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Painlevé - Calogero correpondence. [1]

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Abstract

It is proved that the Painlevé VI equation ($PVI_{\alpha,\beta,\gamma,\delta}$) for the special values of constants ($\alpha = \frac{\nu^2}{4}, \beta = -\frac{\nu^2}{4}, \gamma = \frac{\nu^2}{4}, \delta = \frac{1}{2} - \frac{\nu^2}{4}$) is a reduced hamiltonian system. Its phase space is the set of flat $SL(2, \mathbf{C})$ connections over elliptic curves with a marked point and time of the system is given by the elliptic module. This equation can be derived via reduction procedure from the free infinite hamiltonian system. The phase space of later is the affine space of smooth connections and the "times are the Beltrami differentials. This approach allows to define the associate linear problem, whose isomonodromic deformations is provided by the Painlevé equation and the Lax pair. In addition, it leads to description of solutions by a linear procedure. This scheme can be generalized to G bundles over Riemann curves with marked points, where G is a simple complex Lie group. In some special limit such hamiltonian systems convert into the Hitchin systems. In particular, for SL$(N, \mathbf{C})$ bundles over elliptic curves with a marked point we obtain in this limit the elliptic Calogero N-body system. Relations to the classical limit of the Knizhnik- Zamolodchikov-Bernard equations is discussed.

1 Introduction

1.We learned from Yu. Manin's lectures in MPI (Bonn, 1996) about elliptic form of the famous Painlevé VI equation (PVI) [1]. In this representation PVI looks very similar to the elliptic Calogero-Inozemtsev-Treibich-Verdier (CITV) rank one system [2, 3, 4]. Namely, the both equations are hamiltonian with the same symplectic structure for two dynamical variables and the same Hamiltonians. The only difference is that the time in the PVI system is nothing else as the elliptic module. Therefore, it is non autonomous hamiltonian system, while the CITV Hamiltonian is independent of time. This similarity is not accidental and based on very closed geometric origin of the both systems, which we will elucidate in this talk. It should be confessed from the very beginning that at the

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present time our approach is cover only the one parametric family of $PVI_{\alpha,\beta,\gamma,\delta}$. This family corresponds to the standard two-body elliptic Calogero equation.

2. Painlevé VI and Calogero equations. The Painlevé VI $PVI_{\alpha,\beta,\gamma,\delta}$ equation depends of four free parameters and has the form

$$\frac{d^2X}{dt^2} = \frac{1}{2}(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t})(\frac{dX}{dt})^2 - (\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t})\frac{dX}{dt} +$$
$$+ \frac{X(X-1(X-t)}{t^2(t-1)^2}(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{X-t)^2}) \tag{1.1}$$

It is a hamiltonian systems [5], but we will write the symplectic form and the Hamiltonian below in another variables. Among some distinguish features of this equation we are interesting in its relation to the isomonodromic deformations of linear differential equations. This approach was investigated by Fuchs [6], while first $PVI_{\alpha,\beta,\gamma,\delta}$ was written down by Gambier [7]. The equation has a lot of different applications (see [8]). We shortly present $PVI_{\alpha,\beta,\gamma,\delta}$ in terms of elliptic functions [1].

Let $\wp(u|\tau)$ be the Weiershtrass function on the elliptic curve $T^2_\tau = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$, and $e_i = \wp(\frac{T_i}{2}|\tau)$, $(i = 1,2,3)$ $(T_0,\dots,T_3) = (0,1,\tau,1+\tau)$. Consider instead of (t,X) in (1.1) the new variables

$$(\tau, u) \to (t = \frac{e_3 - e_1}{e_2 - e_1}, X = \frac{\wp(u|\tau) - e_1}{e_2 - e_1}). \tag{1.2}$$

Then $PVI_{\alpha,\beta,\gamma,\delta}$ takes the form.

$$\frac{d^2u}{d\tau^2} = \partial_u U(u|\tau), \quad U(u|\tau) = \frac{1}{(2\pi i)^2}\sum_{j=0}^{3}\alpha_j\wp(u + \frac{T_j}{2}|\tau), \tag{1.3}$$

$(\alpha_0,\dots,\alpha_3) = (\alpha, -\beta, \gamma, \frac{1}{2} - \delta)$. As usual in non autonomous case, the equations of motion (1.3) are derived from the variations of the degenerated symplectic form

$$\omega = \delta v \delta u - \delta H \delta\tau, \quad H = \frac{v^2}{2} + U(u|\tau), \tag{1.4}$$

which is defined over the extended phase space $\mathcal{P} = \{v, u, \tau\}$. The semidirect product of $\mathbf{Z} + \mathbf{Z}\tau$ and the modular group act on the dynamical variables (v, u, τ) preserving (1.4).

Let us introduce the new parameter κ (the level) and instead of (1.4) consider

$$\omega = \delta v \delta u - \frac{1}{\kappa}\delta H \delta\tau. \tag{1.5}$$

It corresponds to the overall rescaling of constants $\alpha_j \to \frac{\alpha_j}{\kappa^2}$. Put $\tau = \tau_0 + \kappa t^H$ and consider the system in the limit $\kappa \to 0$, which is called the critical level. We come to the equation

$$\frac{d^2u}{(dt^H)^2} = \partial_u U(u|\tau_0), \tag{1.6}$$

It is just the rank one $CITV_{\alpha,\beta,\gamma,\delta}$ equation. Thus, we have in this limit
$PVI_{\alpha,\beta,\gamma,\delta} \xrightarrow{\kappa\to 0} CI_{\alpha,\beta,\gamma,\delta}$.

Consider one-parametric family $PVI_{\frac{\nu^2}{4},-\frac{\nu^2}{4},\frac{\nu^2}{4},\frac{1}{2}-\frac{\nu^2}{4}}$ The potential (1.3) takes the form

$$U(u|\tau) = \frac{1}{(4\pi i)^2}\nu^2\wp(2u|\tau), \tag{1.7}$$

We will prove that (1.5) with the potential (1.7) describe the dynamic of flat connections of SL$(2,\mathbf{C})$ bundles over elliptic curves T_τ with one marked point $\Sigma_{1,1}$. In fact, u lies on the Jacobian of T_τ, (v,u) defines a flat bundle, and τ defines a point in the moduli space $\mathcal{M}_{1,1} = \{\Sigma_{1,1}\}$. The choice of the polarization of connections, in other words v and u, depends on complex structure of $\Sigma_{1,1}$. The extended phase space $\mathcal{P}$ includes beside the dynamical variables v and u the "time" τ. It is the bundle over $\mathcal{M}_{1,1}$ with the fibers $\mathcal{R} = \{v,u\}$, which is endowed by the degenerated symplectic structure ω (1.5). This system is derived by a reduction procedure from some free, but infinite hamiltonian system. In this way we obtain the Lax equations, the linear system which monodromies preserve by (1.6) with $U(u|\tau)$ (1.7) and the explicit solutions of the Cauchy problem via the so-called projection method. The discrete symmetries of (1.4) are nothing else as the remnant gauge symmetries. On the critical level it is just two-body elliptic Calogero system. The corresponding quantum system is identified with the KZB equation [9, 10] for the one-vertex correlator on T_τ. In the similar way $PVI_{\frac{\nu^2}{4},-\frac{\nu^2}{4},\frac{\nu^2}{4},\frac{1}{2}-\frac{\nu^2}{4}}$ is the classical limit of the KZB for $\kappa \neq 0$.

3.This particular example has far-reaching generalizations. Consider a phase space, which is the moduli space of flat connections $\mathcal{A}$ of G bundle over Riemann curve $\Sigma_{g,n}$ of genus g with n marked points, where G is a complex simple Lie group. While the flatness is the topological property of bundles, the polarization of connections $\mathcal{A} = (A,\bar{A})$ depends on the choice of complex structure on $\Sigma_{g,n}$. Therefore, we consider a bundle $\mathcal{P}$ over the moduli space $\mathcal{M}_{g,n}$ of curves with flat connections $\mathcal{R} = (A,\bar{A})$ as fibers. The fibers are supplemented by elements of coadjoint orbits $\mathcal{O}_a$ in the marked points x_a. There exists a closed degenerate two-form ω on $\mathcal{P}$, which is non degenerate on the fibers. The equations of motions are defined as variations of the dynamical variables along the null-leaves of this symplectic form. We call them as the *hierarchies of isomonodromic deformations* (HID). They are attended by the *Whitham hierarchies*, which has occurred earlier in [11] as a result of the averaging procedure, and then in [12] as the classical limit of "string equations". Our approach is closed to the Hitchin construction of integrable systems, living on the cotangent bundles to the moduli space of holomorphic G bundles [13], generalized for singular curves in [14, 15]. Namely, the connection $\bar{A}$ plays the same role as in the Hitchin scheme, while A replaces the Higgs field. Essentially, our construction is local - we work over a vicinity of some fixed curve $\Sigma_{g,n}$ in $\mathcal{M}_{g,n}$. The coordinates of tangent vector to $\mathcal{M}_{g,n}$ in this point play role of times, while the Hitchin times have nothing to do with the moduli space. The Hamiltonians are the same quadratic Hitchin Hamiltonians, but now they are time dependent. There is a free parameter κ (*the level*) in our construction. On the critical level ($\kappa = 0$), after rescaling the times, our systems convert into the Hitchin systems. In concrete examples our work is based essentially on [14], which deals with the same systems on the critical level.

As the Hitchin systems, HID can be derived by the symplectic reduction from a free infinite hamiltonian system. In our case the upstairs extended phase space is the space of the affine connections and the Beltrami differentials. We consider its symplectic quotient with respect of gauge action on the connections. In addition, to come to the moduli space $\mathcal{M}_{g,n}$ we need the subsequent factorization under the action of the diffeomorphisms, which effectively acts on the Beltrami differentials only. Apart from the last step, this derivation resembles the construction of the KZB systems in [16], where they are derived as a quantization of the very similar symplectic quotient. Our approach allows to write down the Lax pairs, prove that the HID are consistency conditions of the isomonodromic deformations of the linear Lax equations, and, therefore, justify the notion HID. Moreover, we describe solutions via linear procedures (the projection method). HID are the quasi classical limit of the KZB equations for ($\kappa \neq 0$, as the Hitchin systems are the quasi classical limit of the KZB equations on the critical level [14, 17]. The quantum counterpart of the Whitham hierarchy is the flatness condition, which discussed in [18] within derivation of KZB. The interrelations between quantizations of isomonodromic deformations and the KZB eqs were discussed in [19, 20, 21].

For genus zero our procedure leads to Schlesinger equations. We restrict ourselves to simplest cases with only simple poles of connections. Therefore, we don't include in the phase space the Stokes parameters. This phenomen was investigated in the rational case in detail in [22]. For genus one we obtain a particular case of the Painlevé VI equation (for $SL(2, \mathbf{C}$ bundles with one marked point), generalization of this case on arbitrary simple groups and arbitrary number of marked points.

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2 Symplectic reduction

1. Upstairs extended phase space. Let $\Sigma_{g,n}$ be a Riemann curve of genus g with n marked points. Let us fix the complex structure of $\Sigma_{g,n}$ defining local coordinates $(z, \bar{z})$ in open maps covering $\Sigma_{g,n}$. Assume that the marked points $(x_1, \ldots, x_n)$ are in the generic positions. The deformations of the basic complex structure are determined by the Beltrami differentials μ, which are smooth $(-1, 1)$ differentials on $\Sigma_{g,n}$, $\mu \in \mathcal{A}^{(-1,1)}(\Sigma_{g,n})$. We identify this set with the space of times $\mathcal{N}'$. The Beltrami differentials can be defined in the following way. Consider the chiral diffeomorphisms of $\Sigma_{g,n}$

$$w = z - \epsilon(z, \bar{z}), \quad \bar{w} = \bar{z} \tag{2.1}$$

and the corresponding one-form dw. Up to the conformal factor $1-\partial\epsilon(z,\bar{z})$, it is equal

$$dw = dz - \mu d\bar{z}, \quad \mu = \frac{\bar{\partial}\epsilon(z,\bar{z})}{1-\partial\epsilon(z,\bar{z})}. \tag{2.2}$$

The new holomorphic structure is defined by the deformed antiholomorphic operator annihilating dw, while the antiholomorphic structure is kept unchanged

$$\partial_{\bar{w}} = \bar{\partial} + \mu\partial, \quad \partial_w = \partial.$$

In addition, assume that μ vanishes in the marked points $\mu(z,\bar{z})|_{x_a} = 0$. We consider small deformations of the basic complex structure $(z,\bar{z})$. It allows to replace (2.2) by

$$\mu = \bar{\partial}\epsilon(z,\bar{z}). \tag{2.3}$$

Let $\mathcal{E}$ be a principle stable G bundle over a Riemann curve $\Sigma_{g,n}$. Assume that G is a complex simple Lie group. The phase space $\mathcal{R}'$ is recruited from the following data:
i)the affine space $\{\mathcal{A}\}$ of Lie(G)-valued connection on $\mathcal{E}$.
It has the following component description:
a) C^∞ connection $\{\bar{A}\}$, corresponding to the $d\bar{w} = d\bar{z}$ component of $\mathcal{A}$;
b)The dual to the previous space the space $\{A\}$ of dw components of $\mathcal{A}$. A can have simple poles in the marked points. Moreover, assume that $\bar{A}\mu$ is a C^∞ object ;
ii)cotangent bundles $T^*G_a = \{(p_a, g_a),\ p_a \in \mathrm{Lie}^*(G_a),\ g_a \in G_a\},\ (a = 1,\ldots,n)$ in the points $(x_1,\ldots,x_n)$.
There is the canonical symplectic form on $\mathcal{R}'$

$$\omega_0 = \int_\Sigma < \delta A, \delta\bar{A} > +2\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1}\delta g_a >, \tag{2.4}$$

where $< \ ,\ >$ denotes the Killing form on Lie(G).

Consider the bundle $\mathcal{P}'$ over $\mathcal{N}'$ with $\mathcal{R}'$ as the fibers. It plays role of the extended phase space. There exists the degenerate form on $\mathcal{P}'$

$$\omega = \omega_0 - \frac{1}{\kappa}\int_\Sigma < \delta A, A > \delta\mu. \tag{2.5}$$

Thus, we deal with the infinite set of Hamiltonians $< A, A > (z,\bar{z})$, parametrized by points of $\Sigma_{g,n}$ and corresponding set of times $\mu(z,\bar{z})$. The equations of motion. take the form

$$\frac{\partial A}{\partial\mu}(z,\bar{z}) = 0, \quad \kappa\frac{\partial\bar{A}}{\partial\mu}(z,\bar{z}) = A(z,\bar{z}), \quad \frac{\partial p_b}{\partial\mu} = 0, \quad \frac{\partial g_b}{\partial\mu} = 0. \tag{2.6}$$

We will apply the formalism of hamiltonian reduction to these systems.

2. Symmetries. The form ω (2.5) is invariant with respect to the action of the group $\mathcal{G}_0$ of diffeomorphisms of $\Sigma_{g,n}$, which are trivial in vicinities $\mathcal{U}_a$ of marked points:

$$\mathcal{G}_0 = \{z \to N(z,\bar{z}), \bar{z} \to \bar{N}(z,\bar{z}),),\ N(z,\bar{z}) = z + o(|z - x_a|),\ z \in \mathcal{U}_a\}. \tag{2.7}$$

Another infinite gauge symmetry of the form (2.5) is the group $\mathcal{G}_1 = \{f(z,\bar{z}) \in C^\infty(\Sigma_g, G)\}$ that acts on the dynamical fields as

$$A \to f(A+\kappa\partial)f^{-1}, \quad \bar{A} \to f(\bar{A}+\bar{\partial}+\mu\partial)f^{-1},$$
$$(\bar{A}' \to f(\bar{A}'+\bar{\partial})f^{-1}), \tag{2.8}$$
$$p_a \to f_a p_a f_a^{-1}, \quad g_a \to g_a f_a^{-1}, \quad (f_a = \lim_{z\to x_a} f(z,\bar{z})), \quad \mu \to \mu.$$

In other words, the gauge action of $\mathcal{G}_1$ does not touch the base $\mathcal{N}'$ and transforms only the fibers $\mathcal{R}'$. The whole gauge group is the semidirect product $\mathcal{G}_1 \oslash \mathcal{G}_0$.

3. Symplectic reduction with respect to $\mathcal{G}_1$. Since the symplectic form (2.5) is closed (though is degenerated) one can consider the symplectic quotient of the extended phase space $\mathcal{P}'$ under the action of the gauge transformations (2.8). They are generated by the moment constraints

$$F_{A,\bar{A}}(z,\bar{z}) - 2\pi i \sum_{a=1}^{n} \delta^2(x_a) p_a = 0, \tag{2.9}$$

where $F_{A,\bar{A}} = (\bar{\partial}+\partial\mu)A - \kappa\partial\bar{A} + [\bar{A}.A]$. It means that we deal with the flat connection everywhere on $\Sigma_{g,n}$ except the marked points. The holonomies of $(A,\bar{A})$ around the marked points are conjugate to $\exp 2\pi i p_a$.

Let $(L,\bar{L})$ be the gauge transformed connections

$$\bar{A} = f(\bar{L}+\bar{\partial}+\mu\partial)f^{-1}, \quad A = f(L+\kappa\partial)f^{-1}, \tag{2.10}$$

Then (2.9) takes the form

$$(\bar{\partial}+\partial\mu)L - \kappa\partial\bar{L} + [\bar{L},L] = 2\pi i \sum_{a=1}^{n} \delta^2(x_a) p_a. \tag{2.11}$$

Remark 2.1 *The gauge fixing allows to choose $\bar{A}$ in a such way that $\partial\bar{L} = 0$. Then (2.11) takes the form*

$$(\bar{\partial}+\partial\mu)L + [\bar{L},L] = \sum_{a=1}^{n} \delta^2(x_a) p_a. \tag{2.12}$$

It coincides with the moment equation for the Hitchin systems on singular curves [14].

We can rewrite (2.12) as

$$\partial_{\bar{w}} L + [\bar{L},L] = 2\pi i \sum_{a=1}^{n} \delta^2(x_a) p_a. \tag{2.13}$$

Anyway, by choosing $\bar{L}$ we fix somehow the gauge in generic case. There is additional gauge freedom h_a in the points x_a, which acts on T^*G_a as $p_a \to p_a$, $g_a \to h_a g_a$. It allows to fix p_a on some coadjoint orbit $p_a = g_a^{-1} p_a^{(0)} g_a$ and obtain the symplectic quotient $\mathcal{O}_a = T^*G_a//G_a$. Thus in (2.11) or (2.12) p_a are elements of $\mathcal{O}_a$.

Let $\mathcal{I}_{g,n}$ be the equivalence classes of the connections $(A, \bar{A})$ with respect to the gauge action (2.10) - the moduli space of stable flat G bundles over $\Sigma_{g,n}$. It is a smooth finite dimensional space. Fixing the conjugacy classes of holonomies $(L, \bar{L})$ around marked points (2.11) amounts to choose a symplectic leave $\mathcal{R}$ in $\mathcal{I}_{g,n}$. Thereby we come to the symplectic quotient

$$\mathcal{R} = \mathcal{R}'//\mathcal{G}_1 = \mathcal{J}_1^{-1}(0)/\mathcal{G}_1 \subset \mathcal{I}_{g,n}.$$

The connections $(L, \bar{L})$ in addition to $\mathbf{p} = (p_1, \ldots, p_n)$ depend on a finite even number of free parameters $2r$ $(\mathbf{v}, \mathbf{u})$, $\mathbf{v} = (v_1, \ldots, v_r)$, $\mathbf{u} = (u_1, \ldots, u_r)$.

$$r = \begin{cases} 0 & g = 0, \\ \mathrm{rank} G, & g = 1 \\ (g-1)\dim G, & g \geq 2. \end{cases}$$

The fibers $\mathcal{R}$ are symplectic manifolds with the nondegenerate symplectic form which is the reduction of (2.5)

$$\omega_0 = \int_\Sigma < \delta L, \delta \bar{L} > + 2\pi i \sum_{a=1}^{n} \delta < p_a, g_a \delta g_a^{-1} > . \tag{2.14}$$

On this stage we come to the bundle $\mathcal{P}''$ with the finite-dimensional fibers $\mathcal{R}$ over the infinite-dimensional base $\mathcal{N}'$ with the symplectic form

$$\omega = \omega_0 - \frac{1}{\kappa} \int_\Sigma < L, \delta L > \delta \mu. \tag{2.15}$$

4. Factorization with respect to the diffeomorphisms $\mathcal{G}_0$. We can utilize invariance of ω with respect to $\mathcal{G}_0$ and reduce $\mathcal{N}'$ to the finite-dimensional space $\mathcal{N}$, which is isomorphic to the moduli space $\mathcal{M}_{g,n}$. The crucial point is that for the flat connections the action of diffeomorphisms $\mathcal{G}_0$ on the connection fields is generated by the gauge transforms $\mathcal{G}_1$. But we already have performed the symplectic reduction with respect to $\mathcal{G}_1$. Therefore, we can push ω (2.15) down on the factor space $\mathcal{P}''/\mathcal{G}_0$. Since $\mathcal{G}_0$ acts on $\mathcal{N}'$ only, it can be done by fixing the dependence of μ on the coordinates in the Teichmüller space $\mathcal{T}_{g,n}$. According to (2.3) represent μ as

$$\mu = \sum_{s=1}^{l} \mu_s. \tag{2.16}$$

The Beltrami differential (2.16) defines the tangent vector $\mathbf{t} = (t_1, \ldots, t_l)$, to the Teichmüller space $\mathcal{T}_{g,n}$ at the fixed point of $\mathcal{T}_{g,n}$.

We specify the dependence of μ on the positions of the marked points in the following way. Let $\mathcal{U}'_a \supset \mathcal{U}_a$ be two vicinities of the marked point x_a such that $\mathcal{U}'_a \cap \mathcal{U}'_b = \emptyset$ for $a \neq b$, and $\chi_a(z, \bar{z})$ is a smooth function

$$\chi_a(z, \bar{z}) = \begin{cases} 1, & z \in \mathcal{U}_a \\ 0, & z \in \Sigma_{g,n} \setminus \mathcal{U}'_a. \end{cases}$$

Introduce times related to the positions of the marked points $t_a = x_a - x_a^0$. Then

$$\mu_a = t_a \bar{\partial} n_a(z,\bar{z}), \quad n_a(z,\bar{z}) = (1 + c_a(z - x_a^0))\chi_a(z,\bar{z}). \tag{2.17}$$

The action of $\mathcal{G}_0$ on the phase space $\mathcal{P}''$ reduces the infinite-dimensional component $\mathcal{N}'$ to $\mathcal{T}_{g,n}$. After the reduction we come to the bundle with base $\mathcal{T}_{g,n}$. The symplectic form (2.15) is transformed as follows

$$\omega = \omega_0(\mathbf{v},\mathbf{u},\mathbf{p},\mathbf{t}) - \frac{1}{\kappa}\sum_{s=1}^{l} \delta H_s(\mathbf{v},\mathbf{u},\mathbf{p},\mathbf{t})\delta t_s, \; H_s = \int_\Sigma < L, L > \partial_s \mu \tag{2.18}$$

where ω_0 is defined by (2.14).

In fact, we still have a redundant discrete symmetry, since ω is invariant under the mapping class group $\pi_0(\mathcal{G}_0)$. Eventually, we come to the moduli space $\mathcal{M}_{g,n} = \mathcal{T}_{g,n}/\pi_0(\mathcal{G}_0)$.

The extended phase space $\mathcal{P}$ is the result of the symplectic reduction with respect to the $\mathcal{G}_1$ action and subsequent factorization under the $\mathcal{G}_0$ action. We can write symbolically $\mathcal{P} = (\mathcal{P}''//\mathcal{G}_1)/\mathcal{G}_0$. It is endowed with the symplectic form (2.18).

5. The hierarchies of the isomonodromic deformations (HID). The equations of motion (HID) can be extracted from the symplectic form (2.18). In terms of the local coordinates they take the form

$$\kappa\partial_s \mathbf{v} = \{H_s, \mathbf{v}\}_{\omega_0}, \quad \kappa\partial_s \mathbf{u} = \{H_s, \mathbf{u}\}_{\omega_0}, \quad \kappa\partial_s \mathbf{p} = \{H_s, \mathbf{p}\}_{\omega_0} \tag{2.19}$$

The Poisson bracket $\{\cdot,\cdot\}_{\omega_0}$ is the inverse tensor to ω_0. We also has the Whitham hierarchy accompanying (2.19)

$$\partial_s H_r - \partial_r H_s + \{H_r, H_s\}_{\omega_0} = 0. \tag{2.20}$$

There exists the one form on $\mathcal{M}_{g,n}$ defining *the tau function* of the hierarchy of isomonodromic deformations

$$\delta \log \tau = \delta^{-1}\omega_0 - \frac{1}{\kappa}\sum H_s dt_s. \tag{2.21}$$

The following three statements are valid for the HID (2.19):

Proposition 2.1 *There exists the consistent system of linear equations*

$$(\kappa\partial + L)\Psi = 0, \tag{2.22}$$

$$(\partial_s + M_s)\Psi = 0, \quad (s = 1,\ldots,l = \dim \mathcal{M}_{g,n}) \tag{2.23}$$

$$(\bar{\partial} + \mu\partial + \bar{L})\Psi = 0 \tag{2.24}$$

where M_s is a solution to the linear equation

$$\partial_{\bar{w}} M_s - [M_s, \bar{L}] = \partial_s \bar{L} - \frac{1}{\kappa} L \partial_s \mu. \tag{2.25}$$

Proposition 2.2 *The linear conditions (2.23) provide the isomonodromic deformations of the linear system (2.22), (2.24) with respect to change the "times" on $\mathcal{M}_{g,n}$.*

Therefore, the HID (2.19) are the monodromy preserving conditions for the linear system (2.22),(2.24).

The presence of derivative with respect to the spectral parameter $w \in \Sigma_{g,n}$ in the linear equation (2.22) is a distinguish feature of the monodromy preserving equations. It plagues the application of the inverse scattering method to these types of systems. Nevertheless, in our case we have in some sense the explicit form of solutions:

Proposition 2.3 (The projection method.) *The solution of the Cauchy problem of (2.19) for the initial data $\mathbf{v}^0, \mathbf{u}^0, \mathbf{p}^0$ at the time $\mathbf{t} = \mathbf{t}^0$ is defined in terms of the elements $L^0, \bar{L}^0$ as the gauge transform*

$$\bar{L}(\mathbf{t}) = f^{-1}(L^0(\mu(\mathbf{t}) - \mu(\mathbf{t}^0)) + (\bar{L}^0))f + f^{-1}(\bar{\partial} + \mu(\mathbf{t})\partial)f, \tag{2.26}$$

$$L(\mathbf{t}) = f^{-1}(\partial + L^0)f, \quad \mathbf{p}(\mathbf{t}) = f^{-1}(\mathbf{p}^0)f, \tag{2.27}$$

where f is a smooth G-valued functions on $\Sigma_{g,n}$ fixing the gauge.

3 Relations to the Hitchin systems and the KZB equations.

1. Scaling limit. Consider the HID in the limit $\kappa \to 0$. We will prove that in this limit we come to the Hitchin systems, which are living on the cotangent bundles to the moduli space of holomorphic G-bundles over $\Sigma_{g,n}$ [13]. The critical value $\kappa = 0$ looks singular (see (2.5), (2.18)). To get around we rescale the times $\mathbf{t} = \kappa \mathbf{t}^H$, where $\mathbf{t}^H$ are the "Hitchin times". Therefore, $\delta\mu(\mathbf{t}) = \kappa \sum_s \partial_s \mu(\mathbf{t}^0)\delta t^H$. After this rescaling the forms (2.5),(2.18) become regular in the critical limit. The rescaling procedure means that we blow up a vicinity of the fixed point corresponding to ($\mu = 0$), and the whole dynamic of the Hitchin systems is developed in this vicinity. [2] Denote $\partial_s \mu_o = \partial_s \mu(\mathbf{t})|_{\mathbf{t}=\mathbf{t}^0}$ Then we have instead of (2.5)

$$\omega = \int_\Sigma < \delta A, \delta \bar{A} > +2\pi i \sum_{a=1}^{n} \delta < p_a, g_a^{-1}\delta g_a > - \sum_s \int_\Sigma < \delta A, A > \partial_s \mu(\mathbf{t}^0)\delta t^H. \tag{3.1}$$

If $\kappa = 0$ the connection A behaves as the one-form $A \in \mathcal{A}^{(1,0)}(\Sigma_{g,n}, \mathrm{Lie}(G))$ (see (2.8)). It is so called the Higgs field . An important point is that the Hamiltonians now become the times independent. The form (3.1) is the starting point in the derivation of the Hitchin systems via the symplectic reduction [13, 14]. Essentially, it is the same procedure as described above. Namely, we obtain the same moment constraint (2.12) and the same gauge fixing (2.10). But now we are sitting in a fixed point $\mu(\mathbf{t}^0) = 0$ of the moduli space $\mathcal{M}_{g,n}$ and don't need the factorization under the action of the diffeomorphisms. This only difference between the solutions L and $\bar{L}$ in the Hitchin systems and the hierarchies of isomonodromic deformations.

Propositions 2.1, 2.3 are valid for the Hitchin systems in a slightly modified form.

[2] We are grateful to A.Losev for elucidating this point.

Proposition 3.1 *There exists the consistent system of linear equations*

$$(\lambda + L)\Psi = 0, \quad \lambda \in \mathbf{C} \tag{3.2}$$

$$(\partial_s + M_s)\Psi = 0, \quad \partial_s = \partial_{t_s^H}, \ (s = 1, \ldots, l = \dim \mathcal{M}_{g,n}) \tag{3.3}$$

$$(\bar{\partial} + \bar{L})\Psi = 0, \ \bar{\partial} = \partial_{\bar{z}}, \tag{3.4}$$

where M_s is a solution to the linear equation

$$\bar{\partial} M_s - [M_s, \bar{L}] = \partial_s \bar{L} - L\partial_s \mu(\mathbf{t}^0). \tag{3.5}$$

The parameter λ in (3.2) can be considered as the symbol of ∂ (compare with (2.22)).

When L and M can be find explicitly the simplified form of (2.12) allows to apply "the inverse scattering method" to find solutions of the Hitchin hierarchy as it was done for $\mathrm{SL}(N, \mathbf{C})$ holomorphic bundles over $\Sigma_{1,1}$ [23], corresponding to the elliptic Calogero system with spins. We present the alternative way to describe the solutions:

Proposition 3.2 (The projection method.)

$$\bar{L}(t_s) = f^{-1}(L^0 \partial_s \mu_o (t_s - t_s^0) + \bar{L}^0) f + f^{-1} \bar{\partial} f,$$

$$L(t_s) = f^{-1} L^0 f, \quad p_a(t_s) = f^{-1}(p_a^0) f$$

The degenerate version of these expressions was known for a long time [24].

2. About KZB. The Hitchin systems are the classical limit of the KZB equations on the critical level [14, 17]. The later has the form of the Schrödinger equations, which is the result of geometric quantization of the moduli of flat G bundles [16, 18]. The conformal blocks of the WZW theory on $\Sigma_{g,n}$ with vertex operators in marked points are ground state wave functions

$$\hat{H}_s F = 0, \quad (s = 1, \ldots, l = \dim \mathcal{M}_{g,n}).$$

The classical limit means that one replaces operators on their symbols and finite-dimensional representations in the vertex operators by the corresponding coadjoint orbits. The level κ plays the role of the Planck constant, but in contrast with the limit considerd before, we don't adjust the moduli of complex structures.

Generically, for $\kappa \neq 0$ the KZB equations can be written in the form of the nonstationar Schrödinger equations

$$(\kappa \partial_s + \hat{H}_s) F = 0, \quad (s = 1, \ldots, l = \dim \mathcal{M}_{g,n}).$$

The flatness of this connection (see [18]) is the quantum counterpart of the Whitham equations (2.20). The classical limit in the described above sense leads to the HID. Summarizing, we arrange these quantum and classical systems in the diagram. The vertical arrows denote to the classical limit, while the limit $\kappa \to 0$ on the horizontal arrows includes also the rescaling of the moduli of complex structures. The examples in the bottom of the diagram will be considered in next sections.

$$\left\{\begin{array}{c}\text{KZB eqs.,}\ (\kappa,\mathcal{M}_{g,n},G)\\ (\kappa\partial_{t_a}+\hat{H}_a)F=0,\\ (a=1,\ldots,\dim\mathcal{M}_{g,n})\end{array}\right\} \xrightarrow{\kappa\to 0,\ \mathbf{t}=\kappa\mathbf{t}^H} \left\{\begin{array}{c}\text{KZB eqs. on the critical level,}\\ (\mathcal{M}_{g,n},G),\ (\hat{H}_a)F=0,\\ (a=1,\ldots,\dim\mathcal{M}_{g,n})\end{array}\right\}$$

$$\downarrow{\scriptstyle \kappa\to 0} \qquad\qquad\qquad\qquad \downarrow{\scriptstyle \kappa\to 0}$$

$$\left\{\begin{array}{c}\text{Hierarchies of Isomonodromic}\\ \text{deformations on } \mathcal{M}_{g,n}\end{array}\right\} \xrightarrow{\kappa=0,\ \mathbf{t}=\kappa\mathbf{t}^H} \left\{\begin{array}{c}\text{Hitchin systems}\\ \end{array}\right\}$$

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$$\left\{\begin{array}{c}\text{Schlesinger eqs.}\\ \text{Painlevé type eqs.}\\ \text{Elliptic Schlesinger eqs.}\end{array}\right\} \xrightarrow{\kappa\to 0,\ \mathbf{t}=\kappa\mathbf{t}^H} \left\{\begin{array}{c}\text{Classical Gaudin eqs.}\\ \text{Calogero eqs.}\\ \text{Elliptic Gaudin eqs.}\end{array}\right\}$$

4 Genus zero - Schlesinger's equation.

Consider $\mathbf{C}P^1$ with n punctures $(x_1,\ldots,x_n|x_a\neq x_b)$. The Beltrami differential μ is related only to the positions of marked points (2.17). On $\mathbf{C}P^1$ the gauge transform (2.10) allows to choose $\bar{L}$ to be identically zero. Let $A=f(L+\kappa\partial_w)f^{-1}$. Then the moment equation takes the form

$$(\bar{\partial}+\partial\mu)L=2\pi i\sum_{a=1}^{n}\delta^2(x_a)p_a. \tag{4.1}$$

It allows to find L

$$L=\sum_{a=1}^{n}\frac{p_a}{w-x_a}. \tag{4.2}$$

On the symplectic quotient ω (2.18) takes the form

$$\omega=\delta\sum_{a=1}^{n}<p_a g_a^{-1}\delta g_a>-\frac{1}{\kappa}\sum_{b=1}^{n}(\delta H_{b,1}+\delta H_{b,0})\delta x_b.$$

$$H_{a,1}=\sum_{b\neq a}\frac{<p_a,p_b>}{x_a-x_b},\quad H_{2,a}=c_a<p_a,p_a>. \tag{4.3}$$

$H_{1,a}$ are precisely the Schlesinger's Hamiltonians. Note, that we still have a gauge freedom with respect to the G action. The corresponding moment constraint means that the sum of residues of L vanishes:

$$\sum_{a=1}^{n}p_a=0. \tag{4.4}$$

While $H_{2,a}$ are Casimirs and lead to trivial equations, the equation of motion for $H_{1,a}$ are the Schlesinger equations

$$\kappa\partial_b p_a=\frac{[p_a,p_b]}{x_a-x_b},\ (a\neq b),\quad \kappa\partial_a p_a=-\sum_{b\neq a}\frac{[p_a,p_b]}{x_a-x_b}.$$

As by product, we obtain by this procedure the corresponding linear problem (2.22),(2.23) with L (4.2) and

$$M_{a,1} = -\frac{p_a}{w - x_a}$$

as a solution to (2.25). The tau-function for the Schlesinger equations has the form [22]

$$\delta \log \tau = \sum_{c \neq b} < p_b, p_c > \delta \log(x_c - x_b).$$

5 Genus one - elliptic Schlesinger, Painleve VI...

1. Deformations of elliptic curves. In addition to the moduli coming from the positions of the marked points there is an elliptic module τ, $Im\tau > 0$ on $\Sigma_{1,n}$. As in (2.16),(2.17) we take the Beltrami differential in the form $\mu = \sum_{a=1}^{n} \mu_a + \mu_\tau$, $(\mu_s = t_s \bar{\partial} n_s)$, where $n_a(z, \bar{z})$ is the same as in (2.17) and

$$n_\tau = (\bar{z} - z)(1 - \sum_{a=1}^{n} \chi_a(z, \bar{z})). \tag{5.1}$$

Then

$$\mu_\tau = \tilde{\mu}_\tau (1 - \sum_{a=1}^{n} \chi_a(z, \bar{z})), \quad (\tilde{\mu}_\tau = \frac{t_\tau}{\tau - \tau_0}, \ t_\tau = \tau - \tau_0) \tag{5.2}$$

Here τ_0 defines the reference comlex structure on the curve

$$T_0^2 = \{0 < x \leq 1, \ 0 < y \leq 1, \ z = x + \tau_0 y, \ \bar{z} = x + \bar{\tau}_0 y\}.$$

2. Flat bundles on a family of elliptic curves. Note first, that $\bar{A}$ can be considered as a connection of holomorphic G bundle $\mathcal{E}$ over T_τ^2. For stable bundles $\bar{A}$ can be gauge transformed by (2.10) to the Cartan z-independent form
$\bar{A} = f(\bar{L} + \bar{\partial} + \mu\partial)f^{-1}$, $\bar{L} \in \mathcal{H}$- Cartan subalgebra of Lie(G). Therefore, stable bundle $\mathcal{E}$ is decoposed into the direct sum of line bundles $\mathcal{E} = \oplus_{k=1}^{r} \mathcal{L}_k$, $r = \text{rank}G$. The set of gauge equivalent connections represented by $\{\bar{L}\}$ can be identified with the r power of the Jacobian of T_τ^2, factorized by the action of the Weyl group W of G. Put

$$\bar{L} = 2\pi i \frac{1 - \tilde{\mu}_\tau}{\rho} \mathbf{u}, \quad \mathbf{u} \in \mathcal{H}, \quad (\rho = \tau_0 - \bar{\tau}_0). \tag{5.3}$$

The moment constraints (2.13) leading to the flatness condition take the form

$$\partial_{\bar{w}} L + [\bar{L}, L] = 2\pi i \sum_{a=1}^{n} \delta^2(x_a,)p_a. \tag{5.4}$$

Let $R = \{\alpha\}$ be the root system of of Lie(G) $= \mathcal{G}$ and $\mathcal{G} = \mathcal{H} \oplus_{\alpha \in R} \mathcal{G}_\alpha$ be the root decomposition. Impose the vanishing of the residues in (5.4)

$$\sum_{a=1}^{n} (p_a)_{\mathcal{H}} = 0, \tag{5.5}$$

where $p_a|_{\mathcal{H}}$ is the Cartan component and we have identified $\mathcal{G}$ with its dual space $\mathcal{G}^*$.

We will parametrized the set of its solutions by two elements $\mathbf{v}, \mathbf{u} \in \mathcal{H}$. Define the solutions L to the moment equation (5.4), which is double periodic on the deformed curve T^2_τ. Let $E_1(w)$ be the Eisenstein function

$$E_1(z|\tau) = \partial_z \log \theta(z|\tau),$$

where

$$\theta(z|\tau) = q^{\frac{1}{8}} \sum_{n\in\mathbf{Z}} (-1)^n e^{\pi i(n(n+1)\tau+2nz)}.$$

It is connected with the Weirstrass zeta-function as

$$\zeta(z|\tau) = E_1(z|\tau) + 2\eta_1(\tau)z, \quad (\eta_1(\tau) = \zeta(\frac{1}{2})).$$

Another function we need is

$$\phi(u,z) = \frac{\theta(u+z)\theta'(0)}{\theta(u)\theta(z)} = \exp(-2\eta_1 uz)\frac{\sigma(u+z)}{\sigma(y)\sigma(z)},$$

where $\sigma(z)$ is the Weierstrass sigma function.

Lemma 5.1 *The solutions of the moment constraint equation have the form*

$$L = P + X, \quad P \in \mathcal{H}, \quad X = \sum_{\alpha\in R} X_\alpha. \tag{5.6}$$

$$P = 2\pi i(\frac{\mathbf{v}}{1-\tilde{\mu}_\tau} - \kappa\frac{\mathbf{u}}{\rho} + \sum_{a=1}^{n}(p_a)_{\mathcal{H}} E_1(w-x_a)), \tag{5.7}$$

$$X_\alpha = frac2\pi i1 - \tilde{\mu}_\tau \sum_{a=1}^{n}(p_a)_\alpha \exp 2\pi i\{\frac{(w-x_a)-(\bar{w}-\bar{x}_a)}{\tau-\bar{\tau}_0}\alpha(u)\}\phi(\alpha(u), w-x_a). \tag{5.8}$$

2. Symmetries. The remnant gauge transforms preserve the chosen Cartan subalebra $\mathcal{H} \subset G$. These transformations are generated by the Weyl subgroup W of G and elements $f(w,\bar{w}) \in \mathrm{Map}(T^2_\tau, \mathrm{Cartan}(G))$. Let Π be the system of simple roots, $R^\vee = \{\alpha^\vee = \frac{2\alpha}{(\alpha|\alpha)}\}$, is the dual root system, and $\mathbf{m} = \sum_{\alpha\in\Pi} m_\alpha \alpha^\vee$ be the element from the dual root lattice $\mathbf{Z}R^\vee$. Then the Cartan valued harmonics

$$f_{\mathbf{m},\mathbf{n}} = \exp 2\pi i(\mathbf{m}\frac{w-\bar{w}}{\tau-\tau_0} + \mathbf{n}\frac{\tau\bar{w}-\bar{\tau}_0 w}{\tau-\tau_0}), \quad (\mathbf{m},\mathbf{n} \in R^\vee) \tag{5.9}$$

generate the basis in the space of Cartan gauge transformations. In terms of the variables $\mathbf{v}$ and $\mathbf{u}$ they act as

$$\mathbf{u} \to \mathbf{u} + \mathbf{m} - \mathbf{n}\tau, \quad \mathbf{v} \to \mathbf{v} - \kappa\mathbf{n}, \quad (p_a)_\alpha \to \varphi(m_\alpha, n_\alpha)(p_a)_\alpha. \tag{5.10}$$

Here $\varphi(m_\alpha, n_\alpha) = \exp\frac{4\pi i}{\rho}[(m_\alpha - n_\alpha\bar{\tau}_0 x_a^0) - (m_\alpha - n_\alpha\tau_0\bar{x}_a^0)]$.

The whole discrete gauge symmetry is the semidirect product $\hat{W}$ of the Weyl group W and the lattice $\mathbf{Z}R^\vee \oplus \tau\mathbf{Z}R^\vee$. It is the Bernstein-Schvartsmann complex crystallographic group. The factor space $\mathcal{H}/\hat{W}$ is the genuin space for the "coordinates" $\mathbf{u}$.

According with (2.8) the transformations (5.9) act also on $p_a \in \mathcal{O}_a$. This action leads to the symplectic quotient $\mathcal{O}_a//H$ and generates the moment equation (5.5).

The modular group $\mathrm{PSL}_2(\mathbf{Z})$ is a subgroup of mapping class group for the Teichmüller space $\mathcal{T}_{1,n}$. Its action on τ is the Möbius transform. We summarise the action of symmetries on the dynamical variables:

	W={s}	$\mathbf{Z}R^\vee \oplus \tau\mathbf{Z}R^\vee$	$\mathrm{PSL}_2(\mathbf{Z})$
$\mathbf{v}$	$s\mathbf{v}$	$\mathbf{v} - \kappa\mathbf{n}$	$\mathbf{v}(c\tau + d) - \kappa c\mathbf{u}$
$\mathbf{u}$	$s\mathbf{u}$	$\mathbf{u} + \mathbf{m} - \mathbf{n}\tau$	$\mathbf{u}(c\tau + d)^{-1}$
$(p_a)_\mathcal{H}$	$s(p_a)_\mathcal{H}$	$(p_a)_\mathcal{H}$	$(p_a)_\mathcal{H}$
$(p_a)_\alpha$	$(p_a)_{s\alpha}$	$\varphi(m_\alpha, n_\alpha)(p_a)_\alpha$	$(p_a)_\alpha$
τ	τ	τ	$\frac{a\tau+b}{c\tau+d}$
x_a	x_a	x_a	$\frac{x_a}{c\tau+d}$

3. Symplectic form. The set $(\mathbf{v}, \mathbf{u} \in \mathcal{H}, \mathbf{p} = (p_1, \ldots, p_n))$ of dynamical variables along with the times $\mathbf{t} = (t_\tau, t_1, \ldots, t_n)$ describe the local coordinates in the bundle $\mathcal{P}$. According with the general prescription, we can define the hamiltonian system on this set.

The main statement, formulated in terms of the theta-functions and the Eisenstein functions

$$E_2(z|\tau) = -\partial_z E_1(z|\tau) = \partial_z^2 \log\theta(z|\tau) = \wp(z|\tau) + 2\eta_1(\tau). \tag{5.11}$$

It takes the form

Proposition 5.1 *The symplectic form ω (2.18) on $\mathcal{P}$ is*

$$\frac{1}{4\pi^2}\omega = (\delta\mathbf{v}, \delta\mathbf{u}) + \sum_{a=1}^n \delta < p_a, g_a^{-1}\delta g_a > -\frac{1}{\kappa}(\sum_{a=1}^n \delta H_{2,a} + \delta H_{1,a})\delta t_a - \frac{1}{\kappa}\delta H_\tau \delta\tau, \tag{5.12}$$

with the Hamiltonians

$$H_{2,a} = c_a < p_a, p_a >;$$

$$H_{1,a} == 2(\frac{\mathbf{v}}{1-\tilde{\mu}_\tau} - \kappa\frac{\mathbf{u}}{\rho}, p_a|_\mathcal{H}) + \sum_{b\neq a}(p_a|_\mathcal{H}, p_b|_\mathcal{H})E_1(x_a - x_b) +$$

$$\sum_{b\neq a}\sum_\alpha (p_a|_\alpha, p_b|_{-\alpha})\frac{\theta(-\alpha(\mathbf{u}) + x_a - x_b)\theta'(0)}{\theta(\alpha(\mathbf{u}))\theta(x_a - x_b)};$$

$$H_\tau =$$

$$\frac{(\mathbf{v}, \mathbf{v})}{2} + \{\sum_{a=1}^n \sum_\alpha (p_a|_\alpha, p_a|_{-\alpha})E_2(\alpha(\mathbf{u})) + \sum_{a\neq b}^n (p_a|_\mathcal{H}, p_b|_\mathcal{H})(E_2(x_a - x_b) - E_1^2(x_a - x_b)) +$$

$$\sum_{a\neq b}\sum_\alpha (p_a|_\alpha, p_b|_{-\alpha})\frac{\theta(-\alpha(\mathbf{u}) + x_a - x_b)\theta'(0)}{\theta(\alpha(\mathbf{u}))\theta(x_a - x_b)}(E_1(\alpha(\mathbf{u}) - E_1(x_b - x_a + \alpha(\mathbf{u})) - E_1(x_b - x_a)).$$

Example 1. Consider SL(2, **C**) bundles over the family of $\Sigma_{1,1}$. Then (5.3) takes the form

$$\bar{L} = 2\pi i\frac{1-\tilde{\mu}_\tau}{\rho}\mathrm{diag}(u,-u). \tag{5.13}$$

In this case the position of the maked point is no long the module and we put $x_1 = 0$. Since $\dim \mathcal{O} = 2$ the orbit degrees of freedom can be gauged away by the hamiltonian action of the diagonal group. We assume that $p = \nu[(1,1)^T \otimes (1,1) - Id]$. Then we have from(5.6),(5.7),(5.8)

$$L = 2\pi i \begin{pmatrix} \frac{v}{1-\tilde{\mu}_\tau} - \kappa\frac{u}{\rho} & x(u,w,\bar{w}) \\ x(-u,w,\bar{w}) & -\frac{v}{1-\tilde{\mu}_\tau} + \kappa\frac{u}{\rho} \end{pmatrix}. \tag{5.14}$$

$$x(u,w,\bar{w}) = \frac{\nu}{1-\tilde{\mu}_\tau}\exp 4\pi i\{(w-\bar{w})u\frac{1-\tilde{\mu}_\tau}{\rho}\}\phi(2u,w).$$

The symplectic form (5.12)

$$\frac{1}{4\pi^2}\omega = (\delta v, \delta u) - \frac{1}{\kappa}\delta H_\tau \delta\tau,$$

and

$$H_\tau = v^2 + U(u|\tau),\ U(u|\tau) = -\nu^2 E_2(2u|\tau).$$

It leads to the equation of motion

$$\frac{\partial^2 u}{\partial \tau^2} = \frac{2\nu^2}{\kappa^2}\frac{\partial}{\partial u}E_2(2u|\tau). \tag{5.15}$$

In fact, due to (5.11) we can use $\wp(2u|\tau)$ instead of $E_2(2u|\tau)$ and after rescaling the coupling constant come to (1.3) for special values of constants as in (1.7). The equation (5.15) is the isomonodromic deformation conditions for the linear system (2.22),(2.24) with L (5.14) and $\bar{L}$ (5.13). The lax pair is given by L (5.14) and M_τ

$$M_\tau = \begin{pmatrix} 0 & y(u,w,\bar{w}) \\ y(-u,w,\bar{w}) & 0 \end{pmatrix},$$

where $y(u,w,\bar{w})$ is defined as the convolution integral on T^2_τ

$$y(u,w,\bar{w}) = -\frac{1}{\kappa}x * x(u,w,\bar{w}).$$

The projection method determines solutions of (5.15) as a result of diagonalization of L (5.14) by the gauge transform on the deformed curve T^2_τ. On the critical level ($\kappa = 0$) we come to the two-body elliptic Calogero system.

Example 2. For flat G bundles over $\Sigma_{1,1}$ we obtain Painlevé type equations, related to arbitrary root system. They are described by the system of differential equations for the $\mathbf{u} = (u_1, \ldots, u_r)$, ($r$ =rank$G$) variables. In addition there are the orbit variables $p \in \mathcal{O}(G)$ satisfying the Euler top equations. For SL(N, **C**) bundles the most degenerate orbits $\mathcal{O} = T^*\mathbf{C}P^{N-1}$ has dimension $2N-2$. These variables are gauge away by the diagonal gauge transforms as in the previous example. On the critical level this Painlevé type system degenerates into N-body elliptic Calogero system. For generic orbits we obtain the generalized Calogero-Euler systems.

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