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A Special Case on the Stability and Accuracy for the 1D Heat Equation Using 3-Level and θ-Schemes

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Abstract

We establish the conditions for the compute of the stability restriction and local accuracy on the time step and we prove the consistency and local truncation error by using θ-scheme and 3-level scheme for Heat Equation with smooth initial conditions and for some parameter $\theta \in [0,1]$.

Keywords

Global Truncation, Local Accuracy, Stability Restriction

1. Introduction

In this paper we have considered the heat equation $u_t = u_{xx}$ with $\theta \in [0,1]$. Using θ-scheme and 3-level scheme in space we compute the order of local accuracy in space and time and stability restriction as a function of θ on the time step Δt. Much attention has been paid to the development, analysis and implementation of accurate methods for the numerical solution of this problem in the literature. Many problems are modeled by smooth initial conditions and Dirichlet boundary conditions. A number of procedures have been suggested (see, for instance [1]-[3]). We can say that three classes of solution techniques have emerged for solution of PDE: the finite difference techniques, the finite element methods and the spectral techniques (see [4] and [5]). The last one has the advantage of high accuracy attained by the resulting discretization for a given number of nodes [6]-[8].

We consider Scheme (1) for the 1D heat equation for some parameter $\theta \in [0,1]$. We compute the order of local accuracy in space and time as a function of θ and its the stability restriction. Until $T = 1$, we compute the

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solution with some fixed L^∞ error with the smallest amount of CPU time, and finally we can see this findings producing the relevant convergence and efficiency plot. For the 3-level scheme we consider (11) for the 1D heat equation and we compute the local truncation error. For different values of δ and β we find the stability criterion of the scheme and its accuracy.

2. θ-Scheme

Let

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}=\theta\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\Delta x^2}+(1-\theta)\frac{u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}}{\Delta x^2} \tag{1}$$

be the θ-scheme applied to the one-dimensional heat equation

$$u_t-u_{xx}=0 \tag{2}$$

Now for the order of local accuracy in space and time as a function of θ we write the local truncation error. In time we have

$$\tau=u_j^{n+1}-U^{n+1}$$

where U^{n+1} represents the exact solution of the heat equation. Now we perform Taylor expansion of u_j^{n+1} at t_n.

$$u_j^{n+1}=u_j^n+\Delta t\left(u_j^n\right)_t+\frac{\Delta t^2}{2}\left(u_j^n\right)_{tt}+\frac{\Delta t^3}{6}\left(u_j^n\right)_{ttt}+\mathcal{O}\left(\Delta t^4\right)$$

We can write the LHS of (1) as

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}=\frac{1}{\Delta t}\left(u_j^n+\Delta t\left(u_j^n\right)_t+\frac{1}{2}\Delta t^2\left(u_j^n\right)_{tt}+\frac{\Delta t^3}{6}\left(u_j^n\right)_{ttt}+\mathcal{O}\left(\Delta t^4\right)-u_j^n\right) \tag{3}$$

$$=\left(u_j^n\right)_t+\frac{1}{2}\Delta t\left(u_j^n\right)_{tt}+\frac{\Delta t^2}{6}\left(u_j^n\right)_{ttt}+\mathcal{O}\left(\Delta t^3\right) \tag{4}$$

here $\left(u_j^n\right)_t$ represents the derivative with respect to time, of $u_j(n)$. On the RHS, we have a centered difference approximating second derivative of $u_j(n)$

$$\frac{u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}}{\Delta x^2}=\left(u_j^n\right)_{xx}+\frac{\Delta x^2}{(4!/2)}\left(u_j^n\right)_{x(iv)}+\frac{\Delta x^2}{(6!/2)}\left(u_j^n\right)_{x(vi)}+\mathcal{O}\left(\Delta x^6\right) \tag{5}$$

As we are solving the heat equation, the previous expression is

$$=\left(u_j^n\right)_t+\frac{\Delta x^2}{(4!/2)}\left(u_j^n\right)_{tt}+\frac{\Delta x^4}{(6!/2)}\left(u_j^n\right)_{ttt}+\mathcal{O}\left(\Delta x^6\right) \tag{6}$$

Now, at time $n+1$ we have

$$\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\Delta x^2}=\left(u_j^{n+1}\right)_{xx}+\frac{\Delta x^2}{(4!/2)}\left(u_j^{n+1}\right)_{x(iv)}+\frac{\Delta x^4}{(6!/2)}\left(u_j^{n+1}\right)_{x(vi)}+\mathcal{O}\left(\Delta x^6\right) \tag{7}$$

$$=\left(u_j^{n+1}\right)_t+\frac{\Delta x^2}{(4!/2)}\left(u_j^{n+1}\right)_{tt}+\frac{\Delta x^4}{(6!/2)}\left(u_j^{n+1}\right)_{ttt}+\mathcal{O}\left(\Delta x^6\right) \tag{8}$$

therefore, applying Taylor expansion with respect to Δt we can write

$$\left(u_j^{n+1}\right)_t=\left(u_j^n\right)_t+\Delta t\left(u_j^n\right)_{tt}+\frac{\Delta t^2}{2}\left(u_j^n\right)_{ttt}+\frac{\Delta t^3}{6}\left(u_j^n\right)_{t(iv)}+\mathcal{O}\left(\Delta x^4\right)$$

$$\left(u_j^{n+1}\right)_{tt}=\left(u_j^n\right)_{tt}+\Delta t\left(u_j^n\right)_{ttt}+\frac{\Delta t^2}{2}\left(u_j^n\right)_{t(iv)}+\frac{\Delta t^3}{6}\left(u_j^n\right)_{t(v)}+\mathcal{O}\left(\Delta x^4\right)$$

$$\left(u_j^{n+1}\right)_{ttt}=\left(u_j^n\right)_{ttt}+\Delta t\left(u_j^n\right)_{t(iv)}+\frac{\Delta t^2}{2}\left(u_j^n\right)_{t(v)}+\frac{\Delta t^3}{6}\left(u_j^n\right)_{t(vi)}+\mathcal{O}\left(\Delta x^4\right)$$

So (8) becomes

$$\begin{aligned}&\left(u_j^{n+1}\right)_t+\frac{\Delta x^2}{(4!/2)}\left(u_j^{n+1}\right)_{tt}+\frac{\Delta x^4}{(6!/2)}\left(u_j^{n+1}\right)_{ttt}+\mathcal{O}\left(\Delta x^6\right)\\&=\left(u_j^n\right)_t+\Delta t\left(u_j^n\right)_{tt}+\frac{\Delta t^2}{2}\left(u_j^n\right)_{ttt}+\frac{\Delta t^3}{6}\left(u_j^n\right)_{t(iv)}+\mathcal{O}\left(\Delta x^4\right)\\&\quad+\frac{\Delta x^2}{(4!/2)}\left(\left(u_j^n\right)_{tt}+\Delta t\left(u_j^n\right)_{ttt}+\frac{\Delta t^2}{2}\left(u_j^n\right)_{t(iv)}+\frac{\Delta t^3}{6}\left(u_j^n\right)_{t(v)}+\mathcal{O}\left(\Delta x^4\right)\right)\\&\quad+\frac{\Delta x^2}{(6!/2)}\left(\left(u_j^n\right)_{ttt}+\Delta t\left(u_j^n\right)_{t(iv)}+\frac{\Delta t^2}{2}\left(u_j^n\right)_{t(v)}+\frac{\Delta t^3}{6}\left(u_j^n\right)_{t(vi)}+\mathcal{O}\left(\Delta x^4\right)\right)+\mathcal{O}\left(\Delta x^4\right),\end{aligned}\tag{9}$$

Here RHS of (1) becomes

$$\begin{aligned}&\theta\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\Delta x^2}+(1-\theta)\frac{u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}}{\Delta x^2}\\&=(1-\theta)\left(\left(u_j^n\right)_t+\frac{\Delta x^2}{(4!/2)}\left(u_j^n\right)_{tt}+\frac{\Delta x^4}{(6!/2)}\left(u_j^{n+1}\right)_{ttt}+\mathcal{O}\left(\Delta x^6\right)\right)\\&\quad+\theta\left(\left(u_j^n\right)_t+\Delta t\left(u_j^n\right)_{tt}+\frac{\Delta t^2}{2}\left(u_j^n\right)_{ttt}+\frac{\Delta t^3}{6}\left(u_j^n\right)_{t(iv)}+\mathcal{O}\left(\Delta x^4\right)\right.\\&\quad+\frac{\Delta x^2}{(4!/2)}\left(\left(u_j^n\right)_{tt}+\Delta t\left(u_j^n\right)_{ttt}+\frac{\Delta t^2}{2}\left(u_j^n\right)_{t(iv)}+\frac{\Delta t^3}{6}\left(u_j^n\right)_{t(v)}+\mathcal{O}\left(\Delta x^4\right)\right)\\&\quad\left.+\frac{\Delta x^4}{(6!/2)}\left(\left(u_j^n\right)_{ttt}+\Delta t\left(u_j^n\right)_{t(iv)}+\frac{\Delta t^2}{2}\left(u_j^n\right)_{t(v)}+\frac{\Delta t^3}{6}\left(u_j^n\right)_{t(vi)}+\mathcal{O}\left(\Delta x^4\right)\right)+\mathcal{O}\left(\Delta x^4\right)\right),\end{aligned}$$

After the elimination of some terms we have

$$\begin{aligned}&\left(u_j^n\right)_t+\frac{\Delta x^2}{(4!/2)}\left(u_j^n\right)_{tt}+\frac{\Delta x^4}{(6!/2)}\left(u_j^{n+1}\right)_{ttt}+\mathcal{O}\left(\Delta x^6\right)\\&\quad+\theta\left(\Delta t\left(u_j^n\right)_{tt}+\frac{\Delta t^2}{2}\left(u_j^n\right)_{ttt}+\frac{\Delta t^3}{6}\left(u_j^n\right)_{t(iv)}+\mathcal{O}\left(\Delta x^4\right)+\frac{\Delta t\Delta x^2}{(4!/2)}\left(\left(u_j^n\right)_{ttt}+\frac{\Delta t}{2}\left(u_j^n\right)_{t(iv)}+\frac{\Delta t^2}{6}\left(u_j^n\right)_{t(v)}+\mathcal{O}\left(\Delta x^3\right)\right)\right.\\&\quad\left.+\frac{\Delta t\Delta x^4}{(6!/2)}\left(\left(u_j^n\right)_{t(iv)}+\frac{\Delta t}{2}\left(u_j^n\right)_{t(v)}+\frac{\Delta t^2}{6}\left(u_j^n\right)_{t(vi)}+\mathcal{O}\left(\Delta x^3\right)\right)+\mathcal{O}\left(\Delta x^4\right)\right),\end{aligned}$$

Now simplifying we obtain

$$\begin{aligned}&\left(u_j^n\right)_t+\frac{\Delta t}{2}\left(u_j^n\right)_{tt}+\frac{\Delta t^2}{6}\left(u_j^n\right)_{ttt}+\mathcal{O}\left(\Delta x^3\right)\\&=\left(u_j^n\right)_t+\frac{\Delta x^2}{(4!/2)}\left(u_j^n\right)_{tt}+\frac{\Delta x^4}{(6!/2)}\left(u_j^{n+1}\right)_{ttt}+\mathcal{O}\left(\Delta x^6\right)\\&\quad+\theta\left(\Delta t\left(u_j^n\right)_{tt}+\frac{\Delta t^2}{2}\left(u_j^n\right)_{ttt}+\frac{\Delta t^3}{6}\left(u_j^n\right)_{t(iv)}+\mathcal{O}\left(\Delta x^4\right)+\frac{\Delta t\Delta x^2}{(4!/2)}\left(\left(u_j^n\right)_{ttt}+\frac{\Delta t}{2}\left(u_j^n\right)_{t(iv)}+\frac{\Delta t^2}{6}\left(u_j^n\right)_{t(vi)}+\mathcal{O}\left(\Delta x^3\right)\right)\right.\\&\quad\left.+\frac{\Delta x^4}{(6!/2)}\left(\left(u_j^n\right)_{t(iv)}+\frac{\Delta t}{2}\left(u_j^n\right)_{t(v)}+\frac{\Delta t^2}{6}\left(u_j^n\right)_{t(vi)}+\mathcal{O}\left(\Delta x^3\right)\right)+\mathcal{O}\left(\Delta x^4\right)\right),\end{aligned}$$

Cancelling $\left(u_j^n\right)_t$ and moving all terms to the right side, we get

$$0=\left(u_j^n\right)_{tt}\left(-\frac{\Delta t}{2}+\theta\Delta t+\frac{\Delta x^2}{6}\right)+\left(u_j^n\right)_{ttt}\left(\frac{\Delta x^4}{(6!/2)}+\theta\frac{\Delta t^2}{2}-\frac{\Delta t^2}{6}+\theta\frac{\Delta t\Delta x^2}{6!/2}\right)+\mathcal{O}\left(\Delta x^2\right)+\mathcal{O}\left(\Delta x^4\right) \tag{10}$$

Scheme (10) is first order in time, second order in space. If for example $\theta=1/2$, it becomes second order, this is due to cancellation of the $\Delta\tau$.

Stability Restriction as a Function of θ

Here we will apply Von Neumann stability. Let $u\left(x,t_n\right)=\mathrm{e}^{ikx}$ and $G(k)=\frac{u_j^{n+1}}{u_j^n}$. Then Equation (1) can be written as

$$\frac{u_j^n(G-1)}{\Delta t}=\frac{\theta}{\Delta x^2}\left(\left(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}\right)-\left(u_{j+1}^n-2u_j^n+u_{j-1}^n\right)\right)+\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2}$$

Now dividing by u_j^n we have

$$\frac{(G-1)}{\Delta t}=\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{u_j^n\Delta x^2}+\frac{\theta}{u_j^n\Delta x^2}\left(\left(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}\right)-\left(u_{j+1}^n-2u_j^n+u_{j-1}^n\right)\right)$$

$$G=\Delta t\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{u_j^n\Delta x^2}+\frac{\theta\Delta t}{u_j^n\Delta x^2}\left(\left(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}\right)-\left(u_{j+1}^n-2u_j^n+u_{j-1}^n\right)\right)+1$$

Therefore by using $u_{j\pm1}^n=\mathrm{e}^{ikx\pm ik\Delta x}$ and $r=\frac{\Delta t}{\Delta x^2}$ we can rewrite the expression as

$$G=r\left(\left(\mathrm{e}^{ik\Delta x}-2+\mathrm{e}^{-ik\Delta x}\right)+\frac{\theta}{u_j^n}\left(\left(u_{j+1}^{n+1}-u_{j+1}^n\right)-2\left(u_{j+1}^n-u_j^n\right)+\left(u_{j-1}^{n+1}-u_{j-1}^n\right)\right)\right)+1$$

$$G(k)=r\left(\left(\left(\mathrm{e}^{ik\Delta x}-2+\mathrm{e}^{-ik\Delta x}\right)+\theta(G-1)\right)\left(\mathrm{e}^{ik\Delta x}-2+\mathrm{e}^{-ik\Delta x}\right)\right)+1$$

$$G(k)=r(1-\theta)\left(\mathrm{e}^{ik\Delta x}-2+\mathrm{e}^{-ik\Delta x}\right)+1+r\theta G\left(\mathrm{e}^{ik\Delta x}-2+\mathrm{e}^{-ik\Delta x}\right)$$

$$G(k)\left(1-r\theta\left(\mathrm{e}^{ik\Delta x}-2+\mathrm{e}^{-ik\Delta x}\right)\right)=r(1-\theta)\left(\mathrm{e}^{ik\Delta x}-2+\mathrm{e}^{-ik\Delta x}\right)+1$$

$$G(k)=\frac{r(1-\theta)\left(\mathrm{e}^{ik\Delta x}-2+\mathrm{e}^{-ik\Delta x}\right)+1}{1-r\theta\left(\mathrm{e}^{ik\Delta x}-2+\mathrm{e}^{-ik\Delta x}\right)}=1+\frac{r\left(\mathrm{e}^{ik\Delta x}-2+\mathrm{e}^{-ik\Delta x}\right)}{1-r\theta\left(\mathrm{e}^{ik\Delta x}-2+\mathrm{e}^{-ik\Delta x}\right)}$$

By using the identity $\cos x=\frac{\mathrm{e}^{ix}-\mathrm{e}^{-ix}}{2}$ we have

$$G(k)=1+\frac{-2r\left(1-\cos(k\Delta x)\right)}{1+2r\theta\left(1-\cos(k\Delta x)\right)}$$

We can say this scheme is stable only for $\left|G(k)\right|\le 1$. Now, let $2r\left(1-\cos(k\Delta x)\right)=c$. The inequality is

$$\left|G(k)\right|=\left|1-\frac{c}{1+c\theta}\right|\le 1$$

thus

$$\left|\frac{1+c\theta-c}{1+c\theta}\right|\le 1,\quad \theta,\ c>0$$

Now multiplying by the denominator we have

$$|1+c\theta-c| \le 1+c\theta$$

$$|1+c(\theta-1)| \le 1+c\theta, \quad \theta \in [0,1]$$

The expression in the absolute value becomes

$$1+c(\theta-1) \le 1+c\theta, \quad c \le 2c\theta, \quad \frac{1}{2} \le \theta$$

Therefore by the Von Neumann stability condition, the scheme is stable if $\frac{1}{2} \ge \theta$.

In this case we can say the following about the best combination for Δt, Δx and θ. In order to have both local accuracy and stability, the optimal value of θ is $\frac{1}{2}$ and therefore this scheme represents the Crank-Nicholson scheme. Here Δx and Δt appear in the form $= \frac{\Delta t}{\Delta x^2}$.

In **Figure 1** the convergence plot equation (varying the radio *r*) is

$$\left(I-\frac{\Delta t}{2}A\right)Y^{n+1} = \left(I+\frac{\Delta t}{2}A\right)Y^{n},$$

$$Y^{n+1} = \left(I-\frac{\Delta t}{2}A\right)^{-1}\left(I+\frac{\Delta t}{2}A\right)Y^{n}.$$

with matrix *A* described in the heat equation. We can say the scheme is unconditionally stable. We can see in **Figure 1** that we have a linear convergence with respect to *r*.

3. Three-Level Scheme

We start by computing the stability restriction one has to impose on Δt. We apply Von Neumannstability analysis to the scheme.

Let

$$(1-\gamma)M_x\left(\frac{\Delta u_j^{n+1}}{\Delta t}\right) - \gamma M_x\left(\frac{\Delta u_j^{n}}{\Delta t}\right) - \alpha\left[\beta L_{xx}u_j^{n+1} + (1-\beta)L_{xx}u_j^{n}\right] = 0 \quad (11)$$

where

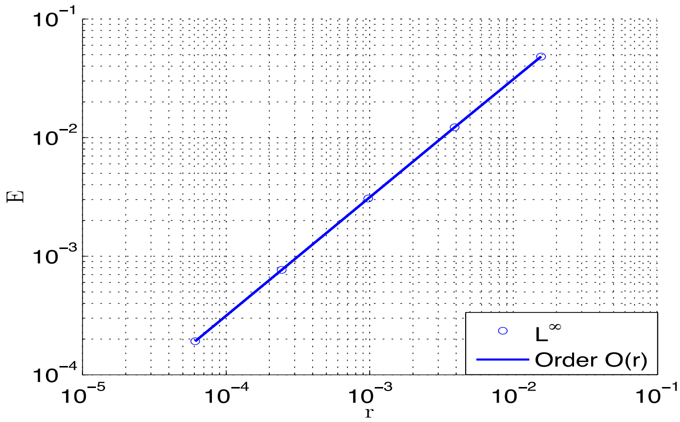

Figure 1. *E* vs. *r* for 1D-heat equation, $u(x,t) = e^{-\pi^2 t}\sin(\pi x)$ with initial temperature $u(x,0) = \sin(\pi x)$ on $[0,1]$.

$$\Delta u_j^{n+1} = u_j^{n+1} - u_j^n, \quad \Delta u_j^{n+1} = u_j^n - u_j^{n-1}, \quad L_{xx} u_j^n = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} \tag{12}$$

and

$$M_x \Delta u_j = \delta \Delta u_{j-1} + (1-2\delta)\Delta u_j + \delta \Delta u_{j+1} \tag{13}$$

By using (12) and (13) we can rewrite (11) as

$$\begin{aligned}
&(1-\gamma)\left(\delta \frac{u_{j-1}^{n+1} - u_{j-1}^n}{\Delta t} + (1-2\delta)\frac{u_j^{n+1} - u_j^n}{\Delta t} + \delta \frac{u_{j+1}^{n+1} - u_{j+1}^n}{\Delta t}\right) \\
&-\gamma\left(\delta \frac{u_{j-1}^n - u_{j-1}^{n-1}}{\Delta t} + (1-2\delta)\frac{u_j^n - u_j^{n-1}}{\Delta t} + \delta \frac{u_{j+1}^n - u_{j+1}^{n-1}}{\Delta t}\right) \\
&-\alpha\left[\beta \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} + (1-\beta)\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\right] = 0,
\end{aligned} \tag{14}$$

or as

$$(1-\gamma) M_x \left(\frac{\Delta u_j^{n+1}}{\Delta t}\right) - \alpha\beta L_{xx} u_j^{n+1} - \gamma M_x \left(\frac{\Delta u_j^n}{\Delta t}\right) + \alpha(\beta - 1) L_{xx} u_j^n = 0$$

The local truncation error for this scheme γ is as follow.

$$\gamma = u_j^{n+1} - U^{n+1}$$

where U^{n+1} represents the exact solution of the heat equation. Therefore we have

$$M_x \left(\frac{\Delta u_j^{n+1}}{\Delta t}\right) = \frac{\gamma}{1-\gamma} M_x \left(\frac{\Delta u_j^n}{\Delta t}\right) + \frac{\alpha}{1-\gamma}\left(\beta L_{xx} u_j^{n+1} + (1-\beta) L_{xx} u_j^n\right) \tag{15}$$

Now expanding M_x operator on the left side, we can isolate the forward difference in time at u_j^n, then

$$\begin{aligned}
(1-2\delta)\frac{u_j^{n+1} - u_j^n}{\Delta t} &= -\delta\left(\frac{u_{j-1}^{n+1} - u_{j-1}^n}{\Delta t} + \frac{u_{j+1}^{n+1} - u_{j+1}^n}{\Delta t}\right) + \frac{\gamma}{1-\gamma} M_x \left(\frac{\Delta u_j^n}{\Delta t}\right) \\
&\quad + \frac{\gamma}{1-\gamma}\left(\beta \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}\right) + (1-\beta)\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\right),
\end{aligned}$$

however,

$$\begin{aligned}
u_j^{n+1} &= u_j^n \Delta t - \frac{\delta \Delta t}{1-2\delta}\left(\frac{u_{j-1}^{n+1} - u_{j-1}^n}{\Delta t} + \frac{u_{j+1}^{n+1} - u_{j+1}^n}{\Delta t}\right) + \frac{\gamma \Delta t}{(1-\gamma)(1-2\delta)} M_x \left(\frac{\Delta u_j^n}{\Delta t}\right) \\
&\quad + \frac{\alpha \Delta t}{(1-\gamma)(1-2\delta)}\left(\beta L_{xx} u_j^{n+1} + (1-\beta) L_{xx} u_j^n\right),
\end{aligned}$$

Expanded this expression becomes

$$\begin{aligned}
u_j^{n+1} &= u_j^n \Delta t - \frac{\delta}{1-2\delta}\left(u_{j-1}^{n+1} - u_{j-1}^n + u_{j+1}^{n+1} - u_{j+1}^n\right) \\
&\quad + \frac{\gamma \Delta t}{(1-\gamma)(1-2\delta)}\left(\delta\left(u_{j-1}^n - u_{j-1}^{n-1}\right) + (1-2\delta)\left(u_j^n - u_j^{n-1}\right) + \delta\left(u_{j+1}^n - u_{j+1}^{n-1}\right)\right) \\
&\quad + \frac{\alpha \Delta t}{(1-\gamma)(1-2\delta)}\left(\beta L_{xx} u_j^{n+1} + (1-\beta) L_{xx} u_j^n\right),
\end{aligned}$$

Finally we have

$$\gamma=\left|u_j^{n+1}-U^{n+1}\right|=\left|u_j^n\Delta t-\frac{\delta}{1-2\delta}\left(u_{j-1}^{n+1}-u_{j-1}^n+u_{j+1}^{n+1}-u_{j+1}^n\right)\right.$$
$$+\frac{\gamma\Delta t}{(1-\gamma)(1-2\delta)}\left(\delta\left(u_{j+1}^n-u_{j+1}^{n-1}\right)+(1-2\delta)\left(u_j^n-u_j^{n-1}\right)+\delta\left(u_{j+1}^n-u_{j+1}^{n-1}\right)\right)$$
$$\left.+\frac{\alpha\Delta t}{(1-\gamma)(1-2\delta)}\left(\beta L_{xx}u_j^{n+1}+(1-\beta)L_{xx}u_j^n\right)-U^{n+1}\right|,$$

4. Stability Criterion for the Three-Level Scheme and Its Accuracy When $\delta=\frac{1}{12}$ and $\beta=\frac{1}{2}+\gamma$

By using Equation (14) we have

$$(1-\gamma)\left(\frac{1}{12}\frac{u_{j-1}^{n+1}-u_{j-1}^n}{\Delta t}+\frac{5}{6}\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{1}{12}\frac{u_{j+1}^{n+1}-u_{j+1}^n}{\Delta t}\right)-\gamma\left(\frac{1}{12}\frac{u_{j-1}^n-u_{j-1}^{n-1}}{\Delta t}+\frac{5}{6}\frac{u_j^n-u_j^{n-1}}{\Delta t}+\frac{1}{12}\frac{u_{j+1}^n-u_{j+1}^{n-1}}{\Delta t}\right)$$
$$-\alpha\left[\left(\frac{1}{2}+\gamma\right)\left(\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\Delta x^2}+\left(\frac{1}{2}-\gamma\right)\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2}\right)\right]=0,$$

Now applying Von Neumann stability again, the aim is to use $u(x,t_{n-1})=\mathrm{e}^{ikx}$, $G(k)=\frac{u_j^n}{u_j^{n-1}}$ and $u_{j\pm1}^{n-1}=\mathrm{e}^{ikx\pm ik\Delta x}$, therefore

$$\frac{(1-\gamma)\mathrm{e}^{ikx}G(G-1)}{\Delta t}\left(\frac{1}{12}\mathrm{e}^{-ik\Delta x}+\frac{5}{6}+\frac{1}{12}\mathrm{e}^{ik\Delta x}\right)-\frac{\gamma\mathrm{e}^{ikx}(G-1)}{\Delta t}\left(\frac{1}{12}\mathrm{e}^{-ik\Delta x}+\frac{5}{6}+\frac{1}{12}\mathrm{e}^{ik\Delta x}\right)$$
$$=\frac{\alpha}{\Delta x^2}\left[\left(\frac{1}{2}+\gamma\right)\left(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}\right)+\left(\frac{1}{2}-\gamma\right)\left(u_{j+1}^n-2u_j^n+u_{j-1}^n\right)\right],$$

Multiplying both sides by $\frac{\Delta t}{\mathrm{e}^{ikx}}$ and write $r=\frac{\Delta t}{\Delta x^2}$ we obtained

$$(1-\gamma)\left(G^2-G\right)\left(\frac{1}{12}\mathrm{e}^{-ik\Delta x}+\frac{5}{6}+\frac{1}{12}\mathrm{e}^{ik\Delta x}\right)-\gamma(G-1)\left(\frac{1}{12}\mathrm{e}^{-ik\Delta x}+\frac{5}{6}+\frac{1}{12}\mathrm{e}^{ik\Delta x}\right)$$
$$=\frac{\alpha r}{\mathrm{e}^{ikx}}\left[\frac{1}{12}\left(u_{j+1}^{n+1}+u_{j+1}^n-2\left(u_j^{n+1}+u_j^n\right)+u_{j-1}^{n+1}+u_{j+1}^n\right)+\gamma\left(u_{j+1}^{n+1}-u_{j+1}^n-2\left(u_j^{n+1}-u_j^n\right)+u_{j-1}^{n+1}-u_{j-1}^n\right)\right]$$
$$=\alpha r\left[\frac{G(G+1)}{2}\left(\mathrm{e}^{ik\Delta x}-2+\mathrm{e}^{-ik\Delta x}\right)+\gamma G(G-1)\left(\mathrm{e}^{ik\Delta x}-2+\mathrm{e}^{-ik\Delta x}\right)\right]$$
$$=\alpha r\left(\mathrm{e}^{ik\Delta x}-2+\mathrm{e}^{-ik\Delta x}\right)\left[G^2\left(\frac{1}{2}+\gamma\right)-G\left(\frac{1}{2}-\gamma\right)\right],$$

Using the cosine identity that $2\cos x=\mathrm{e}^{ik\Delta x}+\mathrm{e}^{-ik\Delta x}$ we have

$$G^2\left(\left(\cos(k\Delta x)+5\right)(1-\gamma)+12\alpha r\left(1-\cos(k\Delta x)\right)\left(\frac{1}{2}+\gamma\right)\right)$$
$$+G\left(-\gamma\left(\cos(k\Delta x)+5\right)+12\alpha r\left(1-\cos(k\Delta x)\right)\left(\frac{1}{2}-\gamma\right)\right)+\gamma\left(\cos(k\Delta x)+5\right)=0$$

We have a quadratic equation in G, where $G^2<1$, therefore

$$\left(\cos(k\Delta x)+5\right)(1-\gamma)+12\alpha r\left(1-\cos(k\Delta x)\right)\left(\frac{1}{2}+\gamma\right)$$
$$+G\left(-\gamma\left(\cos(k\Delta x)+5\right)+12\alpha r\left(1-\cos(k\Delta x)\right)\left(\frac{1}{2}-\gamma\right)\right)+\left(\cos(k\Delta x)+5\right)\le 0,$$

After some cancellations, we can write

$$0\le \cos(k\Delta x)+5+12\alpha r\left(1-\cos(k\Delta x)\right)\beta \le G\left(\gamma\left(\cos(k\Delta x)+5\right)+12\alpha r\left(1-\cos(k\Delta x)\right)(1-\beta)\right)$$

Here, if all $\beta,\ \gamma\in[0,1]$ we need

$$\left(\gamma\left(\cos(k\Delta x)+5\right)\right)>12\alpha r\left(1-\cos(k\Delta x)\right)(1-\beta)$$

$$r<\frac{\left(\gamma\left(\cos(k\Delta x)+5\right)\right)}{12\alpha\left(1-\cos(k\Delta x)\right)(1-\beta)}$$

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