An algebraic approach to promise constraint satisfaction

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Deciding triviality of Mal'cev conditions

A strong Mal'cev condition is a finite conjunction ϕ of identities using variable function symbols $f_1, \ldots, f_n$ and symbols for individual variables.

TRIVID

Given a strong Mal'cev condition ϕ, decide whether it is trivial (i.e., satisfiable by projections).

Clearly, TRIVID $\in$ NP.

TrivId is NP-hard

Reduce from 1-in-3-Sat, i.e., $\mathrm{CSP}(\{0,1\}, R = \{001, 010, 001\})$.

1. Given $\mathcal{I}$ an instance of 1-in-3-Sat with variables $x_1, \ldots, x_n$.
2. Introduce a binary symbol f_i for each x_i.
3. For each constraint $(x_{i_1}, x_{i_2}, x_{i_3}) \in R$, add a new symbol $g_{i_1 i_2 i_3}$ and identities:

$$
\begin{aligned}
g_{i_1 i_2 i_3}(y, x, x) &\approx f_{i_1}(x, y) \\
g_{i_1 i_2 i_3}(x, y, x) &\approx f_{i_2}(x, y) \\
g_{i_1 i_2 i_3}(x, x, y) &\approx f_{i_3}(x, y)
\end{aligned}
$$

Claim. The resulting Mal'cev condition is trivial iff the instance $\mathcal{I}$ has a solution.

Height 1 identities

Definition

An identity is of height 1 if it has exactly one function symbol on either side, i.e., it is of the form

$$f(x_{i_1}, \ldots, x_{i_k}) \approx g(x_{j_1}, \ldots, x_{j_l}).$$

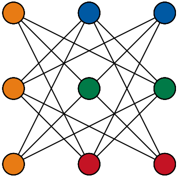
A Mal'cev condition is of height 1 if it involves only height 1 identities.
Example.

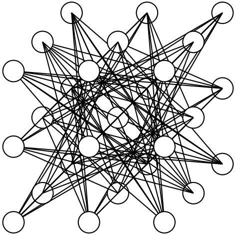
$$\begin{aligned} g_{i_1 i_2 i_3}(y, x, x) &\approx f_{i_1}(x, y) \\ g_{i_1 i_2 i_3}(x, y, x) &\approx f_{i_2}(x, y) \\ g_{i_1 i_2 i_3}(x, x, y) &\approx f_{i_3}(x, y). \end{aligned}$$

Deciding whether a strong height 1 Mal'cev condition is trivial is NP-complete.

From 1-in-3-SAT to graph coloring

Given an instance ϕ of 1-in-3-SAT, we construct a graph $\mathbb{G}$:





- If ϕ is solvable, then $\mathbb{G}$ is 3-colorable; and
- if $\mathbb{G}$ is 4-colorable, then ϕ is solvable.

Hardness of $\mathrm{PCSP}(K_d, K_{2d-2})$

(Brakensiek, Guruswami, 2016)

The following problem is NP-complete:

$(d,2d-2)$-Col

Given a graph, output

- YES if it is d-colorable,
- NO if it is not $(2d-2)$-colorable.

Promise constraint satisfaction

Fix two finite relational structures $\mathbb{D}_1, \mathbb{D}_2$ in the same finite language with a homomorphism $\mathbb{D}_1 \to \mathbb{D}_2$. PCSP$(\mathbb{D}_1, \mathbb{D}_2)$ is the following problem: Given a pp-sentence ψ in the common language, output

- YES if ψ is satisfied in $\mathbb{D}_1$,
- NO if ψ is not satisfied in $\mathbb{D}_2$.

Note. CSP$(\mathbb{D}) \equiv$ PCSP$(\mathbb{D}, \mathbb{D})$ and PCSP$(\mathbb{D}_1, \mathbb{D}_2) \leq$ CSP$(\mathbb{D}_1)$, CSP$(\mathbb{D}_2)$.

A few promise problems

- (d, e)-Col: Complexity not known for $e \geq 2d - 1$.
- $(1, d)$-kSat: Given a kSat instance, output:
  - YES if there is a solution that satisfies at least d literals in each clause,
  - NO if the instance is not solvable.

  NP-hard for $k/d > 2$, in P for $k/d \leq 2$ (Austrin, Guruswami, Håstad, 2013).
- (1-in-3,NAE)-Sat: given a set of triples of variables, output:
  - YES if there is an assignment, s.t., every triple contains exactly one 1;
  - NO if there is every assignment assigns three same values to some triple of variables.

  Complexity not known.

Algebraic approach to CSP

1. The Galois correspondence (Geiger, 1968 & Jeavons et al., 1998+)
   - If $\mathbb{A}$ and $\mathbb{B}$ share a universe and every relation of $\mathbb{B}$ is pp-definable in $\mathbb{A}$ then $\mathrm{CSP}(\mathbb{A}) \leq_{\mathrm{L}} \mathrm{CSP}(\mathbb{B})$.
   - This happens if and only if $\mathrm{Pol}(\mathbb{B}) \subseteq \mathrm{Pol}(\mathbb{A})$.
2. Primitive positive interpretations and Birkhoff's theorem (Bulatov, Jeavons, Krohkin, 2005)
   - If a structure $\mathbb{A}$ is pp-interpretable in $\mathbb{B}$ then $\mathrm{CSP}(\mathbb{A}) \leq_{\mathrm{L}} \mathrm{CSP}(\mathbb{B})$.
   - This happens if and only if there is a clone homomorphism from $\mathrm{Pol}(\mathbb{B})$ to $\mathrm{Pol}(\mathbb{A})$.
3. Homomorphic equivalence
   - If $\mathbb{A}$ and $\mathbb{B}$ are homomorphically equivalent then $\mathrm{CSP}(\mathbb{A}) = \mathrm{CSP}(\mathbb{B})$.

Theorem (Barto, O, Pinsker, 2017)

A structure $\mathbb{A}$ is homomorphically equivalent to a structure pp-interpretable in $\mathbb{B}$ iff there is an h1 clone homomorphism ($\equiv$ a mapping preserving height 1 identities) from $\mathrm{Pol}(\mathbb{B})$ to $\mathrm{Pol}(\mathbb{A})$.

> *"Can you use h1 homomorphisms when composition is not possible?"*
>
> —*Andrei Krohkin*

Algebraic approach to PCSP

1. The Galois correspondence (Pippenger, 2002 & Brakensiek, Guruswami, 2017)
   - If $(\mathbb{A}_1, \mathbb{A}_2)$ and $(\mathbb{B}_1, \mathbb{B}_2)$ share universes and every relation pair of $(\mathbb{B}_1, \mathbb{B}_2)$ is a relaxation of a pp-definable pair in $(\mathbb{A}_1, \mathbb{A}_2)$ then $\mathrm{PCSP}(\mathbb{A}_1, \mathbb{A}_2) \leq_{\mathrm{L}} \mathrm{PCSP}(\mathbb{B}_1, \mathbb{B}_2)$.
   - This happens if and only if $\mathrm{Pol}(\mathbb{B}_1, \mathbb{B}_2) \subseteq \mathrm{Pol}(\mathbb{A}_1, \mathbb{A}_2)$.

We say that (S_1, S_2) is a relaxation of (R_1, R_2) if $S_1 \subseteq R_1$ and $R_2 \subseteq S_2$.

Definition

A function $f\colon A_1^k \to A_2$ is a polymorphism from $\mathbb{A}_1$ to $\mathbb{A}_2$ if it is a homomorphism from $\mathbb{A}_1^k$ to $\mathbb{A}_2$.

The set $\mathrm{Pol}(\mathbb{A}_1, \mathbb{A}_2)$ is not a clone, but it is minor closed (clonoid; closed under composition with projections).

Algebraic approach to PCSP (cont.)

2. Primitive positive interpretations
   - If $(\mathbb{A}_1, \mathbb{A}_2)$ is a pp-power of $(\mathbb{B}_1 \to \mathbb{B}_2)$ then $\mathrm{PCSP}(\mathbb{A}_1, \mathbb{A}_2) \leq_{\mathrm{L}} \mathrm{PCSP}(\mathbb{B}_1, \mathbb{B}_2)$.
3. Bi-homomorphisms
   - If $\mathbb{A}_2 \to \mathbb{A}_1$ and $\mathbb{B}_1 \to \mathbb{B}_2$ then $\mathrm{PCSP}(\mathbb{A}_1, \mathbb{A}_2) = \mathrm{PCSP}(\mathbb{B}_1, \mathbb{B}_2)$.

Theorem (O, 2017)

$(\mathbb{A}_1, \mathbb{A}_2)$ *is a bi-homomorphic image of a pp-power of* $(\mathbb{B}_1, \mathbb{B}_2)$ *iff there is an (h1) minor homomorphism from* $\mathrm{Pol}(\mathbb{B}_1, \mathbb{B}_2)$ *to* $\mathrm{Pol}(\mathbb{A}_1, \mathbb{A}_2)$.

GapLabelCover

(formulation by J. Bulín)

We say that a height 1 Mal'cev condition is bipartite if:

- the function symbols appearing on the left are disjoint with the function symbols appearing on the right, and
- there is no repetition of variables on the right.

GapLabelCover$_\epsilon(L)$

Given a strong bipartite h1 Mal'cev condition involving at most L-ary symbols, output:

- YES if it is trivial,
- NO if every ϵ-subset is non-trivial.

Theorem (Raz, 1995)

For all $\epsilon > 0$ there exists L such that GapLabelCover$_\epsilon(L)$ is NP-hard.

From GapLabelCover to PCSP

There is a natural polynomial time reduction from the following problem to $\mathrm{PCSP}(\mathbb{A}_1, \mathbb{A}_2)$.

$\mathrm{EQ}_L(\mathbb{A}_1, \mathbb{A}_2)$

Given a strong h1 Mal'cev condition of arity at most L, output:

- YES if it is satisfied in $\mathrm{Pol}(\mathbb{A}_1)$,
- NO if it is not satisfied in $\mathrm{Pol}(\mathbb{A}_1, \mathbb{A}_2)$.

Theorem (Austrin, Brakensiek, Guruswami, Håstad, 2014–2016 & O, 2017)

If there exists $\epsilon > 0$ such that every strong h1 Mal'cev condition satisfied in $\mathrm{Pol}(\mathbb{A}_1, \mathbb{A}_2)$ has an ϵ-part which is trivial. Then $\mathrm{PCSP}(\mathbb{A}_1, \mathbb{A}_2)$ is NP-hard.

Problems

In which cases can we use certain CSP algorithms to solve PCSP?

- Some answers by J. Bulín.
- When does local consistency work?
- Is there a collapse in bounded width hierarchy?
- How does the few subpowers algorithm work for PCSP?
- Is there some way to use Zhuk's/Bulatov's algorithm? What are the requirements?

Is there a PCSP dichotomy?

- Is there some other source of hardness except GabLabelCover?
- Is $\mathrm{PCSP} \in \mathrm{P}$ always explained by CSP? (Bulín)

What are the connections to approximation?

- Does UGC play any role?
- Can we prove PCP theorem algebraically?

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