Measuring Power
Power Distribution in Weighted Voting Systems

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1 Abstract

Weighted voting systems are used for games in which the players of the game are not equal. They are often used for votes between shareholders of a corporation, or in representation of the municipalities of a county, as a substitute for redistricting. But how should these weights be assigned in order to be true to the power of each player? How many votes should be required for a win? If weights are assigned based on the fraction of shares or population that a player holds, will this yield the desired distribution of power? The Banzhaf Index of Power indicates that weighted voting systems are not so obvious. And so the question becomes, how can a system be created which will give the desired power distribution?
2 Introduction

Gladwell Toys Inc. is a small, fictitious corporation with a Board of Directors comprised of three shareholders. These board members vote to decide on issues which affect the company. There are exactly one hundred shares, held by the board members as follows: Jerry holds fifty shares, Larry holds forty-nine shares, and George holds just one share. When the board members cast their votes, we would like Jerry to be the most powerful of the three members and George to be the least powerful. More specifically, we would like Jerry to have 50 percent of the power, Larry to have 49 percent, and George to have one percent. A weighted voting system takes into account discrepancies among voters by assigning each voter a weight, which corresponds to the number of votes that he holds. It seems logical then, that by assigning weights to the members of the board based directly on the number of shares that they hold, we should arrive at the desired power distribution.

In 1965, John F. Banzhaf, a professor of law at George Washington University, published a paper titled Weighted Voting Doesn’t Work: A Mathematical Analysis. In the paper, Banzhaf argues that voting power is in fact not proportional to the number of votes that a player holds in a weighted voting system and instead he proposes that power be measured by a voter’s ability to affect the outcome of a vote.

With this, he introduced a method of measuring power distributions, known as the Banzhaf index of power, the study and use of which yield many surprising and important results for the efficacy of weighted voting. And thus the problem for which weighted voting was intended to solve must be reintroduced; that is, given a desired distribution of power among a set of voters, how can a system be defined in order for that distribution to be established?

A perfect example is given by the weighted voting system of the 1965 Nassau County Board of Supervisors. In fact, this is the example used by Banzhaf to support his claim that weighted voting doesn’t work. The six members of the board had weights as follows.

<table>
<thead>
<tr>
<th>District</th>
<th>Number of Votes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hempstead 1</td>
<td>31</td>
</tr>
<tr>
<td>Hempstead 2</td>
<td>31</td>
</tr>
<tr>
<td>Oyster Bay</td>
<td>28</td>
</tr>
<tr>
<td>North Hempstead</td>
<td>21</td>
</tr>
<tr>
<td>Long Beach</td>
<td>2</td>
</tr>
<tr>
<td>Glen Cove</td>
<td>2</td>
</tr>
</tbody>
</table>

The number of votes assigned to each district was proportional to the population of their municipalities and 58 votes were needed to pass an issue. However, using the Banzhaf index of power, Hempstead 1, Hempstead 2, and Oyster Bay each held one-third of the power, leaving the remaining three districts, North Hempstead, Long Beach, and Glen Cove completely powerless. So then what weights should be assigned to each district so that the power distribution in the system reflects the fraction of the total population that each district represents? Can this even be achieved with weighted voting?
This is known as the "inverse problem"; finding a solution to the problem requires a solid understanding of weighted voting systems, and the important differentiations between the Banzhaf indices and the fractional total votes allocated to the players in a system.

3 Weighted Voting Systems

In general, a weighted voting system refers to a yes-no voting system in which the voters are deciding on a single issue. Each voter, or player, \( v_1, v_2, v_3, \ldots, v_n \) (where \( n \) is the number of voters in the system) is assigned a weight, \( w_i \) (where \( w_i \) refers to the number of votes given to voter \( i \)). The quota is the minimum number of votes required to "pass" an issue.

The notation used to describe a system with \( n \) voters and quota \( q \) is

\[ [q : w_1, w_2, \ldots, w_n] \]

where typically \( w_1 \geq w_2 \geq \ldots \geq w_n \).

For example, the system of three players with votes 50, 49, and 1; and quota of 51 is denoted

\[ [51 : 50, 49, 1] \]

where voter \( v_1 \) has weight 50, voter \( v_2 \) has weight 49, and voter \( v_3 \) has weight 1.

A collection of voters is called a coalition. A coalition is winning if the sum of the weights of each voter in the coalition is at least quota. A coalition is losing if the sum is strictly less than quota.

So, for the system \([51: 50, 49, 1]\) there are exactly eight possible coalitions of voters, corresponding to all of the subsets of the set \( \{v_1, v_2, v_3\} \).

\[
\begin{align*}
C_1 & = \emptyset & C_5 & = \{v_1, v_2\} \\
C_2 & = \{v_1\} & C_6 & = \{v_1, v_3\} \\
C_3 & = \{v_2\} & C_7 & = \{v_2, v_3\} \\
C_4 & = \{v_3\} & C_8 & = \{v_1, v_2, v_3\}
\end{align*}
\]

Table 1: Coalitions of \([51: 50, 49, 1]\)

Table 1 gives all eight of the coalitions for \([51: 50, 49, 1]\). The winning coalitions are \( C_5, C_6, \) and \( C_8 \) with total weights of 99, 51 and 100 votes, respectively. Each of the remaining coalitions is losing as the sum of the weights of the players in each coalition is strictly less than quota.

Conventionally, the term yes-no voting system implies that the empty coalition (the set that has no voters; \( C_1 \) in the example above) is always losing and the grand coalition (the set of all voters; \( C_8 \) in the example above) is always winning. That is,

\[ 0 < q \leq w_1 + w_2 + w_3 + \ldots + w_n = \sum_{i=1}^{n} w_i \]
4 Measuring Power

There are several methods for measuring the power of a voter in a weighted voting system including, most notably, the Shapley-Shubik, Banzhaf, Johnston, and Deegan-Packel indices. Each of these computations uses the set of possible coalitions for a given system and the notion of "critical" or "swing" players to determine the power of a voter in a system. The Shapley-Shubik index uses the assumption that a voter is committed to a coalition once they have joined; the Banzhaf index is more accurate for systems where the voters can join and leave coalitions at will. That is, when the order in which voters join coalitions does not matter. The emphasis here will be on the Banzhaf index of power.

4.1 The Banzhaf Index of Power

At first glance, one might expect that the fractional power of a voter in a weighted voting system is equivalent to the fraction of the total number of votes that a voter holds. So for a system in which each voter has a different number of votes, this equivalence would indicate that the voter with the greatest number of votes will have the largest fractional power, and the voter with the least number of votes will have the smallest fractional power, establishing a hierarchy among the voters. The Banzhaf index of power however, uses a player’s ability to affect the outcome of the system as a determination of their power.

In order to compute the index of a player, it is necessary to first determine the number of winning coalitions for which that player is "critical", which is referred to as the Total Banzhaf Power. More specifically, the Total Banzhaf Power of a voter, \( v_i \), denoted \( TBP(v_i) \), is the number of coalitions, \( C \), for which:

1. \( v_i \) is a member of \( C \)
2. \( C \) is winning, and
3. if \( v_i \) is deleted from \( C \), \( C \) is no longer winning

The Banzhaf index of power of a player, \( v_i \), denoted by \( BI(v_i) \), in a system with \( n \) players is then given by

\[
BI(v_i) = \frac{TBP(v_i)}{TBP(v_1) + \ldots + TBP(v_n)} = \frac{TBP(v_i)}{\sum_{j=1}^{n} TBP(v_j)}
\]  

(1)

This is the fraction of total critical instances for which \( v_i \) is critical. So \( 0 \leq BI(v_i) \leq 1 \), and \( \sum_{i=1}^{n} BI(v_i) = 1 \).

Let us return to the system \([51 : 50, 49, 1]\) as an example. In terms of weights, \( v_1 \) holds one-half of the total number of votes, \( v_2 \) holds forty-nine-one-hundredths, and \( v_3 \) holds one-one-hundredth.

In Section 3 we determined that there are exactly three winning coalitions:

\[
C_1 = \{v_1, v_2\} \quad C_2 = \{v_1, v_3\} \quad C_3 = \{v_1, v_2, v_3\}
\]
The total weight of $C_1$, $W_{C_1}$ is
\[ W_{C_1} = w_1 + w_2 = 50 + 49 = 99 \]
The total weight of $C_2$ is
\[ W_{C_2} = w_1 + w_3 = 50 + 1 = 51 \]
The total weight of $C_3$ is
\[ W_{C_3} = w_1 + w_2 + w_3 = 50 + 49 + 1 = 100 \]
Suppose $v_1$ is deleted from $C_1$; then $C_1$ becomes $\{v_2\}$, with a total weight of 49. So $C_1$ is no longer winning and thus $v_1$ is critical to $C_1$. Now suppose $v_1$ is deleted from $C_2$; then $C_2$ becomes $\{v_3\}$, with a total weight of 1. So $C_2$ is no longer winning and $v_1$ is critical to $C_2$. Lastly, suppose $v_1$ is deleted from $C_3$; then $C_3$ becomes $\{v_2, v_3\}$, with a total weight of 50. So $C_3$ is no longer winning and $v_1$ is critical to $C_3$. Then $\text{TBP}(v_1) = 3$.

Now consider $v_2$, who is only a member of two of the three winning coalitions. If $v_2$ was deleted from $C_1$, $C_1$ becomes $\{v_1\}$, with total weight of 50. So $C_1$ is no longer winning and $v_2$ is critical to $C_1$. If $v_2$ is deleted from $C_3$ however, $C_3$ becomes $\{v_1, v_3\}$, which is one of the winning coalitions, and thus $v_1$ is not critical to $C_3$. Then $\text{TBP}(v_2) = 1$.

Similarly, $v_3$ is critical to $C_2$, but not to $C_3$, so $\text{TBP}(v_3) = 1$.

Then the indices of the players are
\[ BI(v_1) = \frac{3}{3 + 1 + 1} = \frac{3}{5} \]
\[ BI(v_2) = \frac{1}{5} \]
\[ BI(v_3) = \frac{1}{5} \]

According to the Banzhaf measurement, $v_2$, the player with 49 votes has fractional power equivalent to that of $v_3$ the player with only 1 vote. Despite the fact that the former holds 49 times as many votes as the latter, each of their abilities to affect the outcome of the system is exactly the same, and thus they have equal power.

### 4.2 Alternative Method for Computing Banzhaf Power

A second method for calculating the Banzhaf index of power uses a player’s membership in winning coalitions to deduce when a player must be critical. The procedure assigns each voter one "point" for each winning coalition for which that voter is a member. A point is subtracted for winning coalitions for which that player is not a member. The sum of these points yields the total Banzhaf power of each voter.

Consider the same system of three voters $[51: 50, 49, 1]$.

In order from least total weight to greatest, the winning coalitions are
\[ \{v_1, v_3\}, \{v_1, v_2\}, \{v_1, v_2, v_3\} \]
Table 2 shows the assignment of points for this system. The results for the Total Banzhaf Powers are the same as those computed in Section 4.1. To understand why this procedure yields the desired result, consider a single voter from the system, $v_i$. Separate the winning coalitions into three groups, A, B, and C, defined by:

A: Winning Coalitions for which $v_i$ is *not* a member.

B: Coalitions from A with $v_i$ added to them.

C: The remaining winning coalitions.

For [51: 50, 49, 1], these sets of coalitions for $v_2$ are

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>${v_1, v_3}$</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>B</td>
<td>${v_1, v_2, v_3}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>${v_1, v_2}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>TBP</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2: Procedure for awarding points for [51: 50, 49, 1]

Because the coalitions in A are winning, the total weight of these coalitions, $W_A$, meet or exceed quota. That is,

$$\sum_{j \neq i} w_j \geq q$$

Since $v_i$ is not a member of the coalitions of A, he is not critical to any C in A. Clearly, any coalition from A will continue to be winning if $v_i$ is added to the coalition, so then $v_i$ will not be critical for any of the coalitions in B. Also note that the number of coalitions in B will be exactly the same as the number of coalitions in A, since the coalitions in B are defined by

$$C_B = \{C_A \cup v_i\}$$

The coalitions in B are winning coalitions for which $v_i$ is a member. Then if $|B| = n$, $n$ points are given to $v_i$. The coalitions in A are winning without $v_i$, so a point is subtracted from $v_i$’s score for every coalition in A. And, because $|A| = |B|$, there will be $n$ points subtracted from $v_i$’s score. So the points assigned for A and B will offset one another, leaving only the points assigned for the coalitions in C.

Now consider C. Any coalition in C, $C_C$, must have $v_i$ as a member. If not, then that coalition would be in A. So then for each $C_C$, $v_i$ is given one point. These points are the only points that remain after A and B offset one another, thus
\[ TBP(v_i) = |C| \]

By definition, \( v_i \) must be critical for every coalition in \( C \), or else \( C \) is winning without \( v_i \) and belongs in \( C \). By determining the weights of the coalitions in \( C \), \( W_C \), we can make some generalizations about the conditions for which a voter \( v_i \) is critical to a coalition \( C \).

Since the coalitions in \( C \) are winning, \( W_C \) meets or exceed the quota. We know the coalitions in \( C \) must be losing when \( v_i \) is deleted (if not, then the coalition belongs in \( B \)). So

\[ W_C - w_i < q \]

and, because \( W_C \geq q \), we have

\[ q - w_i \leq W_C - w_i < q \]

Now, let \( C' \) be the set of coalitions from \( C \) with \( v_i \) taken away. Then we have

\[ q - w_i \leq W'_C < q \]

(2)

Thus in any yes-no voting system, a voter \( v_i \) will be critical for exactly the number of coalitions for which both \( v_i \) is not a member and the inequality in 2 is satisfied.

5 Generating Function

A generating function, \( f(x) \), is a formal power series

\[ f(x) = \sum_{n=0}^{\infty} a_n x^n \]

for which the evaluation at particular values of \( x \) are ignored, and the coefficients of the function yield information about a sequence of numbers. The notation \([x^n] f(x)\) refers to the coefficient \( a_n \) in \( f \).

The generating function for Banzhaf powers is given by

\[ B(x) = (1 + x^{w_1}) \cdot (1 + x^{w_2}) \cdot ... \cdot (1 + x^{w_N}) = \prod_{i=0}^{N} (1 + x^{w_i}) \]  

(3)

where \( N \) is the number of voters in the system. Then each coefficient \([x^n] B(x)\) gives the number of coalitions with total weight \( n \). The generating function produces the total weights of every possible coalition of a system, eliminating the need to explicitly determine every possible coalition of voters for a system.

Even further, by eliminating the term in \( B(x) \) that corresponds to voter \( v_i \) (i.e. the \( i^{th} \) term of the product) from \( B(x) \), we obtain a function whose coefficients yield the number of coalitions with total weight \( n \) that do not include voter \( v_i \). That is, the function

\[ \frac{B(x)}{(1 + x^{w_i})} = (1 + x^{w_1}) \cdot ... \cdot (1 + x^{w_{i-1}}) \cdot (1 + x^{w_{i+1}}) \cdot ... \cdot (1 + x^{w_N}) \]  

(4)
has coefficients \([x^n]\frac{B(x)}{(1+x^{w_i})}\) that give the number of coalitions with total weight \(n\) for which voter \(v_i\) is not a member.

With this, the number of coalitions for which a voter \(v_i\) will be critical is easily determined.

The total Banzhaf power of \(v_i\) is the number of coalitions that do not include \(v_i\) for which the inequality (2) in Section 4.2 is satisfied. Then, returning to the generating function, to determine the total Banzhaf power of \(v_i\), we need only look at the coefficients of \(\frac{B(x)}{(1+x^{w_i})}\) for those powers of \(x\) that satisfy (2). That is,

\[
TBP(v_i) = [x^{q-w_i}] \frac{B(x)}{1 + x^{w_i}} + ... + [x^{q-1}] \frac{B(x)}{1 + x^{w_i}} \tag{5}
\]

For example, consider the system \([3: 2, 2, 1]\)

To confirm Equation 5 we compute the total Banzhaf powers of each voter in the system using the method from Section 4.1 and compare this result to the total Banzhaf powers given by using the generating function.

<table>
<thead>
<tr>
<th>(C_k)</th>
<th>(W_{C_k})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_0 = \emptyset)</td>
<td>0</td>
</tr>
<tr>
<td>(C_1 = {v_3})</td>
<td>1</td>
</tr>
<tr>
<td>(C_2 = {v_1})</td>
<td>2</td>
</tr>
<tr>
<td>(C_3 = {v_2})</td>
<td>2</td>
</tr>
<tr>
<td>(C_4 = {v_1, v_3})</td>
<td>3</td>
</tr>
<tr>
<td>(C_5 = {v_2, v_3})</td>
<td>3</td>
</tr>
<tr>
<td>(C_6 = {v_1, v_2})</td>
<td>4</td>
</tr>
<tr>
<td>(C_7 = {v_1, v_2, v_3})</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 3: All coalitions, \(C_k\), with total weights \(W_{C_k}\) for the system \([3: 2, 2, 1]\).

Table 3 gives all of the possible coalitions for the system, and the corresponding total weights. The table shows that \(C_4, C_5, C_6,\) and \(C_7\) are the winning coalitions, as \(W_{C_4}, W_{C_5}, W_{C_6}, W_{C_7} \geq 3 = q\).

Subtracting \(w_1\) from each of the winning coalitions for which \(v_1\) is a member, \(v_1\) is critical for \(C_4\) and \(C_6\).

Similarly, \(v_2\) is critical for \(C_5\) and \(C_6\) and \(v_3\) is critical for \(C_4\) and \(C_5\).

So, the total Banzhaf powers for the players in this system are

\[TBP(v_1) = 2\quad TBP(v_2) = 2\quad TBP(v_3) = 2\]

Now, using the generating function given in Equation 3 we have

\[
B(x) = \prod_{i=1}^{3} (1 + x^{w_i})
= (1 + x^2)(1 + x^2)(1 + x^1)
= 1 + x + 2x^2 + 2x^3 + x^4 + x^5
\tag{6}
\]
Summing the coefficients of each term of the polynomial given in 6, it is determined that there are eight possible coalitions for this system. More specifically, there is one coalition with total weight zero, one with total weight one, two with total weight two, two with total weight three, one with total weight four, and one with total weight five. These numbers agree with Table 3, which was created by explicitly writing every subset of the set of voters to determine all of the possible coalitions.

Now, to determine the total Banzhaf power of each player, we first produce \( B(x) \) for each player \( v_i \).

\[
\frac{B(x)}{1 + x^{w_1}} = (1 + x^2)(1 + x^1) = 1 + x^2 + x^3 \quad (7)
\]

\[
\frac{B(x)}{1 + x^{w_2}} = (1 + x^2)(1 + x^1) = 1 + x^2 + x^3 \quad (8)
\]

\[
\frac{B(x)}{1 + x^{w_3}} = (1 + x^2)(1 + x^2) = 1 + 2x^2 + x^4 \quad (9)
\]

Finally, using Equation 5, and the polynomials given in 7, 8, and 10, the total Banzhaf power of each player is

\[
TBP(v_1) = \sum_{i=3-1}^{3-2} [x^i] \frac{B(x)}{1 + x^{w_1}} = [x] \frac{B(x)}{1 + x^{w_1}} + [x^2] \frac{B(x)}{1 + x^{w_1}} = 1 + 1 = 2
\]

\[
TBP(v_2) = \sum_{i=3-2}^{3-2} [x^i] \frac{B(x)}{1 + x^{w_2}} = [x] \frac{B(x)}{1 + x^{w_2}} + [x^2] \frac{B(x)}{1 + x^{w_2}} = 1 + 1 = 2
\]

\[
TBP(v_3) = \sum_{i=3-1}^{3-1} [x^i] \frac{B(x)}{1 + x^{w_3}} = [x^2] \frac{B(x)}{1 + x^{w_3}} = 2
\]

These results do indeed agree with the results of the first computation which required producing all of the critical instances. By using the generating function, the Banzhaf power can be computed without this intermediate step.

Also, notice from Equations 4 and 5, and from the example above, that two voters, \( v_i \) and \( v_j \), with the same weight will always have the same total Banzhaf power; 4 will yield the same polynomials for both voters and Equation 5 will give the same TBP for both voters. This may seem like a trivial result; however, it is not obvious from Section 4.1. With the generating function this conclusion has become explicit.

6 Distribution of Power

With the information presented thus far, we begin to gather some results for the possible power distributions of weighted voting systems.

Assume that in every weighted voting system, \( w_i > 0 \) for each \( v_i \).

Consider a system with \( n \) voters and quota such that

\[
q = w_1 + \ldots + w_n = \sum_{i=1}^{n} w_i
\]
Then there is exactly one winning coalition: the coalition with every voter as a member. Clearly, every voter will be critical to that coalition. Thus, for every \( v_i \) \( \text{TBP}(v_i) = 1 \). And, from Equation 1

\[
BI(v_i) = \frac{1}{\sum_{i=1}^{n} 1} = \frac{1}{n}
\]

Then for a weighted voting system for which the quota is exactly the sum of the weights of all of the voters, every voter will have exactly the same power, specifically a fractional power of \( \frac{1}{n} \).

Consider the system of five voters \([104: 100, 1, 1, 1, 1]\)

Despite the fact that \( v_1 \) has one-hundred times as many votes as any other voter, each voter, including \( v_1 \), has the same ability to effect the outcome of a vote and therefore each voter has the same Banzhaf index of power \( \frac{1}{5} \).

Now imagine there is a voting system of \( n \) voters for which each voter has exactly the same weight, say \( w \).

Then the generating function is given by

\[
B(x) = (1 + x^w)^n
\]

And for every voter

\[
\frac{B(x)}{(1+x^w)} = \frac{B(x)}{(1+x^w)} = (1 + x^w)^{n-1}
\]

Then the total Banzhaf power of each voter must be the same by Equation 5, and every voter will have a fractional power of exactly \( \frac{1}{n} \).

### 6.1 Veto Power

Imagine that there is a voter \( v_j \) who is a member of every winning coalition; that is, there are no winning coalitions for which \( v_j \) is not a member. Then voter \( v_j \) is said to have veto power, for, any issue for which \( v_j \) casts a "nay" vote can not be passed. Every coalition without \( v_j \) is losing. Thus

\[
\sum_{i \neq j} w_i < q
\]

So \( v_j \) is critical to every winning coalition because if \( v_j \) is deleted from any winning coalition, the coalition becomes losing. Thus the total Banzhaf power of \( v_j \), is given by the number of winning coalitions.

Now assume \( v_j \) is the only voter with veto power in a system.

If \( w_j < q \), then the total Banzhaf power of \( v_j \) must be greater than the total Banzhaf power of every other voter. If not, there is some voter \( v_k \) with total Banzhaf power equal to \( v_j \). Then \( v_k \) would also be critical to every winning coalition. But then \( v_k \) has veto power, contradicting the assumption.
If \( w_j = q \), then the singleton coalition, consisting of only \( v_j \), is winning. That is, the voter has the ability to pass a motion by them self and every other voter \( v_i \) will not be critical to any coalition. Then the total Banzhaf powers of all voters not \( v_j \) will be zero and thus the index of power for \( v_j \) will be one; \( v_j \) holds all of the power and the rest of the voters are called "dummies".

### 6.2 Dummy Voters

In the previous section, the idea of dummy voters (voters with no fractional power) was introduced by looking at a weighted voting system in which there is a player whose weight is equal to quota.

Now, we make the claim that there need not be a voter with \( w_i = q \) for a system to have dummy voters.

Consider the weighted voting system of the 1965 Nassau County Board of Directors.

\[
[58: 31, 31, 28, 21, 2, 2]
\]

Notice

\[
\sum_{i=3}^{6} w_i = 28 + 21 + 2 + 2 = 53 < 58
\]

This says that the coalition consisting of all of the voters beside \( v_1 \) and \( v_2 \) is losing. And, since adding one of \( v_1 \) or \( v_2 \) will make the coalition a winning one, at least one of \( v_1 \) or \( v_2 \) must be a member of every winning coalition. Because \( w_1 = w_2 \), \( v_1 \) and \( v_2 \) will be critical for the exact same number of coalitions.

Clearly, the grand coalition consisting of all of the voters is winning, as

\[
\sum_{i=1}^{6} w_i = 115
\]

We know that there are no singleton winning coalitions, as no voter has \( w_i = q \).

The smallest total weight of any of the winning coalitions is 59 (notice that there is no combination of voters that gives a total weight of 58). These coalitions are \( \{v_1, v_3\} \), and \( \{v_2, v_3\} \). The smallest total weight of the winning coalitions for which \( v_3 \) is not a member is 62, corresponding to the coalition \( \{v_1, v_2\} \).

As we have determined that at least one of \( v_1 \) or \( v_2 \) must be a member of every winning coalition, these three coalitions, \( \{v_1, v_3\}, \{v_2, v_3\} \) and \( \{v_1, v_2\} \), must be the only two-player winning coalitions.

Now, consider all of the winning coalitions that have the form

\[
\{\{v_1, v_3\} + \{\text{any subset of } \{v_4, v_5, v_6\}\}\}
\]

Computing the Banzhaf index, we get a power distribution as follows:

\[
\text{BI}(v_1) = \text{BI}(v_2) = \text{BI}(v_3) = \frac{1}{3}
\]

\[
\text{BI}(v_4) = \text{BI}(v_5) = \text{BI}(v_6) = 0
\]
6.3 Power Distribution in n-Player Systems

In order to begin answer the inverse problem, we first consider which power distributions, if any, will be impossible, by determining all of the distributions that are possible.

We will require some conditions on the systems being examined. First, assume that the quota is greater than strict majority. That is

\[
\frac{1}{2} \sum_{i=1}^{n} w_i < q \leq \sum_{i=1}^{n} w_i
\]

Although it is not critical to the proof that follows, the reader should note that with this restriction on quota, the complement of a winning coalition is losing. Also, we stipulate that every voter has weight less than quota, and that no voter has veto power.

With these conditions, determining the possible power distributions in a system with three players is relatively simple.

Consider all of the possible winning coalitions in a game with three players. Because \( w_i < q \) for each \( v_i \), we know that there will not be any singleton winning coalitions. However, because no player has veto power, for every voter \( v_i \), it must be true that

\[
\sum_{j \neq i} w_j \geq q
\]

Then every two-player coalition must be winning.

The set of consisting of all three voters is always winning by the assumption made for quota.

Then \( \{v_1, v_2\} \), \( \{v_1, v_3\} \), \( \{v_2, v_3\} \), and \( \{v_1, v_2, v_3\} \) are the coalitions with total weight meeting or exceeding quota.

No voter will be critical for \( \{v_1, v_2, v_3\} \), since deleting any voter results in some two-player coalition, which we know to be winning.

In each of the two-player coalitions, if one of the players is deleted, the result is some singleton set, which we know to be losing. So each voter will be critical to exactly the number of two-player coalitions for which it is a member, which is two for every voter. Therefore the total Banzhaf power for \( v_1 \), \( v_2 \) and \( v_3 \) is two. Consequently,

\[
BI(v_1) = BI(v_2) = BI(v_3) = \frac{2}{6} = \frac{1}{3}
\]

(10)

This completes the proof that for every three-player weighted voting system, in the absence of veto power there is only one possible power distribution, given by Equation 10.

Figure 1 illustrates all of the potential power distributions that may be desired for a three player system by the plane \( x + y + z = 1 \) in three-space. The only distribution that can be achieved lies at \( (x, y, z) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \).

A similar, but slightly more involved procedure proves that there are only five possible power distributions for four-player weighted voting systems with no veto power, none of which establish a strict hierarchy of power among the voters. That is, with no veto power, there is no four-player system for which:

\[
BI(v_1) > BI(v_2) > BI(v_3) > BI(v_4)
\]
In the case of five-player systems, there are exactly 35 possible distributions.
So for an \( n \)-player system, there are finitely many possible power distributions, and consequently, distributions that are not possible.

Returning to the inverse problem, this result says that given a desired power distribution, it will not always be possible to construct a system which produces the desired outcome. So then the question must be amended. We must now ask: how can a system be manipulated in order to produce an outcome of the smallest deviation from the desired distribution?

7 Adjusting a Weighted Voting System

Using the Banzhaf analysis for the purpose of determining the validity of a weighted voting system is a relatively simple process. For systems with several players, the computation of the power indices is a trivial task for a computer (see Appendix for Mathematica code). The operation in reverse however, has proved a very complicated problem.

For example, consider a system in which the power index of the "largest" player, the player with the greatest weight, is above the target; and the "smallest" player has a power index below the target. That is, the largest player is too powerful, and the smallest player is not powerful enough. It seems logical to examine the effect of decreasing the weight of the largest player, to try to bring the power distribution closer to the target.

Figures 2 and 3 show the effect of decreasing the weight of the largest player on the power of the second largest player and the smallest player, respectively, in a twelve-player system. Notice that the power of the second largest player seems to increase almost linearly as the weight of the largest player decreases. However, the effect that this change has on the smallest player is certainly not approximated by a linear model. In fact, it is difficult to recognize any kind of pattern in the graph in Figure 3.

We can also examine the effect of increasing the weight of the smallest player on the same system as a method of decreasing the deviation between the actual Banzhaf indices and the desired power distribution.
Figures 4 and 5 show the effect of increasing the weight of the smallest player on the power of the largest player and the second smallest player, respectively, in the same twelve-player system. Here as the smallest player gains weight, the largest player generally loses power, which is the effect we hope to see. However, notice again that this change has a seemingly unpredictable effect on the power of the player with the second smallest weight.

The results of both methods on their own did not yield, not the whole, the desired result. Then let us appraise the method of changing quota as a tool to adjust the powers of the players in the system.

Figures 6 and 7 illustrate the effect of increasing quota on the power of the smallest and largest player, respectively. Here, the change in power for both the smallest and largest player seems to be unpredictable. Clearly changing quota is also not an effective method of refining the system. It seems that the solution to this problem will not be a one-dimensional. That is, it seems that adjusting just a single-variable of a given system will not be a sufficient or effective solution.
The reader should note that the graphs shown in Figures 2, 3, 4, and 5 are unique to the weighted voting system used to generate these Banzhaf indices, and a system with different weights, quota and/or size will yield different graphs. However, the example used above illustrates the complexity of a general system and the complexity of the task to improve the effectiveness of a system.

There are, of course, more involved methods of adjusting a system which change more than one variable at once and uses the deviation of the actual power indices from the desired power distribution to hypothetically output an improved system. However, without knowing what power distribution the algorithm should actually produce, we cannot know that the program is effective. That is, for systems with more than five players, as we do not have a comprehensive set of points in n-space that make up the possible power distributions for n-players, we cannot be absolutely sure that the output of an algorithm is in fact the point from that set lies closest to the target point.

8 Conclusion

When first asked to create a weighted voting system for the three shareholders, Jerry, Larry, and George, introduced in Section 2, there seems to be an obvious solution. This task, hopefully, now appears much more complicated than this initial assessment suggests. There are several directions to explore in continuance of the search for a solution to the problem that this discussion has posed.

From Section 7 a logical next step might be finding a method to generate all of the possible power distributions for a system of a given size. In order to do so, we need to find all of the possible sets of winning coalitions. An algorithm for this task is presented in a paper titled Using Sets of Winning Coalitions to Generate Feasible Banzhaf Power Distributions. The algorithm makes use of inequality lattices to relate coalitions of a given size.

For example, in a five-player game, the inequality lattice for the two-player coalitions is shown in Figure 8 where the vertices represents coalitions and edges represents a less-than-or-equal-to relationship between two coalitions. The graph is read left to right and
top to bottom, so that the vertex on the top left will have the greatest weight, and the vertex on the bottom right will have the smallest weight. By explicitly defining these relationships, when given a winning coalition size \( k \), we can determine which of the remaining \( k \)-coalitions must be winning. For example, if \( \{2, 4\} \) is a winning coalition, then from Figure 8 we have that \( \{1, 2\}, \{1, 3\}, \{1, 4\}, \) and \( \{2, 3\} \) must be winning coalitions, as they each have weight at least as great as the weight of \( \{2, 4\} \).

A separate, but related extension of this study is to examine the effect that voting blocs have on a weighted voting system. More specifically, how do the power dynamics of a system change as voters group together to form voting blocs.

Figure 9: Predictions for the political party affiliation of each member of the Electoral College for the 2012 presidential election. Blue states indicate democratic votes, red states indicate republican votes, and yellow states indicate undecided states. (Figure taken from http://www.cnn.com/election/2012/electoral-map.html)

Figure 9 shows the predicted political party affiliation of each of the 51 members of the Electoral College in the 2012 presidential election. The Electoral College can certainly be examined as a weighted voting system, where the number of electoral votes held by
each member is their weight, and quota is 270.

Figure 10: Percent of the U.S. population vs. the Banzhaf index held by each member of the Electoral College.

In this system, we hope that the population of each member of the Electoral College is directly proportional to the power that the hold. Figure 10 shows the percent of the population each member holds versus the Banzhaf power of that member in the Electoral College, shown by the red points. The blue line represents the target Banzhaf power given a population. How does this dynamic change if we consider the Democratic states as a single player, or as a voting bloc, and the Republican states as a single player, leaving the undecided, or "battle" states as separate players? Now, instead of a 51-player system, we are considering a 17-player system (with one Democratic bloc, one Republican bloc, and 15 battle players). The quota is unchanged at 270. The new system is

\[270 : 196, 159, 29, 20, 18, 16, 15, 13, 11, 11, 10, 10, 9, 6, 6, 5, 4\]

Figure 11 shows the Banzhaf power of each of the 15 battle states in the modified system with voting blocs (shown by the red points). The blue line represents where the points would lie had the power indices of the battle states not changed from those given by the original 51-player system. Notice that the power of each of the battle states increased under the modified system. Even though the weights of each of the battle states is now much smaller with respect to the greatest weight, the ability of each of the battle members to affect the outcome of a decision increased. About 52 percent of the power is distributed among the battle states in this system, whereas only 34 percent of the power was distributed among the same states in the original system. But this means that on the whole, the power of the states who are members of the voting blocs decreased by forming a bloc. At what point when the bloc is gaining members does this decrease occur? Is this decrease unusual for a typical system with voting blocs?
9 Reflection

Although the discussion of weighted voting systems in this paper focuses on power dynamics primarily through mathematical analyses, the study of these systems has obvious political implications. Consider the prevalent example of the Electoral College. Here the voters are the fifty states and the District of Columbia, and the weights are assigned based on each state’s congressional delegation (two senators for each member along with the representatives apportioned based on population). The quota is given by the absolute majority (currently 270 votes). A large system such as the Electoral College requires computations performed by a computer in order to analyze the Banzhaf power dynamics, as the coalition possibilities are immense; however, the mechanics of the analysis are identical to those of the very simple examples illustrated in Section 4. The theme of this paper suggests that the Electoral College does not satisfy the concept of "one man, one vote", as weights are assigned based, in part, on population sizes. In fact, from Figure 10 it seems that smaller states are favored by this method of assigning votes. In particular, states with less than two percent of the total population benefit. But the two Senate votes that are assigned without consideration to population make up a large percentage of these small territories’ electoral votes, so we might expect the power of these members to be inflated by these two bonus votes.

Without the Senate votes however, electoral votes would be assigned based solely on population, which has proven to be an ineffective method of designing a weighted voting system. Figure 12 shows the population of each member of the Electoral College versus the Banzhaf power of that member without the two Senate votes. Again, the blue line indicates the target power index for a population.

However, by applying the Banzhaf analysis to the system, the assumption is made that every voter is as equally likely to cast a "yea" vote as a "nay" vote; a major simplification
which ignores historic political trends. Thus the validity of this method when used to examine power distribution in complex cases such as the Electoral College is highly criticized. But even through this simplified analysis, the results may change perception of the "fairness" of a given system. In 1971, in \textit{Whitcomb v. Chavis}, the Supreme Court ruled that weighted voting violated the Equal Protection Clause of the Fourteenth Amendment and the Banzhaf index was disaffirmed based on the sensitivity of the assumptions necessary to make the calculations. The New York State Court of Appeals took a different stance on the use of the Banzhaf analysis to affirm or deny the validity of a weighted voting system in \textit{Iannucci v. Board of Supervisors}, arguing that the detachment from political trends and histories was a favorable characteristic of the calculation.

Of course the Banzhaf analysis is only one approach to determine the effectiveness of weighted voting systems, and certainly there are immensely many possible deviations from this means of evaluation. In any case, the ability to draw meaningful conclusions when analyzing weighted voting systems requires a firm understanding of what constitutes a valid logic proof. It is important to recognize that making assumptions about the voting systems that we are considering allows us to draw conclusions that would not be possible without these conditions, but we must be cautious to make explicit that these conclusions are only valid when the conditions are satisfied.

There is also a heavy use of set theory in the study of cooperative games. We have seen the importance of a systematic approach to determining the winning coalitions is for both calculating Banzhaf indices, and to find a method of designing weighted voting systems that satisfy a target power distribution.

There is an obvious connection to game theory, and, as mentioned previously, a different method of analysis may take into account the probability that a player will vote a certain way. A rational voter may weigh the benefit of employing one strategy versus
another. Certainly in practice, there may be deals or payoffs made between players in exchange for casting either "yea" or "nay" vote. In the theoretical analysis employed by the Banzhaf index, these complications are ignored. In practice however, these considerations change the dynamics of these systems, and in some cases, these factors make this simple analysis comparatively one-dimensional. However, by expanding our analytical views of weighted voting systems, we are able to recognize the intricacy involved in the process of allocating votes and, certainly, we depart from the original idea that these systems should be behave in an obvious manner. Undoubtedly there is a higher theme here: we limit our understanding of a subject by limiting the manner in which we view that subject.
10 References


A. Cuttler, A. De Guire, S. Rowell, Using Sets of Winning Coalitions to Generate Feasible Banzhaf Power Distributions, Aug., 2005


11 Appendix

Code used to generate Banzhaf indices:

GenBanz [ quota_, weight_] :=
Module[{number = Length[weight], quota1 = quota - 1,
    c = Table[0, {quota + 1}], i, j,
    swings = Table[0, {Length[weight]}], cs, weightT},
(* NOTE c is indexed from 1 to quota+1,
rather than from 0 to quota *)
c[[1]] = 1;
For[ i = 1, i <= number, i++,
    For[ j = quota1, j >= weight[[i]], j--,
        c[[j + 1]] += c[[j + 1 - weight[[i]]]] ]];
For[ j = 1, j <= quota1, j++, c[[j + 1]] += c[[j]]];
For[ i = 1, i <= number, i++,
    cs = -2; weightT = weight[[i]]; weightSum = quota1 - weightT;
    swings[[i]] = c[[quota]];
    While[ weightSum >= 0, swings[[i]] += cs * c[[weightSum + 1]]];
    cs = - cs; weightSum -= weightT ]
];
Return[ swings]
];

Banzhaf[ quota_, weights_] :=
Module[{swings = GenBanz[ quota, weights]},
    Return[ 100.0 * swings / Apply[Plus, swings]]]