GPU computing with Kaczmarz’s and other iterative algorithms for linear systems

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Abstract

The graphics processing unit (GPU) is used to solve large linear systems derived from partial differential equations. The differential equations studied are strongly convection-dominated, of various sizes, and common to many fields, including computational fluid dynamics, heat transfer, and structural mechanics. The paper presents comparisons between GPU and CPU implementations of several well-known iterative methods, including Kaczmarz’s, Cimmino’s, component averaging, conjugate gradient normal residual (CGNR), symmetric successive overrelaxation-preconditioned conjugate gradient, and conjugate-gradient-accelerated component-averaged row projections (CARP-CG). Computations are performed with dense as well as general banded systems. The results demonstrate that our GPU implementation outperforms CPU implementations of these algorithms, as well as previously studied parallel implementations on Linux clusters and shared memory systems. While the CGNR method had begun to fall out of favor for solving such problems, for the problems studied in this paper, the CGNR method implemented on the GPU performed better than the other methods, including a cluster implementation of the CARP-CG method.

Key words: Scientific computing on GPU; Linear systems; Iterative methods; Convex feasibility problem

1. Introduction

Natural and engineered systems are often modeled by sets of partial differential equations that have no closed-form solution. Discretization of these equations leads to large sparse systems of linear equations. A considerable

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amount of algorithmic and computational work has been performed to develop iterative algorithms for solving these systems as efficiently as possible [1].

The current paper investigates the performance of several iterative solvers on a graphics processing unit (GPU). The algorithms we are interested in include Kaczmarz’s sequential algorithm, as well as several block-parallel algorithms: Cimmino’s method, component averaging (CAV), conjugate gradient normal residual (CGNR), symmetric successive overrelaxation (SSOR)-preconditioned conjugate gradient (CGMN), component-averaged row projections (CARP), and conjugate-gradient-accelerated CARP (CARP-CG).

Given the challenging nature of large-scale systems, parallel computing has been utilized in the development of efficient solution algorithms. In the context of GPUs, recent hardware developments, such as the NVIDIA Tesla C870, offer a massively multithreaded processor architecture that is ideal for high performance computing applications. These innovative designs have led many researchers to implement various linear algebra routines on the GPU. A Jacobi iterative solver was implemented on the GPU by Göddeke et al. [2]. Conjugate gradient and multigrid sparse matrix solvers were implemented on the GPU by Bolz et al. [3]. Krüger and Westermann [4] implemented direct solvers for sparse matrices, and studied their performance using multi-dimensional finite difference equations arising from the 2-D wave equation and the incompressible Navier-Stokes equations. An algorithm to solve dense linear systems using GPUs was implemented on an NVIDIA GeForce 7800 GPU by Galoppo et al. [5]. These are some of several studies recently reported using GPUs to accelerate the solution of various linear algebra problems.

There are many studies reporting the parallel solution of linear systems on hardware other than GPUs. Here, we mention those most closely related to the algorithms studied in this paper. Bramley and Sameh [6] implemented a block-sequential Kaczmarz algorithm with CG acceleration under five different partitioning schemes on structured grids. The disadvantage of the block-sequential approach is that it requires the identification of independent sets of equations which is difficult for unstructured grids. In [7], the same authors extended the work in [6] to include a CG acceleration of a block-Cimmino algorithm and a new projection method called V-RP. Their block-parallel CG-accelerated methods were shown to be robust in practice, but there was no clear “best” partitioning scheme. Arioli et al. [8] studied parallel CG acceleration of block-Cimmino for different block partitionings. The authors restricted their study to block-tridiagonal systems. Gordon and Gordon [9] introduced the CARP algorithm that divides the linear equations into blocks and operates in a block-parallel manner. Kaczmarz row projections are performed within each block in parallel and the results are then merged using component-averaging operations. CARP was shown to be very robust and suitable for unstructured grids.

The main purpose of the paper is to evaluate the performance on the GPU of several parallel algorithms with respect to solving large linear systems arising from the discretization of elliptic convection-diffusion partial differential equations. In particular, we are interested in identifying the best possible algorithm for this architecture, as well as in comparing the GPU against the CPU and
earlier cluster implementations of these algorithms. For this purpose, we will investigate the performance of these algorithms on a set of problems addressed in many earlier works. This set of problems includes six partial differential equations proposed in [7], along with three additional partial differential equations that were investigated in [10, 11]. All nine of these partial differential equations are strongly convection-dominated.

The remainder of the paper is organized as follows. Section 2 provides mathematical preliminaries and an introduction to GPU computing and architectures, including the NVIDIA Tesla C870, which is utilized for computations in this paper. The algorithms considered here are described in Section 3, while the philosophy and drivers behind our GPU implementation of these algorithms are detailed in Section 4. In Section 5, we present extensive computational results, followed by conclusions from this work in Section 6.

2. Preliminaries

2.1. Mathematical background and notation

The problem of finding a point in the intersection of two or more convex sets is referred to as the convex feasibility problem. Let \( C_1, C_2, \ldots, C_n \) be closed convex subsets of a Hilbert space \( X \) with a nonempty intersection
\[
C = C_1 \cap C_2 \cap \ldots \cap C_n \neq \emptyset.
\]
The convex feasibility problem involves finding some \( x \) in \( C \). In image reconstruction, the convex sets \( C_i \) are hyperplanes. The iterative algorithms presented in Section 3 use orthogonal projections on the hyperplanes \( C_i \) to solve these problems. A matrix \( P \in \mathbb{R}^{n \times n} \) is an orthogonal projection onto \( S \subseteq \mathbb{R}^n \) if \( R(P) = S, P^2 = P, \) and \( P^T = P \), where \( R(P) \) is the range of \( P \).

Define a linear system as \( Ax = b \), where \( A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n, \) and \( b \in \mathbb{R}^m \). A consistent linear system is one where \( b \in R(A) \). An inconsistent linear system is such that \( b \notin R(A) \). Simply put, the equations of a linear system are consistent if they possess a common (but not necessarily unique) solution, and inconsistent otherwise.

A linear system with a low condition number is said to be well-conditioned, while a linear system with a relatively high condition number is said to be ill-conditioned. For a linear system with an invertible matrix, the condition number is given by
\[
\kappa(A) = \|A^{-1}\| \|A\|,
\]
where \( \|A\| \) denotes the norm of \( A \). Any norm, \( \|\cdot\| \), without a subscript denotes the Euclidean norm. For rectangular matrices with full column rank, \( i.e., A \in \mathbb{R}^{m \times n} \) and \( \text{rank}(A) = n \), the condition number is given by
\[
\kappa(A) = \frac{\sigma_{\text{max}}(A)}{\sigma_{\text{min}}(A)},
\]
where \( \sigma_{\text{max}}(A) \) denotes the maximum singular value of \( A \) and \( \sigma_{\text{min}}(A) \) denotes the minimum singular value of \( A \).
Two nonzero vectors $u$ and $v$ are conjugate with respect to $A$ if

$$u^T Av = \langle u, v \rangle_A = 0.$$ 

Likewise, it is necessary to define the notation

$$\|x\|^2_S = x^T S x.$$ 

When discussing an iterative algorithm, the $k$th iterate of any vector will always be denoted by a superscript in parenthesis, e.g., $x^{(k)}$. On the other hand, the position in the vector will be denoted with a subscript. For example, $x_j^{(k)}$ would denote the $j$th element of the $k$th iteration of $x$. For an $m \times n$ matrix $A$, $a_{ij}$ denotes the element of $A$ in the $i$th row and $j$th column. Furthermore, $a_i$ is used to denote the $i$th row vector of $A$, unless otherwise noted. The standard basis vectors are written $e_i$.

2.2. The graphics processing unit

This subsection briefly describes the architecture of a GPU from the perspective of a programmer using the compute unified device architecture (CUDA) [12]. CUDA is a hardware and software architecture that issues and manages data-parallel computations on a GPU. The GPU is a single-instruction multiple-data (SIMD) parallel device. The GPU should be viewed as a compute device or coprocessor, while the CPU should be viewed as the host. Since the GPU is a SIMD architecture with a host device, the GPU is utilized for data-parallel and computationally intensive portions of the algorithm or application.

The computationally intensive portions of the algorithm are executed in parallel using thousands of threads. Each thread executes the same set of instructions independently on different data. The GPU instructions are referred to as a kernel, which is downloaded to the device. A block is a group of threads that share data through shared memory and synchronize their execution to coordinate their memory accesses. These synchronization points are specified within the kernel and act as a barrier where all threads in the block are suspended until they reach this barrier. A grid is an organization of thread blocks. Within each grid, there exist a programmer-defined number of blocks.

The memory model for the GPU is provided at the grid level. Read-write access to global memory and read-only access to constant and texture memory is afforded to the entire grid. The host has read-write access with global, constant, and texture memory. Each thread has read-write access to its own set of registers and local memory, while each block has read-write access to designated shared memory.

The multiprocessors on an NVIDIA Tesla C870 are organized in a streaming processor array. Each element of this array is referred to as a texture processor cluster. In the case of the NVIDIA Tesla C870, there are eight texture processor clusters. A texture processor cluster consists of texture memory and two streaming multiprocessors. Each streaming multiprocessor contains instruction and data cache, instruction fetch and dispatch unit, shared memory, eight
streaming processors, and two special function units. The special function units allow for fast single-precision mathematical computations, such as sine, cosine, logarithm, and exponential.

The interested reader can find additional discussions on the Tesla architecture in [13]. More detailed information on GPU computing can be found in two recent special issues of IEEE Proceedings [14] and the Journal of Parallel and Distributed Computing [15].

3. Iterative methods

This section describes the iterative methods utilized in this paper. Interest is placed on solving linear systems of the form $Ax = b$, where $A$ is an $m \times n$ matrix, $x$ is an $n$-vector, and $b$ is an $m$-vector.

3.1. Kaczmarz’s algorithm

Kaczmarz introduced this algorithm in [16]. Kaczmarz’s approach is a projection method that is also referred to as ART (algebraic reconstruction technique) and is used for solving linear systems from image reconstruction problems.

Kaczmarz’s algorithm is sequential. The algorithm sweeps through rows of $A$ in a cyclic manner. At each iteration, the previous iterate is projected orthogonally onto the solution hyperplane $\langle a_i, x \rangle = b_i$. This orthogonal projection leads to the normalized step at iteration $k$

$$s(k) = \lambda_i \frac{b_i - \langle a_i, x(k) \rangle}{\|a_i\|^2} a_i,$$

where $\lambda_i$ is a cyclic relaxation parameter that extends the projections either in front of the hyperplane ($\lambda_i < 1$), exactly on the hyperplane ($\lambda_i = 1$), or beyond the hyperplane ($\lambda_i > 1$), and we assume henceforth that $0 < \lambda < 2$. In this case, $i$ is equal to $k$ modulus $m + 1$, with $k \geq 0$. A randomized version of the algorithm has also been proposed in [17], where the row $i$ is chosen at random rather than sequentially. In either case, each set of $m$ iterations is referred to as a sweep. The iterate progresses as follows

$$x^{(k+1)} = x^{(k)} + s(k)$$

where the step $s^{(k)}$ is defined by (1).

Algorithm 1 is a presentation of Kaczmarz’s method.

3.2. Cimmino’s algorithm

Cimmino introduced his algorithm in [18]. This method is highly parallel and guaranteed to converge in the inconsistent case. In practice, the method is slow, requiring many iterations to reduce the residual error and approach the solution. The algorithm converges to the weighted least-squares solution,
Algorithm 1: Kaczmarz’s algorithm

1 for $k \leftarrow 0$ until convergence or maximum number of iterations met do
2 $i \leftarrow k \bmod m + 1$
3 $s^{(k)} \leftarrow \lambda_i \frac{b_i - \langle a_i, x^{(k)} \rangle}{\|a_i\|^2} a_i$
4 $x^{(k)} \leftarrow x^{(k)} + s^{(k)}$
5 end

which minimizes the weighted sum of the squares of the distances to the sets $C_1, \ldots, C_n$ discussed in Subsection 2.1. This result is due to Combettes [19].

The method of Cimmino involves a simultaneous orthogonal projection onto the set of solution hyperplanes $\langle a_i, x \rangle = b_i$, $i = 1, \ldots, m$. The orthogonal projections lead to a normalized step at iteration $k$

$$s^{(k)}_j = \frac{\lambda_k}{m} \sum_{i=1}^{m} \frac{b_i - \langle a_i, x^{(k)} \rangle}{\|a_i\|^2} a_{ij}, \ j = 1, \ldots, n,$$

where $s^{(k)}_j$ is the $j$th element of the $k$th normalized Cimmino step. The observant reader should recognize that each $s^{(k)}_j$ can be computed in parallel for all $j = 1, \ldots, n$. The iterate progresses as before:

$$x^{(k+1)} = x^{(k)} + s^{(k)}.$$

This algorithm can also be expressed in matrix form as:

$$x^{(k+1)} = x^{(k)} + \lambda_k A^T D \left( b - Ax^{(k)} \right),$$

where

$$D = \frac{1}{m} \text{diag} \left( \frac{1}{\|a_1\|^2}, \frac{1}{\|a_2\|^2}, \ldots, \frac{1}{\|a_m\|^2} \right).$$

Algorithm 2 is a presentation of Cimmino’s algorithm.

Algorithm 2: Cimmino’s algorithm

1 $D \leftarrow \frac{1}{m} \text{diag} \left( \frac{1}{\|a_1\|^2}, \frac{1}{\|a_2\|^2}, \ldots, \frac{1}{\|a_m\|^2} \right)$
2 for $k \leftarrow 0$ until convergence or maximum number of iterations met do
3 $x^{(k+1)} \leftarrow x^{(k)} + \lambda_k A^T D \left( b - Ax^{(k)} \right)$
4 end
3.3. Component-averaging

Component-averaging (CAV) was introduced by Censor et al. [20]. CAV projects the current iterate onto all the system’s hyperplanes in parallel, just like Cimmino’s algorithm. CAV retains the desired convergence properties of Cimmino’s method, in the sense that it converges in the inconsistent case. Furthermore, CAV exhibits significantly faster numerical convergence for large sparse systems in practice. The exhibited acceleration is noticeable on a single processor as well as parallel architectures.

The scalar weighting of Cimmino’s method is replaced by diagonal sparsity weighting. That is, the weights are inversely proportional to the number of nonzeros in each column. CAV’s orthogonal projections lead to a normalized step at iteration $k$

$$s_j^{(k)} = \frac{\lambda_k}{m} \sum_{i=1}^{m} \frac{b_i - \langle a_i, x^{(k)} \rangle}{\sum_{p=1}^{n} s_p a_{ip}^2} a_{ij}, \quad j = 1, \ldots, n,$$  \hspace{1cm} (4)

where $s_p$ is the number of nonzeros in column $p$, for $p = 1, \ldots, n$. As with the previously introduced iterative methods, the iterate progresses using the formula

$$x^{(k+1)} = x^{(k)} + s^{(k)}.$$  

As in Cimino’s algorithm, CAV can be expressed in matrix form

$$x^{(k+1)} = x^{(k)} + \lambda_k A^T D_S \left( b - A x^{(k)} \right),$$

where

$$D_S = \text{diag} \left( \frac{1}{\|a_1\|_S^2}, \frac{1}{\|a_2\|_S^2}, \ldots, \frac{1}{\|a_m\|_S^2} \right).$$  \hspace{1cm} (5)

The reader will notice only a subtle difference between CAV and Cimmino’s method, namely (3) versus (5), that is, the difference between the sparsity-oriented versus scalar weighting schemes which are captured by equations (3) and (5).

Lastly, CAV is proven to converge to a minimizer of a certain proximity function regardless of $x_0$ and independent of the consistency of the underlying linear system. For more details on these convergence results, the reader is referred to [20].

Algorithm 3 is a presentation of CAV.

3.4. Conjugate gradient normal residual (CGNR)

The CGNR algorithm was introduced by Hestens and Stiefel in [21]. In order to solve the linear system $Ax = b$ when $A$ is nonsymmetric, CGNR attempts to solve the equivalent system

$$A^T A x = A^T b.$$  \hspace{1cm} (6)
Algorithm 3: Component-averaging (CAV)

1. Let $s_j$ denote the number of nonzeros in column $j$
2. $S \leftarrow \text{diag}(s_1, s_2, \ldots, s_n)$
3. $D_S \leftarrow \text{diag}\left(\frac{1}{\|a_1\|_S^2}, \frac{1}{\|a_2\|_S^2}, \ldots, \frac{1}{\|a_m\|_S^2}\right)$
4. for $k \leftarrow 0$ until convergence or maximum number of iterations met do
5. \[ x^{(k+1)} \leftarrow x^{(k)} + \lambda_k A^T D_S (b - Ax^{(k)}) \]
6. end

By construction, this system is symmetric positive semidefinite. CGNR works
by applying the conjugate gradient method to the normal equations (6). Conju-
gate gradient can be applied to (6) because $A^T A$ is symmetric positive semidef-
inite.

The algorithm does not store $A^T A$ explicitly in memory. This fact makes
CNGR a natural choice for sparse matrices, since operations such as matrix-
vector multiplication are usually very efficient. In addition, the implicit use
of $A^T A$ leads to relatively large diagonal elements. This contribu-
tes to the algorithm’s relative robustness.

The downside of using CGNR is the fact that $\kappa(A^T A) = \kappa(A)^2$. The rela-
tively large condition number of (6) leads to a slow rate of convergence, depend-
ing, of course, on the condition number of the original system. This important
fact makes a preconditioner all the more important for this algorithm.

Algorithm 4 is a presentation of the CGNR algorithm.

Algorithm 4: Conjugate gradient normal residual (CNGR)

1. $r^{(0)} \leftarrow b - Ax^{(0)}$
2. $z^{(0)} \leftarrow A^T r^{(0)}$
3. $p^{(0)} \leftarrow z^{(0)}$
4. for $i \leftarrow 0$ until convergence or maximum number of iterations met do
5. \[ w^{(i)} \leftarrow A p^{(i)} \]
6. \[ \alpha^{(i)} \leftarrow \|z^{(i)}\|^2 / \|w^{(i)}\|^2 \]
7. \[ x^{(i+1)} \leftarrow x^{(i)} + \alpha^{(i)} p^{(i)} \]
8. \[ r^{(i+1)} \leftarrow r^{(i)} - \alpha^{(i)} w^{(i)} \]
9. \[ z^{(i+1)} \leftarrow A^T p^{(i+1)} \]
10. \[ \beta^{(i)} \leftarrow \|z^{(i+1)}\|^2 / \|z^{(i)}\|^2 \]
11. \[ p^{(i+1)} \leftarrow z^{(i+1)} + \beta^{(i)} p^{(i)} \]
12. end
3.5. Symmetric successive overrelaxation (SSOR)-preconditioned conjugate gradient (CGMN)

CGMN is a conjugate gradient acceleration of the SSOR algorithm. CGMN was introduced by Björck and Elfving in [22], where it was referred to as an SSOR preconditioning of CG. It will become clear that CGMN can also be considered a CG acceleration of Kaczmarz’s algorithm. Similar to CGNR, it is advantageous to utilize CGMN when $A$ is sparse. CGMN is a CG acceleration of the SSOR algorithm applied to the system

$$x = A^T y, A A^T y = b.$$  \hfill (7)

Therefore, a relatively large condition number can negatively impact the rate of convergence since $\kappa(A^T A) = \kappa(A)^2$.

Define a Kaczmarz sweep as before

$$x^{(i+1)} = x^{(i)} + \lambda_i \frac{b_i - a_i^T x^{(i)}}{\|a_i\|^2} a_i, \quad i = 1, \ldots, m. \hfill (8)$$

Applying Gauss-Seidel’s method to (7), it is evident that the $i$th minor step, $y^{(i)}$, is updated by

$$s^{(i)}_y = e_i \left( b_i - a_i^T A^T y^{(i)} \right) / a_i^T a_i.$$  

By $x = A^T y$,

$$x^{(i+1)} = x^{(i)} + A^T s^{(i)}_y$$

$$= x^{(i)} + A^T e_i \left( b_i - a_i^T A^T y^{(i)} \right) / a_i^T a_i$$

$$= x^{(i)} + a_i \frac{b_i - a_i^T x^{(i)}}{\|a_i\|^2}.$$  

Therefore, Gauss-Seidel’s method is equivalent to Kaczmarz’s method for (7).

Since CGMN involves an SSOR relaxation parameter, (8) becomes

$$x^{(i+1)} = x^{(i)} + \lambda_i \frac{b_i - a_i^T x^{(i)}}{\|a_i\|^2} a_i, \quad i = 1, \ldots, m, \hfill (9)$$

where $\lambda$ is the relaxation parameter. If the relaxation parameter is $0 < \lambda < 2$, then the algorithm is consistent with Kaczmarz’s method. SSOR implies a forward and backward sweep of Kaczmarz. Therefore, (9) becomes

$$x^{(i+1)} = x^{(i)} + \lambda_j \frac{b_j - a_j^T x^{(i)}}{\|a_j\|^2} a_j, \quad i = 1, \ldots, 2m; j = \min(i, 2m + 1 - i).$$

It is now evident that the forward sweep is followed by the backward sweep. Furthermore, the double sweep can be transformed into

$$x^{(k+1)} = Q_{SSOR} x^{(k)} + R_{SSOR} b,$$
where

\[ Q_{\text{SSOR}} = Q_1 Q_2 \ldots Q_m Q_m \ldots Q_2 Q_1, \]

and

\[ Q_i = I - \lambda a_i a_i^T / \| a_i \|^2. \]

\( Q_{\text{SSOR}} \) is symmetric, since \( Q_i \) is symmetric for all \( i \).

Conjugate gradient is then applied to the system

\[ (I - Q_{\text{SSOR}}) x = R_{\text{SSOR}} b. \]

Define

\[ c = R_{\text{SSOR}} b \quad \text{and} \quad B = (I - Q_{\text{SSOR}}). \]

Then, the conjugate gradient method is applied to solve the symmetric positive semidefinite system \( Bx = c \), where \( B \) is symmetric and positive semidefinite and \( c \) is in the range of \( B \). For an arbitrary \( x^{(0)} \), define

\[
\begin{align*}
p^{(0)} &= r^{(0)} = c - Bx^{(0)}, \\
q^{(k)} &= Bp^{(k)}, \\
\alpha^{(k)} &= \| r^{(k)} \|^2 / (p^{(k)})^T q^{(k)}, \\
x^{(k+1)} &= x^{(k)} + \alpha^{(k)} p^{(k)}, \\
r^{(k+1)} &= r^{(k)} - \alpha^{(k)} q^{(k)}, \\
\beta^{(k)} &= \| r^{(k+1)} \|^2 / \| r^{(k)} \|^2, \\
p^{(k+1)} &= r^{(k+1)} + \beta^{(k)} p^{(k)}.
\end{align*}
\]

The algorithm proceeds by computing \( x^{(k+1)} \), \( r^{(k+1)} \), and \( p^{(k+1)} \) for each \( k = 0, 1, 2, \ldots \), where \( q^{(k)} = p^{(k)} - Q_{\text{SSOR}} p^{(k)} \). To compute \( p^{(0)} \), it is necessary to perform a full forward and backward Kaczmarz sweep as defined by (8) on the system \( Bx = c \). To compute \( q^{(k)} \), let \( x^{(1)} = p^{(k)} \) and again use the forward and backward Kaczmarz sweep with \( b = 0 \), which leads us to \( q^{(k)} = p^{(k)} - Q_{\text{SSOR}} p^{(k)} \).

Notice that there is no \( R_{\text{SSOR}} \) term since \( b = 0 \).

Algorithm 5 is a presentation of CGMN.

3.6. Conjugate-gradient-accelerated component-averaged row projections

Component-averaged row projections (CARP), which was introduced by Gordon and Gordon [10], divides the set of \( m \) equations into blocks. Each processor performs Kaczmarz iterations in parallel. The results of these Kaczmarz iterations are then merged together by averaging to form the next iterate.

In [10], the authors proved that CARP is equivalent to Kaczmarz’s algorithm with cyclic relaxation parameters in some superspace \( \mathbb{R}^s \), where \( s = \sum_{j=1}^{n} s_j \) and \( s_j \) denotes the number of nonzeros in column \( j \). There is a mapping that exists between \( \mathbb{R}^s \) and \( \mathbb{R}^n \), such that every \( x \in \mathbb{R}^n \) maps to some \( y \in \mathbb{R}^s \). That is, every component \( x_j \) of \( x \) maps to \( s_j \) components of \( y \).
Algorithm 5: Symmetric successive overrelaxation (SSOR)-preconditioned conjugate gradient (CGMN)

\begin{algorithm}
\begin{algorithmic}[1]
\State $t \leftarrow x^{(0)}$
\For{$i \leftarrow 1$ to $2m$}
\State $j \leftarrow \min (i, 2m + 1 - i)$
\State $t \leftarrow t + \lambda \frac{b_j - a^T_j t}{\|a_j\|^2} a_j$
\EndFor
\State $p^{(0)} \leftarrow t - x^{(0)}$
\State $r^{(0)} \leftarrow p^{(0)}$
\For{$k \leftarrow 0$ until convergence or maximum number of iterations met}
\For{$i \leftarrow 1$ to $2m$}
\State $j \leftarrow \min (i, 2m + 1 - i)$
\State $t \leftarrow t + \lambda \frac{a_j^T t}{\|a_j\|^2} a_j$
\EndFor
\State $q^{(k)} \leftarrow p^{(k)} - t$
\State $\alpha^{(k)} \leftarrow \|r^{(k)}\|^2 / \langle p^{(k)}, q^{(k)} \rangle$
\State $x^{(k+1)} \leftarrow x^{(k)} + \alpha^{(k)} p^{(k)}$
\State $r^{(k+1)} \leftarrow r^{(k)} - \alpha^{(k)} q^{(k)}$
\State $\beta^{(k)} \leftarrow \|r^{(k+1)}\|^2 / \|r^{(k)}\|^2$
\State $p^{(k+1)} \leftarrow r^{(k+1)} + \beta^{(k)} p^{(k)}$
\EndFor
\end{algorithmic}
\end{algorithm}
The fact that CARP is equivalent to Kaczmarz’s algorithm in some superspace $\mathbb{R}^s$ allows for the construction of a CG-accelerated CARP. The CG acceleration is performed in much the same manner as the CG acceleration of SSOR, which created CGMN. A forward followed by a backward sweep of CARP is performed, rather than the forward and backward sweep of Kaczmarz. Due to the superspace equivalency, it is shown in [10] that the CG acceleration of CARP always converges, even when the linear system is inconsistent and/or rectangular. The details are available in [10].

Algorithm 6 is a presentation of CARP-CG.

**Algorithm 6**: Conjugate gradient accelerated component-averaged row projections (CARP-CG)

1. $t \leftarrow x^{(0)}$
2. Perform the following Kaczmarz double sweep using CARP
   3. for $i \leftarrow 1$ to $2m$
      4. $j \leftarrow \min (i, 2m + 1 - i)$
      5. $t \leftarrow t + \lambda \frac{b_j - a_j^T t}{\|a_j\|^2} a_j$
   6. end
7. $p^{(0)} \leftarrow t - x^{(0)}$
8. $r^{(0)} \leftarrow p^{(0)}$
9. for $k \leftarrow 0$ until convergence or maximum number of iterations met
   10. $t \leftarrow p^{(k)}$
    11. Perform the following Kaczmarz double sweep using CARP
        12. for $i \leftarrow 1$ to $2m$
           13. $j \leftarrow \min (i, 2m + 1 - i)$
           14. $t \leftarrow t + \lambda \frac{a_j^T t}{\|a_j\|^2} a_j$
        15. end
    16. $q^{(k)} \leftarrow p^{(k)} - t$
    17. $\alpha^{(k)} \leftarrow \|r^{(k)}\|^2 / \langle p^{(k)}, q^{(k)} \rangle$
    18. $x^{(k+1)} \leftarrow x^{(k)} + \alpha^{(k)} p^{(k)}$
    19. $r^{(k+1)} \leftarrow r^{(k)} - \alpha^{(k)} q^{(k)}$
    20. $\beta^{(k)} \leftarrow \|r^{(k+1)}\|^2 / \|r^{(k)}\|^2$
    21. $p^{(k+1)} \leftarrow r^{(k+1)} + \beta^{(k)} p^{(k)}$
22. end

4. GPU implementation

4.1. Preliminary tests

The parallelization of linear algebra algorithms has been under investigation for the last few decades, and led to the proposal of a number of implementations
that utilize parallel computers. Depending on the specific parallel architecture, there are appropriate algorithms that match the characteristics of the underlying hardware such as memory organization and size, node communication bandwidth, and processing model (MIMD, SIMD). In many cases, a specific algorithm does not lend itself to a particular architecture, and it is required to develop a parallel version of the algorithm to achieve the desired speedup.

With this in mind, Gordon and Gordon [9] proposed the CARP algorithm, a block-parallel version of the Kaczmarz algorithm implemented on a 16-node Linux cluster. In addition to exhibiting increased robustness for the problems studied in [9], CARP also exhibited a better or comparable solution time relative to other well-established algorithms that had been proposed for the same kind of problems. In implementations with a single general-purpose processor, CARP and CARP-CG, being block-parallel algorithms, are not suitable; thus Gordon and Gordon in [11] show that the CGMN algorithm gives the best robustness and performance, with CGNR offering the same robustness but lower performance on the same set of problems as in [10]. Note that CARP-CG reduces to CGMN, when the number of blocks is equal to one.

In the preliminary search of the most appropriate iterative algorithms to implement on the GPU, it became clear to us that Kaczmarz’s algorithm may not be the best possible algorithm for the GPU architecture. We first investigated Kaczmarz’s algorithm in the form presented in [16]. It is important to keep in mind that while a thread is very cheap to originate on a GPU, allowing for performance gains in operations as simple as level-one BLAS routines, the problem must be large enough to mitigate the overhead of launching the thread, transferring the data to and from the GPU, and any interruptions necessary to check for convergence on the CPU. It was quickly discovered that, due to this fact and because the only available operations of this algorithm that can be parallelized are the dot product and vector addition in (1), this algorithm could only exhibit superior performance on the GPU when studying very large, dense linear systems. For details on how one can use a GPU to implement reduction-based parallel operations and other BLAS routines, the reader is referred to [12]. The performance of the implementation is shown in Table 1. In this table, the problems studied are one hundred percent dense $n \times n$ systems, where the elements of $A$ and $b$ are drawn randomly from a normal distribution with a mean of zero and standard deviation of one. The reader can clearly see that the memory initialization and memory copy necessary to load the data on the GPU can be costly for dense systems depending on the total number of iterations required. However, this cost is amortized by the superior cost per iteration as the problem size grows and/or the number of required iterations increases.

In the case of Cimmino’s algorithm, we quickly discovered two facts. First, this algorithm exhibited very poor solution times due to the number of necessary iterations for the particular PDEs studied in this paper. Additionally, the algorithm was very costly at each iteration, regardless of whether it was implemented on the GPU or the CPU. We implemented Cimmino’s algorithm for dense linear systems in order to get a better idea of the best-case speedup that could be expected by placing this algorithm on the GPU. The results are
Table 1: Run times of Kaczmarz’s algorithm for dense \( n \times n \) systems

<table>
<thead>
<tr>
<th>Problem size (n)</th>
<th>GPU time (sec)</th>
<th>CPU time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Memory init.</td>
<td>Per iter.</td>
</tr>
<tr>
<td>1000</td>
<td>0.003</td>
<td>0.034</td>
</tr>
<tr>
<td>2500</td>
<td>0.016</td>
<td>0.036</td>
</tr>
<tr>
<td>5000</td>
<td>0.061</td>
<td>0.039</td>
</tr>
<tr>
<td>7500</td>
<td>0.137</td>
<td>0.044</td>
</tr>
<tr>
<td>10000</td>
<td>0.245</td>
<td>0.043</td>
</tr>
<tr>
<td>15000</td>
<td>0.547</td>
<td>0.055</td>
</tr>
</tbody>
</table>

Table 2: Run times of Cimmino’s algorithm for dense \( n \times n \) systems

<table>
<thead>
<tr>
<th>Problem size (n)</th>
<th>GPU time (sec)</th>
<th>CPU time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Memory init.</td>
<td>Per iter.</td>
</tr>
<tr>
<td>1000</td>
<td>0.004</td>
<td>0.624</td>
</tr>
<tr>
<td>2500</td>
<td>0.018</td>
<td>5.347</td>
</tr>
<tr>
<td>5000</td>
<td>0.063</td>
<td>15.496</td>
</tr>
<tr>
<td>10000</td>
<td>0.265</td>
<td>28.634</td>
</tr>
</tbody>
</table>

presented in Table 2. Cimmino can be parallelized on the GPU by implementing slightly specialized level-two BLAS routines. For more details on how one might implement level-two BLAS routines for dense matrices, the reader is referred to [12]. Ultimately, the low potential for improvement in the cost per iteration and the number of required iterations led us to reject this method for the particular PDE problems studied.

In addition, CAV was studied. Due to the fact that CAV is equivalent to Cimmino’s algorithm when applied to dense problems, CAV was preliminarily tested on the GPU using banded linear systems (also generated randomly). We found that the method had potential and mapped well to the GPU. However, as was the case with Cimmino’s algorithm, it appeared that CAV requires too many iterations. Based on the results demonstrated in [20], albeit on different problems, and the potential for speedup on the GPU (see Table 3 and Table 4), we decided it was appropriate to at least try this method on the PDEs in Section 5. In Table 3, the number of diagonals above and below the main diagonal were held constant at twenty-five, while the size of the matrix itself was varied. In Table 4, the size of the matrix was fixed at 25,000 \( \times \) 25,000, while the number of nonzero diagonals above and below the main diagonal varied from one to sixty-four. The relatively poor convergence witnessed here is due to the small variance in the nonzero elements of the PDEs studied. This relatively small variance causes CAV to perform no better than Cimmino.

4.2. GPU implementation

We have tailored the kernel to the seven bands at fixed offsets of the \( A \) matrix. The reads and writes can be perfectly coalesced with the restriction that the number of the discretization points on the \( x \) direction is a multiple of 16 to achieve coalesced global memory accessing. In other words, if we desire
40 discretization points on the $x$ axis, it is better to choose 48, which will give higher resolution and better performance than the 40-point discretization.

For a desired tolerance of $10^{-7}$, the arithmetic results need to be at least double precision. There are two ways to achieve this accuracy. The first is to exclusively use double-precision floating-point arithmetic for all operations. The second is to use single-precision floating point with iterative refinement in order to achieve the required double-precision termination criterion.

Using exclusively double precision for all operations can be necessary depending on the size of the problem, the algorithm that is used, and the termination criterion. The drawback of using double-precision floating point is the lower speed of the floating-point unit relative to single-precision, and the requirement to load and store 64-bit data, compared to the 32-bit data of single-precision floating point. When it is necessary or desired to use single-precision arithmetic, iterative refinement reduces the roundoff errors in the computed solution of a system of linear equations. The basic steps of solving $Ax = b$ with iterative refinement are given in Algorithm 7.

Since the Tesla C870 GPU is only capable of single precision, the double-precision refinement is done on the CPU, which involves the transfer of the data from the GPU to the CPU and vice versa in each refinement iteration. This extra data transfer is unavoidable with the Tesla C870. Newer GPU models, such as the Tesla C1060 have double- and single-precision capabilities, which would eliminate these data transfers. In our current implementation, the single-precision step of the algorithm is carried out on the GPU and involves the
Algorithm 7: CPU-GPU iterative refinement

1. for $i \leftarrow 0$ to (Maximum-CPU-Iterations − 1) do
2.   Compute $d = b - Ax$ in double precision on the CPU
3.   Transfer $d$ to the GPU in single precision
4.   Solve $Ac = d$ approximately on the GPU
5.   \[ \|c^{(0)}\| = \text{Maximum Representable Number} \]
6.   for $k \leftarrow 1$ to (Maximum-GPU-Iterations − 1) do
7.     Carry out one iteration of CGNR, CGMN, CARP-CG, or CAV on the GPU
8.     If $\|c^{(k)}\|/\|c^{(k-1)}\| > 1.0$ terminate loop
9.   end
10. Transfer $c$ to the CPU
11. Update $x = x + c$ in double precision on the CPU
12. If $res^{(i)}/res^{(0)} < \text{TOL}$, terminate; else, iterate
13. end

solution of $Ac = d$ with one of the iterative methods CGNR, CGMN, CARP-CG, or CAV. At the end of each internal iteration, we test whether the solution of $Ac = d$ has been improved or not by checking whether $\|c^{(k)}\|/\|c^{(k-1)}\| > 1.0$. In case this condition is not met, the $Ac = d$ loop is terminated and the iterative-refinement step updates the solution of $x = x + c$ in double precision.

Even though a double-precision GPU could carry out all the operations in double precision, thereby avoiding the refinement steps, it is not guaranteed that the overall performance would be better than using iterative refinement [23]. Although the exclusive use of double precision requires fewer iterations to achieve convergence, this performance benefit is assuaged by the increased memory bandwidth and the reduced performance of double-precision arithmetic. For example, it is notable that the peak single-precision performance of the Tesla C1060 is 933 GFLOPS, while its peak double-precision performance is only 78 GFLOPS.

5. Computational results

The test problems used here are the same used in [9, 10, 11], and are given below:

1. $\Delta u + 1000 = F$
2. $\Delta u + 1000e^{xyz}(ux + uy - uz) = F$
3. $\Delta u + 100xux + yu_y + zu_z + 100(x + y + z)u/xyz = F$
4. $\Delta u - 10^5x^2(ux + uy + uz) = F$
5. $\Delta u - 1000(1 + x^2)ux + 100(uy + uz) = F$
6. $\Delta u - 1000[(1 - 2x)ux + (1 - 2y)uy + (1 - 2z)uz] = F$
7. $\Delta u - 1000x^2ux + 1000u = F$
8. \( \Delta u - \partial(10e^{xy}u) / \partial x - \partial(10e^{-xy}u) / \partial y = F \)
9. \( \Delta u - \partial(1000e^{xy}u) / \partial x - \partial(1000e^{-xy}u) / \partial y = F \)

Here, we use the notation \( \Delta = u_{xx} + u_{yy} + u_{zz} \). Problems 1–7 have the following analytical solution:

Problem 1: \( u(x, y, z) = xyz(1 - x)(1 - y)(1 - z) \)
Problem 2: \( u(x, y, z) = x + y + z \)
Problem 3–7: \( u(x, y, z) = e^{xyz} \sin(\pi x) \sin(\pi y) \sin(\pi z) \).

The expression for the right-hand side function \( F \) in problems 1–7 is computed analytically, using the preassigned solution. For problems 8 and 9, the right-hand-side function \( F \) is irrelevant, because the equation system was set up by first computing the system matrix \( A \), and then computing \( b = Av \), where \( v \) was chosen as \( v = (1, 1, \ldots, 1)^T \) (similar to the approach taken in [1, Problem F3D]). All the test problems were solved on the unit cube domain \([0,1] \times [0,1] \times [0,1]\), with the Dirichlet boundary condition \( u = 0 \). The problems were discretized using a uniform mesh with the same number of mesh points in each direction, and the equations were obtained by using a seven-point centered difference scheme. Test runs were made for problems of size \( 80 \times 80 \times 80 = 512,000 \) equations. Additional tests were also made on smaller grid sizes in order to study how the various algorithms perform as the mesh is gradually refined.

Since the results in this section are compared to the results in [10, 11], an identical termination criterion is chosen, which is based on the relative residual:

\[
\text{res/\text{res}^{(0)}} = \frac{\| b - Ax \|}{\| b - Ax^{(0)} \|} < 10^{-7}, \quad \text{with} \quad x^{(0)} = 0.
\]

For problems 3 and 7, this criterion was unattainable. Instead, we used the same termination criteria used in [10, 11], i.e., \( 10^{-4} \) and \( 5 \cdot 10^{-4} \), respectively. Since this criterion depends on the scaling of the equations, we first normalized the equations (for all the tested methods) by dividing each equation by the \( L_2 \)-norm of its coefficients.

We investigate the GPU implementation of the most robust methods (CGNR, CGMN, CARP-CG) for the nine PDE problems, and we compare their performance to a single CPU implementation of CGNR and CGMN, and to the 16-node cluster implementation of CARP-CG [10]. The CARP-CG is a block-parallel algorithm and is suitable only for parallel computers, thus its CPU implementation is not considered. Moreover, we show the results of the GPU implementation of the CAV algorithm for the sparse PDE problems due to the fact that CAV achieves the highest GFLOPS performance.

Figures 1–9 present the convergence of each method for each problem with respect to time. It is clear from these figures that, for all problems, the GPU implementation of CGNR achieves the best performance relative to the other methods on both the CPU and the GPU. On the CPU, CGMN is consistently better than CGNR, as expected [11]. Although CARP-CG performs the best on a 16-node Linux cluster [10], the proposed implementation does not map well on the GPU architecture. The same can be observed for CGMN, which lends itself to a single CPU architecture, but is not the best choice for a GPU implementation. The relaxation parameters, \( \lambda \), for CGMN are identical to the ones in [11].
Figure 1: Convergence results for Problem 1.

Figure 2: Convergence results for Problem 2.
Figure 3: Convergence results for Problem 3.

Figure 4: Convergence results for Problem 4.
Figure 5: Convergence results for Problem 5.

Figure 6: Convergence results for Problem 6.
Figure 7: Convergence results for Problem 7.

Figure 8: Convergence results for Problem 8.
In general, GPUs are suitable for algorithms, such as BLAS, that exhibit fine grain parallelism. There is a trade-off between using small blocks to take advantage of the fine grain parallelism available on the GPU and the number of iterations required, since one of the basic underlying assumptions in CARP-CG is that the block boundary points are just a small fraction of the total number of points [10]. The proposed GPU CARP-CG is a direct implementation of the algorithm of [10] with uniformly divided grids and scaling-factor values that give the smallest number of iterations. The Kaczmarz sweeps can be executed on the GPU or on the CPU. Despite the additional burden of transferring the data to the CPU, executing the Kaczmarz sweeps on the CPU is a better choice. For the same reason, CGMN does not map well on the GPU, as the Kaczmarz sweeps slow down the performance of the algorithm. CAV is the algorithm that achieves the highest GFLOPS performance on the GPU, as can be seen in Table 5. However, its extremely slow rate of convergence makes CAV the slowest algorithm among the ones studied in this paper.

The algorithm that exhibits the best balance between parallelization and rate of convergence with iterative refinement is CGNR. In Tables 6 and 7, we display the number of iterations and times to solution, respectively, for each architecture (GPU, CPU, 16-node cluster) and algorithm. Although the GPU CGNR requires more iterations to achieve the same level of precision relative to GPU CGMN and GPU CARP-CG, we observe that GPU CGNR exhibits the best solution time due to its superior utilization of the fine-grain parallelism available on the GPU architecture. The very large number of iterations required
### Table 5: Average GFLOPS performance over the nine problems. For GPU-CAV, the average is over the six problems that it does not fail.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>GFLOPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>GPU-CGNR</td>
<td>11.82</td>
</tr>
<tr>
<td>GPU-CG3MN</td>
<td>0.32</td>
</tr>
<tr>
<td>GPU-CARP-CG</td>
<td>0.66</td>
</tr>
<tr>
<td>GPU-CAV</td>
<td>15.26</td>
</tr>
<tr>
<td>CPU-CG3MN</td>
<td>0.2</td>
</tr>
<tr>
<td>CPU-CGNR</td>
<td>0.31</td>
</tr>
</tbody>
</table>

by GPU CAV results in its poor performance and its failure to converge for Problems 2, 4, and 8.

The robustness of the algorithms of [10, 11] is preserved here despite the single-precision arithmetic, largely due to iterative refinement, which guarantees convergence to a solution within the desired accuracy. Moreover, the CGNR algorithm with single-precision iterative refinement is much more robust than CGMN and CARP-CG, in terms of required iterations to convergence. In Table 8, we present the ratio of iterative-refinement iterations for each algorithm over the double-precision iterations. It can be observed from this table that CGNR is affected less by the use of single-precision floating-point arithmetic. This is demonstrated further in Table 9, where it can be seen that the number of iterative-refinement iterations is always smaller for CGNR in every problem except for Problem 5. Moreover, CGNR does not involve a relaxation parameter that needs to be adjusted as in CARP-CG and CGMN.

Finally, in Figure 10, we show the speedup obtained by the GPU in comparison to the CPU CGMN and the 16-node cluster CARP-CG implementations. The GPU implementation is 5–20 times faster than the CPU, depending on the particular problem. Compared to the 16-node cluster, the GPU implementation is 1–3 times faster. The achieved speedup for CGNR is roughly a factor of 30 to 40.
<table>
<thead>
<tr>
<th>Problem</th>
<th>GPU</th>
<th>CPU</th>
<th>Cluster [10]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.01</td>
<td>7.79</td>
<td>110.54</td>
</tr>
<tr>
<td>2</td>
<td>2.45</td>
<td>15.95</td>
<td>2.45</td>
</tr>
<tr>
<td>3</td>
<td>1.26</td>
<td>4.32</td>
<td>8.80</td>
</tr>
<tr>
<td>4</td>
<td>3.26</td>
<td>46.29</td>
<td>45.11</td>
</tr>
<tr>
<td>5</td>
<td>1.27</td>
<td>9.40</td>
<td>7.45</td>
</tr>
<tr>
<td>6</td>
<td>0.73</td>
<td>5.56</td>
<td>6.48</td>
</tr>
<tr>
<td>7</td>
<td>0.65</td>
<td>4.34</td>
<td>4.91</td>
</tr>
<tr>
<td>8</td>
<td>16.68</td>
<td>220.06</td>
<td>115.36</td>
</tr>
<tr>
<td>9</td>
<td>2.02</td>
<td>10.70</td>
<td>32.31</td>
</tr>
</tbody>
</table>

Table 7: Runtimes (sec) for Problems 1–9

<table>
<thead>
<tr>
<th>Problem</th>
<th>CGNR</th>
<th>CGMN</th>
<th>CARP-CG</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.22</td>
<td>1.14</td>
<td>1.61</td>
</tr>
<tr>
<td>2</td>
<td>1.03</td>
<td>1.04</td>
<td>6.53</td>
</tr>
<tr>
<td>3</td>
<td>1.00</td>
<td>1.00</td>
<td>0.65</td>
</tr>
<tr>
<td>4</td>
<td>1.01</td>
<td>1.00</td>
<td>1.86</td>
</tr>
<tr>
<td>5</td>
<td>1.01</td>
<td>1.04</td>
<td>1.57</td>
</tr>
<tr>
<td>6</td>
<td>1.02</td>
<td>1.05</td>
<td>2.32</td>
</tr>
<tr>
<td>7</td>
<td>1.00</td>
<td>1.39</td>
<td>1.37</td>
</tr>
<tr>
<td>8</td>
<td>1.15</td>
<td>1.75</td>
<td>2.08</td>
</tr>
<tr>
<td>9</td>
<td>1.02</td>
<td>0.83</td>
<td>4.42</td>
</tr>
</tbody>
</table>

Table 8: Ratio of iterative-refinement iterations over double-precision iterations for each method

<table>
<thead>
<tr>
<th>Problem</th>
<th>CGNR</th>
<th>CGMN</th>
<th>CARP-CG</th>
<th>CAV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>78</td>
<td>−</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>−</td>
</tr>
<tr>
<td>5</td>
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<td>3</td>
<td>9</td>
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<td>2</td>
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<td>0</td>
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<tr>
<td>8</td>
<td>2</td>
<td>99</td>
<td>23</td>
<td>−</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>2</td>
<td>57</td>
<td>14</td>
</tr>
</tbody>
</table>

Table 9: Number of iterative refinements for Problems 1–9
6. Conclusions

Among the algorithms considered in this paper, CGNR was shown to be the most efficient for solving the investigated partial differential equations on the GPU. This finding is in contrast to earlier works on the CPU, where CGMN was found to be consistently better than CGNR. Moreover, CGNR was the most robust method under the GPU’s single-precision floating-point arithmetic. CAV was able to achieve the highest GFLOPS but slowest solution time due to poor convergence.

The GPU has a clear advantage when compared to the CPU. In particular, the computational results demonstrated that the GPU implementation is five to twenty times faster than the CPU, depending on the particular problem. Compared to a 16-node cluster, the GPU implementation was one to three times faster. When the cost of the hardware, its power consumption, and its portability are accounted for, it becomes apparent that a single GPU offers a clear advantage over a 16-node Linux cluster.

While algorithm performance is known to be architecture-specific, our computational results specifically demonstrate that the GPU offers a low-cost and high-performance computing solution for solving large-scale partial differential equations. Continued improvements, such as performance and double precision, will only make the GPU more attractive as a high-performance computing device for scientific computing applications.
Acknowledgement

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References


