ADAPTIVE MULTIPLE KNOT B-SPLINE WAVELETS
FOR SOLVING SAINT-VENANT EQUATIONS

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Solving the Saint-Venant equations by numerical methods like finite element and finite
difference methods yields an unstable solution for a fairly large open channel. Multiple
Knot B-Spline Wavelets (MKBSW) are in the class of semi-orthogonal wavelets that have
compact support. Hence, these basis functions are suitable for solving the Saint-Venant
equations. However, solving the Saint-Venant equations by MKBSW method requires
a long CPU time. In this paper, we present an adaptive wavelet method to solve the
Saint-Venant equation in a fairly short time. In fact, we first solve the problem in a few
first moments and then by statistical methods of time series and regression, where the
active wavelets are predicted in the next moments. Moreover, by this adaptive method,
the cumulative errors (that are produced by solving the discretized system, numerically)
decrease for large open channels. Two numerical examples are given to support our
results.

Keywords: B-spline wavelet; hyperbolic partial differential equation; Saint-Venant
equations.

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1. Introduction

Solving initial-boundary value problems (IBVP) such as the Saint-Venant equations
and Biot’s model7 by Galerkin methods leads to very large systems Ax = b. Then,
for numerical implementation, it is required to form a sparse matrix $A$. For this
sake, the basis functions with locally compact support are suitable. In particular,
onorthonormal basis functions with locally compact support cut down on the expenses
of numerical computations. However, construction of orthonormal basis functions
with local support is not easy. Although, a family of wavelet orthonormal basis
functions has been given by Daubechies et al. in Ref. 4, but there is no explicit
formula for Daubechies wavelet. Hence, many researchers have tried to construct
semi-orthogonal basis wavelets with locally compact support and explicit formulas.
This can be done by multiple knot B-spline wavelets (MKBSW)\(^2,3\). The MKBSW
as basis functions of Galerkin’s method work well for solving IBVP, but for Saint-
Venant equations, it requires a long CPU time. In this work, according to MKBSW
we present an adaptive wavelet method to solve the Saint-Venant equation in a
fairly short time.

Let us first recall the notions of refinable function and multiresolution analysis
as introduced by Urban\(^9\). For a function $\phi \in L^2(\Omega)$, let a reference subspace $V_0$ be
generated as the $L^2$-closure of the integer translates of $\phi$, namely:

$$V_0 := \text{clos}_{L^2}(\phi(-k) : k \in \mathbb{I}_0),$$

and consider the other subspaces $V_j := \text{clos}_{L^2}(\phi_{j,k} : k \in \mathbb{I}_j)$, $j \geq 0$, where $\phi_{j,k} := 2^{j/2} \phi(2^j \cdot -k)$, $j \geq 0$, $k \in \mathbb{I}_j$ where $\langle F \rangle$ and $\mathbb{I}_j$ denote the space spanned by $F$ and
some appropriate set of indices, respectively.

**Definition 1.1.** A function $\phi \in L^2(\Omega)$ is said to generate a multiresolution analysis
(MRA) if it generates a nested sequence of closed subspace $V_j$ ($j \in \mathbb{Z}^+$) that satisfy

(i) $V_0 \subset V_1 \subset \cdots$;
(ii) $\text{clos}_{L^2}(\cup_{j \geq 0} V_j) = L^2(\Omega)$;
(iii) $\cap_{j \in \mathbb{Z}} V_j = \{0\}$;
(iv) $f \in V_j \iff f(2 \cdot) \in V_{j+1}$;
(v) $f \in V_0 \iff f(-k) \in V_0$, $k \in \mathbb{Z}$;
(vi) $\{\phi_{j,k}\}_{k \in \mathbb{I}_j}$ forms a Riesz basis for $V_j$, i.e.

there are constants $A$ and $B$ with $0 < A \leq B < \infty$ such that

$$A \sum_{k \in \mathbb{I}_j} |c_k|^2 \leq \left\| \sum_{k \in \mathbb{I}_j} c_{j,k} \phi_{j,k} \right\|_{L^2(\Omega)}^2 \leq B \sum_{k \in \mathbb{I}_j} |c_k|^2,$$

independent of $j$.

If $\phi$ generates an MRA, then $\phi$ is called a refinable function. In case that the
different integer translates of $\phi$ are orthogonal ($\phi(-k) \perp \phi(-\tilde{k})$ for $k \neq \tilde{k}$), the
refinable function is called an orthogonal refinable function.

Since the subspaces $V_j$ are nested, there exists a subspace $W_j$, such that

$$V_{j+1} = V_j \oplus W_j, \quad j \in \mathbb{Z},$$
where \( W_j \) is some direct summand, not necessarily the orthogonal one. Then, the problem of constructing the spaces \( W_j \) means to find a stable system of functions \( \Psi_j = \{ \psi_{j,k} : k \in \Gamma_j \} \), such that \( W_j = \text{clos}_{L^2} \{ \psi_{j,k} : k \in \Gamma_j \} \), \( j \geq 0 \) where \( \psi_{j,k} := 2^{j/2} \psi(2^j \cdot -k), k \in \Gamma_j \), \( j \geq 0 \). For abbreviation, we set \( W_{-1} = V_0, \Gamma_{-1} = \mathbb{I}_0 \) and hence this gives rise to a decomposition of \( V_j \), namely,

\[
V_j = \bigoplus_{k=-1}^{j-1} W_k.
\]

**Definition 1.2.** The elements of \( \Psi_j \) are called a set of wavelet if the system \( \Psi = \bigcup_{j=-1}^{\infty} \Psi_j \) forms a Riesz basis for \( L^2(\Omega) \), i.e.

\[
\left\| \sum_{j \geq -1} \sum_{k \in \Gamma_j} d_{j,k} \psi_{j,k} \right\|_{L^2(\Omega)}^2 \sim \sum_{j \geq -1} \sum_{k \in \Gamma_j} |d_{j,k}|^2.
\]

If \( (\psi_{j,k}, \psi_{j,k}) = \delta_{k,m}, m \in \Gamma_j \) where \( (f,g) = \int_{\Omega} f(x) \overline{g}(x) dx \) is the standard inner product, then \( \psi \) is called an orthonormal wavelet. The wavelets \( \psi_{j,k} \) are called semi-orthogonal, if \( (\psi_{j,k}, \psi_{j,k}) = 0 \); \( j \neq \tilde{j} \) for all \( j, \tilde{j} \geq -1, k \in \Gamma_j, \tilde{k} \in \Gamma_{\tilde{j}} \).

We denote the multiresolution space \( V_j \) and the wavelet space \( W_j \) on bounded interval \([0, 1]\) by \( V_j^{[0,1]} \) and \( W_j^{[0,1]} \) for any \( j \geq 0 \), respectively.

## 2. Multiple Knot B-Spline Wavelets for \( L^2[0, 1] \)

In this section we give the semi-orthogonal spline wavelets in \( L^2[0, 1] \) that has been constructed by Chui and Quak.\(^3\) Let \( m \in \mathbb{N} \) be fixed throughout this section.

**Definition 2.1.** For \( j \in \mathbb{Z}_+ \), let a knot sequence on \([0, 1]\) be given by \( t^{(j)} := t_m^{(j)} := \{ t_k^{(j)} \}_{k=0}^{2^j} \), with

\[
\begin{align*}
t^{(j)}_{m+1} &= t^{(j)}_{m+2} = \ldots = t^{(j)}_0 = 0, \\
t^{(j)}_k &= 2^{j-k}, \quad (k = 1, \ldots, 2^j - 1), \\
t^{(j)}_{2^j} &= t^{(j)}_{2^j+1} = \ldots = t^{(j)}_{2^j+m-1} = 1.
\end{align*}
\]

The spline space of order \( m \) for the knot sequence \( t_m^{(j)} \) is defined by

\[
S_{m,j} := S_{m,t^{(j)}} := \{ s \in C^{m-2}[0, 1] : s_{(t_k^{(j)}, t^{(j)}_{k+1})} \in \Pi_{m-1}, k = 0, \ldots, 2^j - 1 \}.
\]

The sequence of subspaces \( V_j^{[0,1]} \) is given by

\[
V_j^{[0,1]} = S_{m,j}, \quad V_0^{[0,1]} := \Pi_{m-1}.
\]

By standard spline theory one can establish the following:

**Theorem 2.1.** A basis for \( V_j^{[0,1]} \) is given by the B-spline \( B_{i,m,j}, i = -m + 1, \ldots, 2^j - 1 \) and thus \( \dim V_j^{[0,1]} = 2^j + m - 1 \). Here,

\[
B_{i,m,j}(x) := (t_{i+m} - t^{(j)}_i)[t^{(j)}_i, t^{(j)}_{i+1}, \ldots, t^{(j)}_{i+m}](t-x)^{m-1}.
\]
where \([\ldots, \cdot]_t\) is the \(m\)th divided difference of \((t - x)^{m-1}_+\) with respect to the variable \(t\). The support of \(B_{i,m,j}\) is \([t_i^j, t_{i+m}^j]\).

For \(i = -m + 1, \ldots, -1\), the knot sequence defining \(B_{i,m,j}\) contains a multiple knot at 0, and for \(i = 2^j - m + 1, \ldots, 2^j - 1\), a multiple knot at 1. The inner ones \((i = 0, \ldots, 2^j - m\) for \(2^j \geq m\)) are just dilation and translation of the cardinal B-spline \(N_m(x) = m[0,1,\ldots,m]_t(t-x)^{m-1}_+\) used as the refinable function for \(L^2(\mathbb{R})\), namely:

\[
B_{i,m,j} = N_m(2^i - i), \quad i = 0, \ldots, 2^j - m.
\]

To find suitable wavelet function spanning \(W_j^{[0,1]}\) in the orthogonal decomposition \(V_j^{[0,1]} = V_j^{[0,1]} \oplus W_j^{[0,1]}\), we use an argument that identifies the wavelet space \(W_j^{[0,1]}\) with a subspace of a spline of order \(2m\). For each \(m \in \mathbb{N}\), define the spline space

\[
\bar{S}_{2m} = \{ s \in S_j^{2m} : s(t_k^{(j)}) = 0, k = 0, \ldots, 2^j - 1 \}
\]

and its subspace

\[
\bar{S}_{0}^{2m} = \{ s \in S_j^{2m} : s(t_k^{(j)}) = 0, k = 0, \ldots, 2^j - 1 \}
\]

of all splines in \(\bar{S}_{2m}\) that vanish on the coarse knot sequence \(t_k^{(j)}\).

**Lemma 2.1.** For all \(j \in \mathbb{N}\) such that \(2^j \geq 2m - 1\), there exists \(2^j - 2m + 2\) linearly independent inner wavelet \(\psi_{j,i}\), \(i = 0, \ldots, 2^j - 2m + 1\), in \(W_j^{[0,1]}\) which are given by

\[
\psi_{j,i} = \frac{1}{2^{2m-1}} \sum_{k=0}^{2m-2} (-1)^k N_{2m}(k + 1) B_{2i+k,2m,t_k^{(j+1)}}^{(m)}.
\]

**Proof.** See Ref. 3.

By Lemma 2.1, there exist \(2^j - 2m + 2\) inner wavelets and, consequently, \(2m - 2\) boundary wavelets need to be constructed. This task can be split into the construction of \(m-1\) so-called 0-boundary wavelets, i.e. wavelets whose support contains the left endpoint of the interval \([0,1]\). By symmetry, the so-called 1-boundary wavelets are obtained from the 0-boundary wavelets by an index transformation \(i \leftrightarrow 2^i - 2m + 1 - i\) and \(x \leftrightarrow 1 - x\). We have the following lemma for 0-boundary wavelets:

**Lemma 2.2.** For \(j \in \mathbb{Z}_+\), if \(2^j \geq 2m - 1\), there exist \(m - 1\) wavelets on the 0-boundary which can be write as

\[
\psi_{j,i} = \frac{1}{2^{2m-1}} \sum_{k=-m+1}^{-1} \alpha_{k} B_{k,2m,t_k^{(j+1)}}^{(m)}
\]

\[
+ \frac{1}{2^{2m-1}} \sum_{k=0}^{2m-2+2i} (-1)^k N_{2m}(k + 1 - 2i) B_{k,2m,t_k^{(j+1)}}^{(m)}
\]

(2.3)
with supports \([0, (2m - 1 + i)2^{-j}]\) for \(i = -m + 1, \ldots, -1\). The coefficients \(\alpha_i := (\alpha_{i,k})_{1 \leq k \leq m-1}^T\) are derived by solving the linear system

\[B\alpha_i = r_i^T\]

with \(B := (B_{-k,2m}(j,i))_{1 \leq k, \ell \leq m-1}\) and \(r_i := (r_i,\ell)_{1 \leq \ell \leq m-1}\) where

\[r_i,\ell = - \sum_{k=0}^{2m-2+2i} (-1)^k N_{2m}^1(k+1-2i)N_{2m}^2(2\ell - k),\]

for \(\ell = 1, \ldots, m-1\) and \(\alpha_i := (\alpha_i,k)_{1 \leq k \leq m-1}^T\).

**Proof.** See Ref. 3.

**Theorem 2.2.** For \(2^j \geq 2m - 1\), the wavelet basis \(\psi_{j,i}^N\) at level \(j\) by

\[\psi_{j,i} := \begin{cases} \psi_{j,i} - \alpha_{j,i}B_{-m+1,m,j} & i = -m + 1, \ldots, -1, \\ \psi_{j,i} & i = 0, \ldots, 2^j - 2m + 1, \\ \psi_{j,i} - \beta_{j,i}B_{2^j-1,m,j} & i = 2^j - 2m + 2, \ldots, 2^j - m, \end{cases}\]

satisfies the homogenous Dirichlet boundary conditions where

\[\alpha_{j,i} := \frac{\psi_{j,i}(0)}{B_{-m+1,m,j}(0)}, \quad \beta_{j,i} := \frac{\psi_{j,i}(1)}{B_{2^j-1,m,j}(1)}\]

**Proof.** See Ref. 6.

**3. Saint-Venant Equation**

In this section, we present a solution to solve the Saint-Venant equations by the modified multiple knot B-spline wavelets given in Sec. 2. To do this, we consider the initial-boundary value Saint-Venant problem for unsteady flow in an open channel having no lateral inflow or outflow for one dimensional as (see Refs. 8 and 10):

\[
\begin{align*}
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) + gA \frac{\partial h}{\partial x} + \frac{g\nu^2 Q |Q|}{R^{5/3} A} &= 0, \\
\frac{\partial h}{\partial t} + \frac{1}{B} \frac{\partial Q}{\partial x} &= 0, \\
Q(x,0) &= Q^0 \quad 0 \leq x < L, \\
h(x,0) &= h^0 \quad 0 \leq x \leq L, \\
Q(L,t) &= 0 \quad t \geq 0, \\
h(0,t) &= h_0 \quad t > 0,
\end{align*}
\]

(3.1)

in which \(x\) = distance along the channel length, \(t\) = time, \(A\) = flow area, \(B\) = top water surface width, \(g\) = acceleration due to gravity, \(Q\) = discharge, \(h\) = water surface elevation, \(R\) = hydraulic radius, \(\nu\) = Manning coefficient, and \(L\) = length of channel, also \(h^0, h_0\) and \(Q^0\) are positive constant scalers. In general, \(A\) and \(R\) are the functions of \(h\) (i.e. \(A = A(h), R = R(h)\)).
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In order to present the variational form of the Saint-Venant equation (3.1), we first take a positive integer $N$, let $\Delta t$ denote the corresponding time-step: $\Delta t = T/N$ and $t_n$ the subdivisions of $[0, T]$:

$$t_n = n\Delta t, \quad 0 \leq n \leq N.$$  

Since the value of $h$ is nonzero in the first of channel (i.e. the constant $h_0$), then one can define

$$H(x, t_{n+1}) := h(x, t_{n+1}) - h_0.$$  

Now, the variational form of problem (3.1) is that\(^8,10\):

Find $Q(x, t_{n+1}) \in V = \{Q(x, t_k) \in H^1(\Omega) : Q(L, t_k) = 0, k = 0, \ldots, N\}$, and $H(x, t_{n+1}) \in S = \{H(x, t_k) \in H^1(\Omega) : H(0, t_k) = 0, k = 1, \ldots, N\}$ such that

$$d(H, v) + m(Q, v) + b(Q, v) = (\alpha, v)_{0} \quad \forall v \in S,$$

$$s(H, e) + w(Q, e) = (\beta, e)_{0} \quad \forall e \in V,$$  

where $\Omega = [0, L]$, $(\cdot, \cdot)_0$ is an inner product in the $L_2(\Omega)$ space, and the bilinear forms on $V \times S$ are given respectively by

$$m(Q, v) = \int_{\Omega} \left( \frac{1}{\Delta t} + \frac{gn^2|Q(x, t_n)|}{R^{4/3}(x, t_n)A(x, t_n)} \right) Q(x, t_{n+1})vdx,$$

$$b(Q, v) = -2\int_{\Omega} \frac{Q(x, t_n)}{A(x, t_n)}Q(x, t_{n+1})v'dx + \frac{2Q(x, t_n)Q(x, t_{n+1})v}{A(x, t_n)}|_{\partial\Omega},$$

$$d(H, v) = -g\int_{\Omega} H(x, t_{n+1})(A(x, t_n)v')dx + gA(x, t_n)H(x, t_{n+1})v|_{\partial\Omega},$$

$$s(H, e) = \frac{1}{\Delta t}\int_{\Omega} H(x, t_{n+1})edx,$$

$$w(Q, e) = -\frac{1}{B}\int_{\Omega} Q(x, t_{n+1})e'dx + \frac{1}{B}Q(x, t_{n+1})e|_{\partial\Omega},$$

$$(\alpha, v) = \int_{\Omega} \alpha vdx,$$

where $\partial \Omega$ is the boundary of $\Omega$ and $v|_{\partial\Omega}$ the restriction of $v$ on $\partial\Omega$.

**Remark 3.1.** Solving the Saint-Venant equations by numerical schemes like finite difference and finite element methods lead to some non-favorite oscillations for water surface elevation. The reason for these oscillations lies in the method of approximation for the nonlinear terms. One of the ways to smooth these oscillations is adding artificial viscosity to the scheme.\(^1\) Also, average rule is another method to eliminate oscillations. One can apply the average of nonlinear terms in space to eliminate oscillations\(^5\) or the average of $H(x, t_n)$ and $Q(x, t_n)$ in space for time $t = t_{n+1}$ for finite element method.\(^8,10\) Since using the above heuristic techniques ruin the stability for large scales, we did not use them and apply multiple knot B-spline wavelet directly.
4. Adaptive Schemes

In order to obtain an approximated solution as much as exact of Saint-Venant equations, derived by the multiple knot B-spline wavelet method, we need to consider a fine level as much possible as. But this takes a long CPU time. In addition, if the channel length is large, the system resulted by discretization is not directly solvable, and one should use the iterative methods. Therefore, total error is the sum of errors resulted by discretization and the numerical solution of the system. We present an adaptive method to cut down on both CPU time and error. Our strategy is as follows:

By considering $W_j^N = \text{clos}_{L_2}(\psi_{j,k}^N : k \in \Gamma_j)$, $j \geq 0$, decomposition of the approximation space $V_j$ in level $j$ is determined by $V_j = V_{j_0} \oplus \bigoplus_{k=j_0}^j W_k^N$ where $j_0$ is a natural number and satisfies $2^{j_0} \geq 2m - 1$.

By Theorem 2.1, Lemmas 2.1 and 2.2, there exist $2^{j_0} + m - 1$ refrible functions and $\sum_{k=j_0}^j 2^k$ wavelet basis functions. Since the solution of Saint-Venant equations are smooth, then most of the wavelet coefficients are tiny. Hence, if we remove the wavelet bases associated to the tiny wavelet coefficients from

$$\{B_{i,m,j} \}_{-m+1 \leq i \leq 2^{j-1}} \cup \{\psi_{j,k}^N \}_{-m+1 \leq i \leq 2^{j-m}}$$

the solution would be almost unchanged. Moreover, the CPU time would be decreased significantly. Also, the resulted system can be solved directly that yields no cumulative errors. Therefore, we first solve the problem in a few first moments, $t = t_1, \ldots, t_r$. Let the solution at time $t = t_n$ be as follows:

$$H^{(n)} = \sum_i c^m_{i,j} B_{i,m,j} + \sum_{|d^m_{j,k}| \geq \varepsilon} d^m_{j,k} \psi^N_{j,k} + \sum_{|d^m_{j,k}| < \varepsilon} d^m_{j,k} \psi^N_{j,k},$$

for $n = 1, \ldots, r$ where $\varepsilon$ is a given tolerance. Since, $\psi^N_{j,k}$ has local support and $\varepsilon$ is enough small, then we have:

$$H^{(n)} \approx \sum_i c^m_{i,j} B_{i,m,j} + \sum_{|d^m_{j,k}| \geq \varepsilon} d^m_{j,k} \psi^N_{j,k}, \quad n = 1, \ldots, r.$$

According to the wavelet coefficients $d^m_{j,k}$ at times $t_n = t_1, \ldots, t_r$ which $|d^m_{j,k}| \geq \varepsilon$, we will predict the position of wavelet coefficients $d^{r+1}_{j,k}$ at time $t = t_{r+1}$ with $|d^{r+1}_{j,k}| \geq \varepsilon$. To this end, we define

$$D_t = \{\ell^m_n : j = j_0, \ldots, J, n = 1, \ldots, r\}, \quad (4.1)$$

$$D_m = \{m^r_n : j = j_0, \ldots, J, n = 1, \ldots, r\}, \quad (4.2)$$

where

$$\ell^m_n = \min_k \{k : |d^m_{j,k}| \geq \varepsilon\}, \quad (4.3)$$

$$m^r_n = \max_k \{k : |d^{r+1}_{j,k}| \geq \varepsilon\}. \quad (4.4)$$

Now, by $D_t$ and $D_m$, we can predict the position of lowest and highest coefficients of wavelet at time $t = t_{r+1}$, respectively, i.e. $\ell^{r+1}_j$ and $m^{r+1}_j$. This can be done by...
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regression or time series methods. Finally, we solve the problem at time \( t = t_{r+1} \), with refinable basis functions and the wavelet bases \( \psi_{j,k}^N \) that \( |d_{j,k}^{r+1}| \geq \varepsilon \).

The following algorithm describes the method:

**Algorithm 1. Adaptive scheme**

1. **Input** \( (m, J, \Delta t, r, \varepsilon) \).
2. Solve the variational form of Saint-Venant equations (3.2) up to a few first moments \( (t = \Delta t \text{ to } t = r \Delta t) \).
3. Take \( d_{s,j,k} \) as the wavelet coefficient of the basis \( \psi_{j,k}^N \) at time \( t = t_s \).
4. \( E = \emptyset \)
5. For \( s = 1 : r \)
   6. For \( j = j_0 : J \)
   7. For \( k = -m + 1 : 2^j - m \)
   8. If \( |d_{s,j,k}| \geq \varepsilon \)
   9. \( E = E \cup \{d_{s,j,k}^j\} \)
10. Endif
11. Endfor
12. Endfor
13. Endfor
14. Form \( D_{\ell} \) and \( D_{m_j} \) as (4.1) and (4.2), respectively.
15. Predict the active wavelet coefficients at time \( t = t_{m+1} \).
16. Solve the Saint-Venant equation at time \( t = t_{r+1} \) with the refinable functions and wavelet basis functions corresponding to the active wavelet coefficients.

In Step 1, the order of MKBS \( (m) \), the finest level \( (J) \), time step \( (\Delta t) \), \( r \) initial time step and tolerance parameter for active wavelet coefficients \( (\varepsilon) \) are taken as initial data. In Step 2, by using MKBSW as basis functions of Galerkin’s method the variational form of (3.2) is solved for a few first moments. All wavelet coefficients in a few first moments are saved in Step 3. In Steps 4 to 13, the active wavelet coefficients are saved in the set \( E \). Active wavelets contain those which coefficients are not lower than the given tolerance \( \varepsilon \). In Step 14, for \( j = 1, 2, \ldots, J \) and \( t = t_1, t_2, \ldots, t_r \), the minimum and maximum active wavelet coefficients of \( E \) are saved in \( D_{\ell} \) and \( D_{m_j} \), respectively. By statistical methods, the active wavelet coefficients at time \( t = t_{r+1} \) are predicted in Step 15. In Step 16, the Saint-Venant equation (3.1) is solved by Galerkin’s method. The basis functions are selected from the MKBS functions at level \( j_0 \) with MKBSW functions at level \( j = j_0, j_0 + 1, \ldots, J \) corresponding to the active wavelet coefficients.

**5. Numerical Experiments**

In this section, we consider two examples that predict the solution of Saint-Venant equations by statistical methods. These examples are given from Refs. 8 and 10.
Example 5.1. The task of estimating the movement of a surge (or shock) or a dam-break wave, resulting from the sudden up-stream opening (or the sudden downstream closure) of a sluice gate for emergencies or dam failures, has occupied the attention of researchers as well as practicing engineers for several decades. The determination of the surge height at different locations along the channel provides important information for the design of the bank height. A dreadful disaster due to dam-break flood waves reminds the decision makers to take heed on the dam-safety problem. We consider an open channel with rectangular cross section that its bottom width is 6.1 m. The bottom slope is 0.00008, Manning coefficient $n = 0.013$ and the length of channel is 20 m. The initial conditions in the channel are 5.79 m-depth and a steady discharge of $126 \text{m}^3/\text{s}$. The water surface level in reservoir is constant at the up-stream end and also the sluice gate at the downstream end of the channel is suddenly closed at time $t = 0$. Here, we used the multiple knot B-spline wavelet of degree 2 and $J = 2$. Moreover, by $\Delta t = 1/20$ we first solved the problem for $t = \Delta t$ to $t = 6 \Delta t$ and second we predicted the solution at $t = 7 \Delta t$ to $t = 10 \Delta t = 0.5 \text{sec}$.

Based on the time series plot of the first six times, Trend model with linear Trend is a suitable model for this data. The equation
\[
d = 157.533 - 1.77143t
\]
as a linear regression equation is fitted for the data which $d$ is the same $\ell^{\phi}$ in (4.3) and shows the subscript of lowest coefficient of wavelet at time $t$. The estimation of the model is verified by least square error method. Here, the results of the time series analysis and regression model are the same. Figures 1 and 2 show the subscript of lowest coefficient of wavelet at time $t$. In Fig. 1, actual points show $\ell_{k}^{\phi}$ for $k = 1, \ldots, 6$. Also, fits points show the same points that are estimated by the time series

![Graph showing the position of wavelet coefficient with time series method.](image-url)
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Fig. 2. The position of wavelet coefficient with regression method (color online).

Fig. 3. The flow depth in the channel at time $t = 0.5$ sec by regression method.

Table 1. CPU times of times series and MKBSW methods for different times.

<table>
<thead>
<tr>
<th>Method</th>
<th>$t = 7\Delta t$</th>
<th>$t = 8\Delta t$</th>
<th>$t = 9\Delta t$</th>
<th>$t = 10\Delta t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MKBSW</td>
<td>2657</td>
<td>2754</td>
<td>2782</td>
<td>2846</td>
</tr>
<tr>
<td>Time series</td>
<td>547</td>
<td>602</td>
<td>657</td>
<td>718</td>
</tr>
</tbody>
</table>

model and finally, forecast points show the predicted points for $t = 7\Delta t, \ldots, 10\Delta t$. Figure 3 shows the flow depth in the channel at time $t = 0.5$ sec by regression method. In addition, we give the CPU time for both two methods at time $t = 0.5$ sec in Table 1.
Example 5.2. We consider an open rectangular channel, having a bottom width of 6.1 m, is carrying a flow of 126 m$^3$/s. The bottom slope is 0.04, with a Manning coefficient of $n = 0.00008$ and the channel length is 20 m. We consider flow depth $h^0 = 6.5949$ m. Suppose that a sluice gate at the downstream end is suddenly closed at time $t = 0$. We solved this problem by multiple knot B-spline wavelet and regression methods with $m = 2, J = 3$. Figure 4 shows the flow depth in the
channel at time $t = 0.3\text{sec}$ and Fig. 5 shows the flow depth in the channel at time $t = 0.3\text{sec}$ with multiple knot B-spline wavelet.

6. Conclusion

In general, solving the Saint-Venant equations by Galerkin methods takes a long CPU time. We discretized the Saint-Venant equations by Galerkin method with multiple knot B-spline wavelet basis. By statistical methods, we proposed an adaptive method to predict the solution. In this adaptive method, we used this reality that most of the wavelet coefficients are very small, so the basis functions associated with these small coefficients can be removed. In other words, we can solve the Saint-Venant problem by Galerkin method on an approximation space with small dimension.

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References