Robust FDI filters and fault sensitivity analysis
in continuous-time descriptor systems

M. L. Corradini, A. Cristofaro, S. Pettinari

Abstract—This paper addresses the robust (in the disturbance de-coupling sense) design problem of fault detection and isolation observers for uncertain continuous-time linear descriptor systems. In the considered set-up, the only available signal is the output variable. A bank of disturbance-decoupled filters are designed to distinguish actuator from sensor faults. A fault sensitivity analysis is discussed for three different classes of faults affecting both the actuators and the sensors.

Index Terms—Robust Fault Detection Filters, Fault isolation, Descriptor Systems, Fault sensitivity.

I. INTRODUCTION

Descriptor systems have been widely employed to model practical plants such as circuits, power systems and networks; during the last years the theory of singular systems has been extensively developed and several control problems have been addressed (see for example [5], [6], [8], [13], [17], [18], [19], [20]). The impulsive behavior due to the presence of infinite poles makes descriptor systems a natural tool for the modeling of phenomena having both spatially and temporally varying behavior.

In order to improve reliability of practical control systems it is crucial to detect fault and isolation techniques for dynamic systems. During the past several decades, a great amount of results were reported particularly for state-space dynamic systems, e.g., see [2], [3], [9], and [12]. The basic idea behind the observer-based FDI approaches is to generate diagnosis signals, called residuals, by using observers based only on the outputs which are available for measurements. The problem of designing observers and detection filters for descriptor systems have been studied by several authors ([1], [11], [15], [16], [21] among many others). As for the case of non-singular plants ([3], [4], [10], [14]), the design of fault detection filters able to decouple disturbances affecting the plant with respect to eventual faults occurring in the system is a major issue.

Referring to the observer structure proposed by [21] in which unperturbed descriptor systems are considered, the author presented some methods for the design of robust fault detection observers for uncertain continuous-time linear descriptor systems in [7]. This paper addresses a fault isolation strategy for continuous-time linear descriptor systems using some of the techniques discussed in [7]. Adapting a classical strategy for non singular systems (see for instance [3]), a bank of independent observers are employed to distinguish actuator faults from sensor faults. In addition a fault sensitivity analysis provides structural conditions on the continuous-time system and the observers parameters which guarantee the detectability of three different classes of faults both in the actuators and in the sensors. The paper is organized as follows: in Section II the model of the system is presented, some technical assumptions are stated and the observer is introduced. The observer properties are analyzed in Section III, underlining conditions that ensure the estimation error to be completely disturbance-decoupled; the fault detection approach reported in [7] is briefly summarized and the fault isolation technique is presented as well. Some algebraic conditions for asymptotic fault detectability are discussed in Section IV both for actuator and sensor faults. Finally in Section V some numerical simulations support the theoretical results and illustrate the efficiency of the proposed methods.

II. PROBLEM STATEMENT

Let us consider the uncertain single input continuous-time linear descriptor system

\[
\begin{align*}
E \dot{x}(t) &= Ax(t) + Bu(t) + B_d d(t) + F \eta(t) \\
y(t) &= Cx(t) + G \xi(t)
\end{align*}
\]

where \( x \in \mathbb{R}^n \), \( u, d \in \mathbb{R} \), \( \eta \in \mathbb{R} \), \( y \in \mathbb{R}^p \), \( \xi \in \mathbb{R} \), the matrices \( E, A, B_u, B_d, F, C, G \) have appropriate dimension, \( \text{rank}(C) = p \), \( \text{rank}(B_d) = m \leq p \) and \( \text{rank}(E) = r \) with \( n - p \leq r < n \).

Assumption 1. The state \( x \) is not available for direct measurement and only the output variable is used for control and/or detection purposes.

We recall that the singular system \((E, A)\) is called regular if there exists \( s \in \mathbb{C} \) with

\[
\det(sE - A) \neq 0.
\]

Assumption 2. The descriptor system (1) is regular.

Assumption 3. The matrices \( E, C \) satisfy

\[
\text{rank}[E^T C^T] = n.
\]

The perturbation term \( d(\cdot) \) represents plant uncertainties, while the unknown terms \( \eta(\cdot), \xi(\cdot) \), which are supposed to have known input matrices \( F \in \mathbb{R}^{n \times h_a} \) and \( G \in \mathbb{R}^{p \times h_s} \), represents faults affecting the actuators and the sensors of the system respectively.

Assumption 4. The disturbance term and the actuator
faults are supposed to enter the system with different distribution matrices, i.e.
\[ \text{rank}[B_{d} F] > \max\{m,h\} \]
Following the approach described in [21] let us consider a pair of matrices \( R \in \mathbb{R}^{n \times n}, V \in \mathbb{R}^{n \times p} \) satisfying the following identity:
\[ [R \ V] \cdot \begin{bmatrix} E \\ C \end{bmatrix} = RE + VC = I_{n \times n}. \] (3)
Define the following full-order observer:
\[
\begin{align*}
\dot{z}(t) &= Nz(t) + Ly(t) + Hu(t) \\
\dot{x}(t) &= V y(t) - z(t)
\end{align*}
\] (4)
with
\[
\begin{align*}
N &= RA + KC \\
L &= K - NV \\
H &= -RB_{a}
\end{align*}
\]
where the feedback gain \( K \) is chosen in order to stabilize the matrix \( N \).

**Remark 2.1:** We point out that the eigenvalues of the matrix \( N \) can be properly assigned by the feedback gain \( K \) if and only if \((C,RA)\) is an observable pair.
The evolution of the estimation error \( e(t) = x(t) - \hat{x}(t) \) is given by
\[
\dot{e}(t) = N e(t) + R B_{d} d(t) + R F \eta(t) + K G \xi(t) - V G \dot{\xi}(t); \] (5)
which depends on the state \( x \) that is not measurable and for this reason we are lead to consider the output error:
\[ e_{y}(t) = C e(t) + G \xi(t). \] (6)

### III. Observer design and residual generation

In this section some issues regarding the observer design are discussed. Only the critical case \( n = p + r \) is presented, being the one which involves the most restrictive conditions; on the other hand, the general framework \( n < p + r \) can be easily reduced to the critical case by neglecting a suitable subset of the output space.

Let us prove the following technical result:

**Proposition 3.1:** If \( n = p + r \) and the matrices \( R, V \) satisfy the identity (3), then the following conditions hold:
\[
\begin{align*}
CV &= I_{p \times p} \\
CRE &= 0
\end{align*}
\] (7) (8)

**Proof:** Let us suppose for sake of simplicity that the first \( r \) columns of the matrix \( E \) are linearly independent:
\[ r = \text{rank}(E) = \text{rank}[E_{1} \ldots E_{n}] = \text{rank}[E_{1} \ldots E_{r}] \] (9)
Setting \( C = [C_{1} \ldots C_{n}] \) and denoting by \( e_{1}, \ldots, e_{n} \) the (column) vectors of the canonical basis of \( \mathbb{R}^{n} \), identity (3) can be rewritten as
\[ RE_{j} + VC_{j} = e_{j} \quad \forall \quad j = 1, \ldots, n \]
With a left multiplication by the matrix \( C \) one gets
\[ CRE_{j} + (CV)C_{j} = C e_{j}, \quad j = 1, \ldots, n \]
Observing that \( C e_{j} = C_{j} \), the above equality can be reformulated as follows
\[ CRE_{j} = (I_{p \times p} - CV)C_{j}, \quad j = 1, \ldots, n \] (10)
Since \( \text{rank}(E) = r \) and due to assumption (9), there exist a family of real coefficients \( \lambda_{j}^{(k)} \) such that
\[ E_{k} = \sum_{j=1}^{r} \lambda_{j}^{(k)} E_{j}, \quad k = r + 1, \ldots, n. \] (11)
Using the above expressions in (10), we get for \( k \geq r + 1 \)
\[ CRE_{k} = CR \sum_{j=1}^{r} \lambda_{j}^{(k)} E_{j} = (I - CV) C_{k} = (I - CV) \sum_{j=1}^{r} \lambda_{j}^{(k)} C_{j} \]
and therefore
\[ (I - CV) \left( C_{k} - \sum_{j=1}^{r} \lambda_{j}^{(k)} C_{j} \right) = 0, \quad k = r + 1, \ldots, n. \] (12)
Now, if \((I - CV) \neq 0\), that is \( 1 \leq \text{rank}(I - CV) \leq p \), the \( p \) equations given in (12) must be linearly dependent. Assume for simplicity that
\[
C_{n} - \sum_{j=1}^{r} \lambda_{j}^{(n)} C_{j} = \sum_{\ell=1}^{p-1} \mu_{\ell} \left( C_{n-\ell} - \sum_{j=1}^{r} \lambda_{j}^{(n-\ell)} C_{j} \right)
\]
for some coefficients \( \mu_{\ell} \in \mathbb{R} \). Inserting the above expression for \( C_{n} \) into the matrix \([E^{T} C^{T}]^{T}\) and observing that by (11), for any choice of the coefficients \( \mu_{\ell} \in \mathbb{R} \),
\[
f(E_{1}, \ldots, E_{n-1}) := \sum_{\ell=1}^{p-1} \mu_{\ell} \left( E_{n-\ell} - \sum_{j=1}^{r} \lambda_{j}^{(n-\ell)} E_{j} \right) = 0,
\]
we have
\[
\begin{bmatrix} E \\ C \end{bmatrix} = \begin{bmatrix} E_{1} \cdots E_{n-1} & f(E_{1}, \ldots, E_{n-1}) + \sum_{j=1}^{r} \lambda_{j}^{(n)} E_{j} \\ C_{1} \cdots C_{n-1} & f(C_{1}, \ldots, C_{n-1}) + \sum_{j=1}^{r} \lambda_{j}^{(n)} C_{j} \end{bmatrix}
\]
and consequently \( \text{rank} \begin{bmatrix} E \\ C \end{bmatrix} \leq n - 1 \).
The latter condition contradicts the identity (3); in conclusion one must have \((I_{p \times p} - CV) = 0\) and this condition together with equation (10) implies also \( CRE = 0 \).

**A. Disturbance decoupling**

The aim of this subsection is to design the observer such that \( RB_{d} = 0 \) in order to ensure that the estimation error is completely independent of the disturbance term. This requires \( m < p \); by contradiction, let us suppose that \( m = p \) with \( RB_{d} = 0 \). This means that
\[ \dim(\ker(R)) \geq p = n - r. \]
On the other hand, from (8), we have \( \ker(C) \subset \text{Im}(R) \) with \( \dim(\text{Im}(R)) \geq \text{rank}(E) = r = n - p \). We can deduce that
\[ \ker(C) = \text{Im}(R) \perp \ker(R^{T}) = \text{Im}(C^{T}) \]
and as a consequence
\[ R^{T} C^{T} = (CR)^{T} = 0. \] (13)
If the above condition is verified then the eigenvalues of \( N = RA + KC \) cannot be arbitrarily assigned since the pair \((C, RA)\) is not observable. Conversely if \( m < p \), the observer parameters can always be chosen such that
\[
RB_d = 0. \tag{14}
\]
It is worth to note that, in order to ensure the compatibility of the above condition with identity (3), one must require that
\[
Im(B_d) \cap Im(E) = 0.
\]
To complete the design of the observer it remains only to check the observability of the pair \((C, RA)\) in order to ensure the existence of a stabilizing gain \( K \) for the matrix \( N \). To this purpose we point out that a necessary condition for the
observability of \((C, RA)\)

\[
Im(A) \neq [Im(B_d) + Im(E)],
\]
where \( Im(A) \neq Im(E) \) by definition, since the system has been supposed to be regular. A sufficient condition for observability is given in [7].

\section*{B. Fault detection}

Let us consider the output error \( e_y(t) \): assuming that the observer is disturbance-decoupled, i.e. \( RB_d = 0 \), the evolution is given by the equality
\[
e_y(t) = C \begin{bmatrix} e^N(t)(x(0) - \hat{x}(0)) + \int_{t_0}^t e^N(t-s) R F \eta(s) ds \end{bmatrix} + C \begin{bmatrix} \int_{t_0}^t e^N(t-s)(K G \xi(s) - V G \xi(s)) ds \end{bmatrix} + G \xi(t).
\]
Assuming to know a reference initial condition \( x_0 \) and a bound parameter \( \epsilon_0 > 0 \) such that \(|x(0) - x_0| \leq \epsilon_0\), in the absence of faults (i.e. \( \eta(t) \equiv \xi(t) \equiv 0 \)) the following decay estimate holds for the norm of \( e_y(t) \):
\[
\|e_y(t)\| \leq \epsilon_0 \|C\| \cdot \|e^N\| =: \sigma(t). \tag{15}
\]
Let us recall that, being \( N \) a Hurwitz matrix by construction, one has
\[
\lim_{t \to \infty} \sigma(t) = 0.
\]
A sufficient condition to detect the presence of a fault in the system can be detected is to check when the inequality
\[
\|e_y(t)\| \leq \sigma(t)
\]
is violated. More precisely one can define the residual signal
\[
r(t) := (\|e_y(t)\| - \sigma(t))^+ \tag{16},
\]
where \((\cdot)^+\) stands for the positive part of a real function: if one has
\[
r(t_1) > 0
\]
for some \( t_1 > 0 \), then a fault has necessarily occurred at a time instant \( t_0 \leq t_1 \).

\textbf{Remark 3.1}: It is worth to note that the reference value \( \sigma(t) \) is time-variant and in particular it is exponentially vanishing, this improving the fault detection efficiency with respect to standard methods based on fixed reference thresholds.

\section*{C. Fault isolation}

We will discuss now the design of a pair of independent observers with fault isolation purposes: the main goal is to generate two different residual signals, say \( r(s)(t) \), \( r(a)(t) \), with the property of being invariant with respect to actuator and sensor faults respectively.

We start by studying the case of an observer insensitive to actuator faults. Since the actuator faults enter the systems as perturbation (disturbances), it is sufficient to repeat the steps of disturbance decoupling (see Section III.A). In particular, if \( p > m + h \), the matrix \( R \) can be designed such that
\[
R [B_d F] = 0. \tag{17}
\]
If together with the above condition the observability (or at least the detectability) of the pair \((C, RA)\) can be ensured, than the resulting observer is invariant with respect to actuator faults. We introduce a superscript \( (s) \) to the observer matrices \( R(s), K(s), V(s), N(s) \) in order to emphasize the sensitivity exclusively to the sensor fault \( \xi(t) \); the estimation error, denoted by \( e(s) \), is
\[
e(s)(t) = N(s)e_y(s)(t) + K(s)G\xi(s) - V(s)G\xi(s). \]

Let us present now the design of an observer of type (3)-(4) which is robust with respect to sensor faults. Integrating by parts, the terms in \( e_y(t) \) which depend on \( \xi(t) \) can be rewritten as
\[
C \begin{bmatrix} \int_{t_0}^t e^{N(t-s)}[K G \xi(s) - V G \xi(s)] ds \end{bmatrix} + G \xi(t)
\]
\[
= (I - CV)G\xi(t) + C \int_{t_0}^t e^{N(t-s)}[K G - V N G] \xi(s) ds,
\]
where \( t_0 \) is the time instant corresponding to the fault occurrence. By condition (7) one has \((I - CV)G = 0\); moreover if the stabilizing feedback gain \( K \) can be designed such that
\[
(K - NV)G = LG = 0, \tag{18}
\]
than the only fault depending term affecting the system is \( \xi(t_0)C e^{N(t-t_0)} VG \), which is asymptotically vanishing indeed. In conclusion, if (14) and (18) hold, the output error is disturbance-decoupled and asymptotically invariant to sensor faults; as before, we denote by \( N(a), K(a), R(a), V(a) \) the matrices of the observer and by \( e(a)(t) \) the corresponding estimation error.

\textbf{Remark 3.2}: If the fault \( \xi(t) \) enters the system continuously, meaning that \( \lim_{t \to +\infty} \xi(t) = 0 \), then the output error \( e_y(s)(t) \) is completely decoupled from the effects of the sensor fault.

Summarizing, we have designed two independent observers to be used in combination, say \((N(s), R(s), K(s), V(s))\) and \((N(a), R(a), K(a), V(a))\); checking the positivity of the associated residual signals
\[
r(s)(t) := (\|e_y(s)(t)\| - \sigma(s)(t))^+,
\]
where \( \sigma(s)(t) \) is the time variant function.
\[ r^{(a)}(t) := \left( ||e^{(a)}(t)|| - \sigma^{(a)}(t) \right) ^+ , \]

equipped with the following definition

Definition 1: Given the residual signal \( r(t) \) defined by (16), the fault \( f(t) \) with \( ||f(t)|| > 0 \) for \( t \geq t_0 \) is said

- Locally or weakly detectable: if there exists \( t_1 \geq t_0 \) such that \( r(t_1) > 0 \).
- Definitely detectable: if for any \( t_1 \geq t_0 \) there exists \( t_2 \geq t_1 \) with \( r(t_2) > 0 \).
- Asymptotically or strongly detectable: if there exists \( t_1 \geq t_0 \) such that \( r(t) > 0 \) for any \( t > t_1 \).

The cases of actuator faults and sensor faults are presented separately and three simplified classes of faults are defined:

- Abrupt faults (stepwise):
\[ \mathcal{F}_A = \{ f(t) : \exists \alpha \in \mathbb{R}, t_0 > 0 : f(t) = 0 \text{ for } t < t_0, f(t) = \alpha \text{ for } t \geq t_0 \} \]

- Incipient faults (drift-like):
\[ \mathcal{F}_I = \{ f(t) : \exists \beta \in \mathbb{R}, t_0 > 0 : f(t) = 0 \text{ for } t < t_0, f(t) = \beta \text{ for } t \geq t_0 \} \]

- Oscillating faults (sinusoidal):
\[ \mathcal{F}_O = \{ f(t) : \exists \gamma, \omega \in \mathbb{R}, t_0 > 0 : f(t) = 0 \text{ for } t < t_0, f(t) = \gamma \sin \omega t \text{ for } t \geq t_0 \} \]

We point out that the combination of faults in the first two classes can be treated in a standard way as well as the more general case of higher degree polynomial faults; this will be clarified in the computation below. For sake of simplicity only the single input case is presented, i.e. it is assumed that \( F \in \mathbb{R}^{n \times 1} \) and \( G \in \mathbb{R}^{p \times 1} \). The general case follows straightforward considering the columns of the input matrices separately.

A. Actuator faults analysis

Let us start considering an abrupt fault \( \eta(t) \in \mathcal{F}_A \), with \( \eta(t) \equiv \alpha \) for \( t \geq t_0 \); the dynamics of the estimation error is given by
\[ e_y(t) = C \left( e^{N t}(x_0 - \hat{x}_0) + \int_{t_0}^{t} e^{N(t-s)} R \omega ds \right) , \quad t \geq t_0. \]

Since \( N \) is a Hurwitz matrix by design, the inverse \( N^{-1} \) is well defined and the integral in the right-hand side can be rewritten as
\[ \int_{t_0}^{t} e^{N(t-s)} R \omega ds = \alpha e^{Nt} \left( \int_{t_0}^{t} e^{-N s} \omega ds \right) R \]
\[ = \alpha e^{Nt} \left( -N^{-1} e^{-N t} + N^{-1} e^{-N t_0} \right) R \]
\[ = -\alpha N^{-1} R + \alpha N^{-1} e^{N(t-t_0)} e^{-N(t-t_0)} R. \]

Now \( \lim_{t\to\infty} ||e_y(t)|| = ||\alpha CN^{-1} R|| \) and \( \lim_{t\to\infty} r(t) = 0 \); hence we can state the following sufficient condition for fault detectability.

Proposition 4.1: The presence of an abrupt fault \( \eta(t) \in \mathcal{F}_A \) in the actuator can be checked if the following algebraic condition hold:
\[ CN^{-1} R \neq 0. \]

We point out that, since \( \lim_{t\to\infty} ||N^{-1} e^{N(t-t_0)} R|| = 0 \), the above condition is also necessary if one is interested in strong (asymptotic) fault detectability.

With analogous computations the case of an incipient fault \( \eta(t) = \beta t \) for \( t \geq t_0 \) can be treated; integrating by parts, the forced response of the error satisfies
\[ \int_{t_0}^{t} e^{N(t-s)} R \omega \omega ds = \beta e^{N t} \left( \int_{t_0}^{t} e^{-N s} \omega \omega ds \right) R \]
\[ = \beta N^{-1} e^{N t} \left( -t e^{-N t} + t_0 e^{-N t_0} + \int_{t_0}^{t} e^{-N s} \omega \omega ds \right) R \]
\[ = -\beta (t N^{-1} + N^{-2} R + \beta (t_0 N^{-1} + N^{-2} e^{N(t-t_0)} R). \]

Proposition 4.2: A sufficient condition for the detectability of an incipient fault \( \eta(t) \in \mathcal{F}_I \) in the actuator is that

\[ either \ CN^{-1} R \neq 0 \ or \ CN^{-2} R \neq 0 \]

The above condition can be generalized to the case of polynomial faults \( \eta(t) = \sum_{j=0}^{k} a_j t^j \) for \( t \geq t_0 \).

Proposition 4.3: The fault \( \eta(t) = \sum_{j=0}^{k} a_j t^j \), \( t \geq t_0 \), is detectable if there exists an integer \( j \in \{ 0, 1, ..., k \} \) such that
\[ CN^{-1-j} R \neq 0. \]

Finally the class of oscillating faults is considered; in this case the analysis results to be slightly more complicated and it requires more work. Let \( \eta(t) = \gamma \sin \omega t \) for \( t \geq t_0 \). With a recursive integration by parts, one has
\[ \int_{t_0}^{t} e^{N(t-s)} R \gamma \sin \omega \omega ds = \gamma e^{N t} \left( \int_{t_0}^{t} e^{-N s} \sin \omega \omega ds \right) R \]
\[ = -\gamma e^{N t} (I + \omega^2 N^{-2})^{-1} N^{-1} \left( e^{-N t} \sin \omega t - e^{-N t_0} \sin \omega t_0 + e^{-N t} \omega N^{-1} \cos \omega t - e^{-N t_0} \omega N^{-1} \cos \omega t_0 \right) R. \]

Since the matrices \( e^{-N t} \) and \( (I + \omega^2 N^{-2})^{-1} \) commute, the expression in the right-hand side of the above chain of equalities can be simplified as
\[ -\gamma (N + \omega^2 N^{-1})^{-1} (\sin \omega t I + \omega \cos \omega N^{-1}) R \]
\[ + \gamma (N + \omega^2 N^{-1})^{-1} (\sin \omega t_0 I + \omega \cos \omega t_0 N^{-1}) e^{N(t-t_0)} R. \]

For any fixed \( \omega \neq 0 \), as the parameter \( t \) varies in \( [0, 2\pi] \) the vector \( \sin \omega t I + \omega \cos \omega N^{-1} R \) describes the whole set of (non trivial) directions obtained as \{Span\{N^{-1} R}\} + \delta R\} with \( \delta \in \{0, 1\} \); hence, without loss of generality, we can restrict our analysis to the vectors \( R \) and \( N^{-1} R \).

It follows that a sufficient condition for detectability is that, for any choice of \( \omega \neq 0 \), one has either \( C((N + \omega^2 N^{-1})^{-1} R \neq 0 \) or \( C((N + \omega^2 N^{-1})^{-1} N^{-1} R \neq 0 \); such property is reformulated in the statement of the next proposition independently of the parameter \( \omega \).

Proposition 4.4: The actuator oscillating fault is detectable if at least one of the following conditions is satisfied:

1) \( R \notin [N \ker(C) + N^{-1} \ker(C)] \)
2) $RF \notin [N^2 \ker(C) + \ker(C)]$

Remark 4.1: If the condition above are verified simultaneously, than the fault can be asymptotically detected; on the other hand, even if one of the above conditions is not satisfied, for any fixed $\omega$ there exists only a discrete set of time instants $t_1, t_2, \ldots, t_k, \ldots$, depending on $\omega$, such that

$$C(N + \omega^2 N^{-1})^{-1}(\sin \omega t I + \omega \cos \omega t N^{-1})RF = 0.$$ 

This means that any oscillating fault can be definitely detected, even if for a countable set of time instants the residual may be insensible to it.

Remark 4.2: It is worth to note that the conditions of Proposition 4.4 are not necessary indeed: given a set of distinct positive real numbers $\omega_1^2, \omega_2^2, \ldots, \omega_n^2$, for any fixed $v \in \mathbb{R}^n$ with $Nv \notin \text{Span}\{v\}$, the vectors

$$(N + \omega_j^2 N^{-1})^{-1}v, \ j = 1, \ldots, n$$

are linearly independent. This means that, even if none of the conditions in Proposition 4.4 are verified, the residual will be insensible to oscillating faults only for particular choices of the parameter $\omega$.

B. Sensor faults analysis

Let us suppose now that the sensor of the system undergoes a fault $\xi(t)$ for $t \geq t_0$. The analysis is equivalent to the case of actuator faults, with $LG$ instead of $RF$. For completeness we summarize in the next statements the main results on sensor faults detectability.

Proposition 4.5: The presence of a polynomial sensor fault $\xi(t) = \sum_{j=0}^{k} \beta_j t^j$ for $t \geq t_0$ can be checked if for some $j \in \{0, 1, \ldots, k\}$ the following condition is satisfied

$$CN^{-1-j}LG \neq 0.$$ 

Proposition 4.6: The presence of an oscillating sensor fault $\xi(t) = \gamma \sin(\omega t)$ for $t \geq t_0$ can be checked if at least one of the following conditions is verified

1) $LG \notin [N \ker(C) + N^{-1} \ker(C)]$

2) $LG \notin [N^2 \ker(C) + \ker(C)]$

V. NUMERICAL SIMULATIONS

We consider the uncertain descriptor system defined by the following matrices:

$$E = \begin{bmatrix} 1 & 1 & -2 & 1 & 6 \\ 0 & 1 & -3 & 2 & 3 \\ 1 & 1 & -2 & 1 & 6 \\ 0 & 2 & -6 & 4 & 6 \\ 1 & 1 & -2 & 1 & 6 \end{bmatrix}, \ A = \begin{bmatrix} 3 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & -2 & 1 & 6 \\ 0 & 2 & -6 & 4 & 6 \\ 1 & 1 & -2 & 1 & 6 \end{bmatrix}, \ B_u = B_d = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ F = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \ G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$ 

We point out that $\text{rank}(E) = 2$ and that the nominal system is stabilizable by the static output feedback

$$u(t) = SCx(t), \ S := [0 \ 0 \ 5]. \quad (19)$$

According to the notation introduced in Section III, we can compute the matrices of the observers as showed below:

$$R(s) = \begin{bmatrix} 0 & 0.6 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & -1 & 0.3 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{bmatrix}, \ V(s) = \begin{bmatrix} 0 & -0.3 & -4 \\ 0.5 & 0.5 & 0 \\ 0.5 & -0.16 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ K(s) = \begin{bmatrix} -0.3 & 0.3 & 1.2 \\ -3 & -1.8 & 5.1 \\ -4.4 & 3.7 & 2.2 \\ -0.6 & 1.2 & -12.2 \end{bmatrix}, \ N(s) = R(s)A + K(s)C$$

and

$$R(o) = \begin{bmatrix} 1 & 0.6 & 1 & -0.6 & 1 \\ 3 & 0 & -3 & 0 & 0 \\ -1 & 0.3 & 1 & -0.3 & 0 \\ -2 & 0 & 2 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 \end{bmatrix}, \ V(o) = \begin{bmatrix} 0 & -0.3 & -4 \\ 0.5 & 0.5 & 0 \\ 0.5 & -0.16 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ K(o) = \begin{bmatrix} 2.3 & 8.8 & -11.7 \\ 4.5 & -0.4 & -11 \\ -1.3 & -25.1 & 19 \\ 3 & -6.5 & -5.2 \end{bmatrix}, \ N(o) = R(o)A + K(o)C.$$ 

By construction we have

$$R(s)B_d = R(o)B_d = 0, \ R(s)F = 0, \ K(o)G = N(o)V(o)G,$$

so the estimation error (5) is completely independent of the disturbance term, and hence the FDI technique is robust. The system is supposed to be affected by a disturbance term $d(t) = 0.2 \sin(1.3t)$; the true initial condition is set as $x_0 = (1.21, 0.18, 1.19, 0.21, -1.17)$, while the reference initial condition and the bound parameter are defined as $x_0 = (1, 0, 1, 0, -1)$ and $\epsilon_0 = 0.5$.

Example 1: In the first simulation we consider the oscillating fault $\eta(t) = -0.07 \sin(3.2t)$ affecting the actuator of the system for $t \geq t_0 := 4.5$. Since the observer matrices are designed such that $R(s)F = R(s)B_d = 0$, the disturbance as well as the actuator fault do not affect the residual $r^{(o)}(t)$; on the other hand, the effects of the faults on the output error are very evident from the evolution of the disturbance-decoupled residual signal $r^{(o)}(t)$ and thus the fault can be properly isolated. Figure 2 and Figure 3 illustrate the behavior of the residual signals $r^{(s)}(t)$ and $r^{(o)}(t)$ respectively.
Fig. 1 - Evolution of the residual signal $r^{(a)}(t)$ subject to an oscillating actuator fault for $t \geq 4.5$

Fig. 2 - Evolution of the residual signal $r^{(a)}(t)$ subject to an incipient sensor fault for $t \geq 4.5$

Example 2: The second example is dedicated to show the performance of the proposed approach in the presence of an incipient sensor fault $\xi(t) = 0.1t$ for $t \geq t_0^* = 1.4$. The presence of the fault can be detected from the behavior of the residual signal $r^{(a)}(t)$, which is asymptotically increasing. In this case, due to the presence of the term $\xi(t_0^*) \neq 0$, the residual signal $r^{(a)}(t)$ is temporarily affected by the fault; on the other hand its maximum value is very close to zero and after a transient it vanishes exponentially, this leading to a straightforward fault isolation. The behavior of the residual signals is depicted in Figure 3 and Figure 4.

Fig. 3 - Evolution of the residual signal $r^{(a)}(t)$ subject to an incipient sensor fault for $t \geq 1.4$

Fig. 4 - Evolution of the residual signal $r^{(a)}(t)$ subject to an incipient sensor fault for $t \geq 1.4$

REFERENCES