|t)$,

$z(N+1|t) = \bar{z}$,

$x(t) \in \{z(t|t)\} \oplus \Omega$,

$$\sum_{k=t}^{N+1-1} h(z(k|t), v(k|t)) \in \mathbb{Y}_t.$$

We apply this optimization problem within the following algorithm which is a modified version of Algorithm 1.

Algorithm 2 REMPC Algorithm with Free $z(t|0)$

given: initial state $x(0)$

for $t = 0, 1, 2, \ldots$ do

solve ($P_{\text{mod}}$)

apply $u(t) = \nu^*(z|t), x(t), z^*(t|t)$ to system (1)

end for

The optimal nominal initial state is denoted by $z^*(t|t)$. The linear error feedback applied to the real system results in $u^*(t) = \nu^*(t|t) + K(x(t) - z^*(t|t))$. When optimizing over the nominal initial state, the sequence of nominal states $\{z^*(0|0), z^*(1|1), \ldots\}$ is not necessarily a trajectory of the nominal system. Despite this characteristic, one can show that the results presented in Theorem 1 also hold for the modified optimization problem ($P_{\text{mod}}$) with the closed loop

$x(t+1) = Ax(t) + Bu^*(t) + w(t), \quad x(0) = x_0$, \hspace{1cm} (13a)

$y(t) = h(x(t), u^*(t))$. \hspace{1cm} (13b)

Theorem 3: Let the optimization problem ($P_{\text{mod}}$) be feasible at time $t = 0$. Then it is feasible for all $t \in \mathbb{N}_{\geq 0}$ and it holds for the closed-loop system (13) that

$A_v[\ell^\text{int}(z, \nu)] \subseteq (-\infty, \ell^\text{int}(\bar{z}, \bar{\nu}))$, 

$(x(t), u(t)) \in \mathbb{Z}, \quad \forall t \in \mathbb{N}_{\geq 0}$,

$A_v[y] \subseteq \mathbb{Y}$.

Proof: The proof is analogous to that of Theorem 1. \[ \blacksquare \]

Remark 4: Since the sequence $\{z^*(0|0), z^*(1|1), \ldots\}$ generated by Algorithm 2 is not a nominal solution of (1) in the sense of (6), it may not satisfy the property from Definition 2.

IV. NUMERICAL EXAMPLE

In the following, we apply the approaches to an example from literature in order to compare the two approaches for averagely constrained robust economic MPC. In [16], a model for a room heating system is given by

$x(t+1) = 0.5x(t) + 15u(t) - 7.5 + w(t)$,

where the state $x \in \mathbb{X} = \{x \in \mathbb{R} : |x| \leq 15\}$ represents the deviation to a desired temperature, while the input $u \in \mathbb{U} = \{u \in \mathbb{R} : 0 \leq u \leq 1\}$ represents the opening position of a valve where 0 represents a closed and 1 a completely
open valve. The disturbance \( w \) is bounded to the set \( \mathbb{W} = \{ w \in \mathbb{R} : |w| \leq 2 \} \). The prediction horizon is chosen to be \( N = 20 \). We use \( K = -1/30 \) within the linear control law such that the minimal RPI is given by \( \Omega = \mathbb{W} \). Given the cost function

\[
\ell(x, u) = \begin{cases} 
50u^2 & \text{for } x \geq 0 \\
0.25x^2 + 50u^2 + 3.34xu & \text{for } x < 0.
\end{cases}
\]

In addition, we introduce the auxiliary output \( y = u \) and introduce the average constraint restricting \( Av[y] \in [0, 0.3] \) in order to keep the flow through the valve small. The robust optimal steady-state for this system under the given average constraint is at \( \bar{z}_s = -8 \). Note that the associated input \( \bar{v}_s \) has to satisfy the tightened average constraint.

Optimizing over the nominal initial state has a significant effect on the performance (averaged over 10 simulations); the auxiliary output \( y = h(x, u) \) satisfies its average constraint for both approaches as shown in the table below.

<table>
<thead>
<tr>
<th>Avg. performance</th>
<th>MPC with ( (P_{\text{LinAvg}}) )</th>
<th>MPC with ( (P_{\text{mod}}) )</th>
<th>MPC with ( (P_{\text{REMPC}}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avg. output</td>
<td>12.78702</td>
<td>7.93695</td>
<td>6.12073</td>
</tr>
<tr>
<td>Avg. output</td>
<td>0.23408</td>
<td>0.29886</td>
<td>0.36735</td>
</tr>
</tbody>
</table>

For comparison, the results achieved with MPC with \( (P_{\text{REMPC}}) \) are provided as well. Since this algorithm does not handle average constraints, the performance is better, but the average constraint is violated. When considering the stage cost, one can find that there exist states in the neighborhood of \( \bar{z}_s \) which are “cheaper”, but which do not satisfy the tightened average constraint. However, as denoted by the green cross in Figure 1, the nominal closed loop determined with Algorithm 1 and \( (P_{\text{LinAvg}}) \) stays at \( \bar{z}_s \). By the additional degree of freedom provided within \( (P_{\text{mod}}) \), the optimization can choose the initial state of the nominal system \( \bar{z}(t) \); cyan crosses in Figure 1 independent of the dynamics of the nominal system, and hence, can get closer to the bound of the average constraint without violating it. For this example, this provides a better performance compared to the performance of MPC with \( (P_{\text{LinAvg}}) \).

V. CONCLUSION

In this paper, a previously presented idea on tube-based economic MPC was extended considering linear average constraints in the problem setup. Based on invariant sets for the error, a tightening for the average constraints was derived such that by guaranteeing these tightened average constraints with the nominal system, the real system satisfies its original average constraints. Moreover, we have shown that taking the nominal initial state as an optimization variable at each iteration provides the same feasibility results but induces a better performance as shown in the numerical example. On the other hand, optimal operation at the steady-state was only provable when the sequence of nominal initial states is a closed-loop trajectory of the nominal system.

Further research is in the direction of considering nonlinear average constraints and nonlinear systems.

ACKNOWLEDGMENTS

The authors would like to thank Matthias A. Müller for the helpful discussions.

REFERENCES