LETTER

A Simple Chaotic Spiking Oscillator Having Piecewise Constant Characteristics

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SUMMARY  This paper studies a chaotic spiking oscillator consisting of two capacitors, two voltage-controlled current sources of signum shape and one impulsive switch. The vector field of circuit equation is piecewise constant and embedded return map is piecewise linear. Using the map parameter condition for chaos generation is given. Using a simple test circuit typical phenomena can be confirmed experimentally.

key words: chaos, bifurcation, switched dynamical systems, switched capacitor

1. Introduction

Design of simple chaotic circuits has been studied extensively for both fundamental study and applications. Chaotic circuits are important real physical systems to understand interesting nonlinear phenomena. Also, they are basic building blocks for engineering applications including data encryption, secure communication and EMI improvement [1]–[6]. We have also studied simple chaotic spiking oscillators (CSOs) with integrate-and-fire switch (IFSW) [7], [8]. The CSOs are included in switched dynamical systems having rich bifurcation phenomena [9]. The IFSW is a key nonlinear element in basic neuron models. Using plural neuron models we can construct pulse-coupled neural networks (PCNNs) having applications including associative memories and image segmentation [10]–[12].

This paper presents a simple CSO consisting of two capacitors, two voltage-controlled current sources (VCCSs) of signum shape and one IFSW. The circuit equation has piecewise constant (PWC [13]) vector field and it is well suited for theoretical analysis. The signum VCCSs grow oscillation and the IFSW resets a capacitor voltage to a base level if it reaches a threshold: the VCCSs and IFSW realize stretching and folding mechanism for chaos generation, respectively. The embedded 1D return maps are PWL that can be described explicitly.

Motivations for studying such circuits include at least three points. First, the circuit model is very simple and the implementation is easy. The signum VCCS is almost equivalent to OTA. Second, the IFSW can cause interesting phenomena and relates deeply to neuron models. Our circuit may be developed into PCNNs having rich synchronization phenomena and the phenomena may be applicable to effective signal processing [8], [10]. Third, the circuit equation has PWC vector field and it is well suited for theoretical analysis of chaos and bifurcation phenomena.

2. The Piecewise Constant Chaotic Spiking Oscillator

Figure 1 shows the PWC chaotic spiking oscillator consisting of two VCCSs and the IFSW S where \( r_1 \) and \( r_2 \) denote parasitic resistors. Two VCCSs have signum characteristics:

\[
\begin{align*}
I_1 &= I_1 \text{sgn}(v_1) \\
I_2 &= I_2 \text{sgn}(v_1 - v_2)
\end{align*}
\]

As a preparation to define operation of \( S \) we consider dynamics in the case where the \( S \) is open all the time. For simplicity, we assume that \( r_1 \) and \( r_2 \) are large enough and open them. In this case the circuit dynamics is described by

\[
\begin{align*}
\frac{dv_1}{dt} &= I_2 \text{sgn}(v_1 - v_2), \\
\frac{dv_2}{dt} &= I_1 \text{sgn}(v_1)
\end{align*}
\]

For simplicity we assume the following conditions:

\[ I_1 > 0, \quad I_2 > 0, \quad C_1 I_1 > C_2 I_2 > 0. \]

Equation (2) defines PWL vector fields divided by two border lines \( v_1 = 0 \) and \( v_2 = v_1 \). This PWL vector field gives PWL trajectories and PWL return map. As shown in Fig. 2, we consider the trajectory starting from a point \( \alpha \) in

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the first region $v_1 > 0$ and $v_2 > v_2$. In this region $\frac{dv_2}{dt} > 0$ and $\frac{d}{dt}v_1 > 0$ are satisfied hence both $v_1$ and $v_2$ increase. Then the trajectory hits the line $v_1 = v_2$ (point $\beta$) and enters into the second region $v_1 > 0$ and $v_1 < v_2$. In this second region, $\frac{dv_2}{dt} > 0$ and $\frac{d}{dt}v_1 < 0$ are satisfied, the trajectory hits the line $v_1 = 0$ (point $\gamma$) and enters into the next region. Repeating in this manner, the trajectory vibrates divergently around the origin and draws “rect-spiral.” Condition (3) and continuity property of capacitor voltages guarantee the connection of trajectories on the border lines. It should be noted that this system exhibits unstable spiral if signum VCCSs are replaced with linear ones [8].

We then define operation of the $S$ as shown in Fig. 3(a): if $v_1$ reaches the threshold voltage $V_T > 0$ then $S$ is closed and $v_1$ is reset to the base voltage $E$:

$$(v_1(t+), v_2(t+)) = (E, v_2(t)) \text{ if } v_1(t) = V_T.$$  \hfill (4)

For simplicity, we assume that reset of $v_1$ is instantaneous without delay and continuity property of $v_2$ is held. We also assume $E < 0 < V_T$. Figures 3(b) and (c) show a trajectory in the phase space and chaotic attractor. Roughly speaking the instantaneous switching by IFSW and amplitude increase by the two VCCSs correspond to folding and stretching mechanism to generate chaos, respectively.

3. Dimensionless Equation and Return Map

In order to analyze the chaotic attractor we normalize the circuit equation. Using the following dimensionless variables and parameters:

$$\tau = \frac{I_2 t}{C_1 V_T}, \quad x = \frac{v_1}{V_T}, \quad y = \frac{v_2}{a V_T}, \quad a = \frac{C_1 I_1}{C_2 I_2}, \quad q = \frac{E}{V_T}.$$  

Equations (2) and (4) are transformed into Eq. (5).

$$\begin{align*}
\frac{dx}{d\tau} &= \text{sgn}(x - ay), \\
\frac{dy}{d\tau} &= \text{sgn}(x), \\
(x(\tau+), y(\tau+)) &= (q, y(\tau)) \text{ if } x(\tau) = 1. \quad \text{for } S = \text{off.} 
\end{align*}$$  \hfill (5)

It should be noted that this normalized equation is characterized by two parameters $a$ and $q$. Following Condition (3) and $E < 0$, the parameters satisfy

$$a > 1, \quad q < 0.$$  \hfill (6)

Figure 4(a) illustrates PWC vector field that is proportional to Fig. 2 (border line $y = a^{-1}x$ corresponds to $v_1 = v_2$). Figures 4(b) to (d) show typical behavior in this normalized phase space.

In order to analyze these phenomena we derive return map. As shown in Fig. 5, let $L = \{(x, y) | x = 0\}$ and let a point on $L$ be represented by its $y$-coordinate. Since the trajectories starting from $L$ return to $L$ at some positive time, we can define 1-D return map $f$ from $L$ to $L$ itself and the dynamics can be integrated into iteration of $f$:

$$f : L \mapsto L, \quad y_{n+1} = f(y_n).$$  \hfill (7)

![Fig. 2](image1.png) Typical waveform and vector field for $S = \text{off.}$

![Fig. 3](image2.png) Basic circuit dynamics. (a) Time-domain waveform, (b) Phase plane, (c) Chaotic attractor.

![Fig. 4](image3.png) Basic dynamics of normalized equation for $a = 4.7$. (a) PWC vector field. (b) Chaotic attractor for $q = -0.2$. (c) Chaotic attractor for $q = -0.68$. (d) Divergence trajectory for $q = -1.4$. 

![Fig. 5](image4.png)
where \( y_0 \) is an initial point on \( L \) and \( y_n \) is the \( n \)-th return point on \( L \). Since trajectories are PWL, the return map is also PWL and can be described explicitly. For the description we define two key points shown in Fig. 5. The first one is \( Y_a \in L \) such that a trajectory starting from \( Y_a \) passes the intersection of the threshold \( x = 1 \) and border \( y = \frac{q}{a} \). The second one is \( Y_d \in L \) such that a trajectory starting from \( Y_d \) reaches \( x = 1 \) and jumps to the intersection of the base \( x = q \) and border \( y = \frac{q}{a} \). These points are given by

\[
Y_a = \frac{1}{a} - 1, \quad Y_d = \frac{q}{a} - 1.
\]

Let us consider a trajectory starting from \( L \) in the following three cases. (T1) If \( y_n \geq Y_a \), the trajectory returns to \( L \) without reaching the threshold \( x = 1 \). (T2) If \( Y_a > y_n > Y_d \), the trajectory reaches \( x = 1 \), jumps to \( x = q \), intersects \( y = \frac{q}{a} \) and returns to \( L \). (T3) If \( y_n \leq Y_d \), the trajectory reaches \( x = 1 \), jumps to \( x = q \) and returns to \( L \) without intersecting the line \( y = \frac{q}{a} \). Using PWL solution, we can give explicit description of the return map for the three cases.

\[
f(y_n) = \begin{cases} 
\frac{a + 1}{a - 1} y_n & \text{for } y_n \geq Y_a \\
\frac{a + 1}{a - 1} (y_n + 1 - q) & \text{for } Y_a > y_n > Y_d \\
y_n + 1 + q & \text{for } y_n \leq Y_d.
\end{cases}
\]

Using this map we can give theoretical results in the following three cases.

Case 1: \( 1 < a \, \text{and} \, -1 < q \). The map shape is as shown in Fig. 6(b) where \( Y_b \equiv f(Y_a) \) and \( Y_c \equiv f(Y_b) \). Equation (9) guarantees \( Y_c < Y_b < Y_n \), and \(-1 < q \) guarantees \( f(y_n) > y_n \) for \( y_n \leq Y_d \). Hence \( J_1 \equiv [Y_c, Y_b] \) is to be an invariant interval such that \( f(J_1) \subseteq J_1 \) and there exists some positive integer \( m \) such that \( \frac{df^m}{dy}(y_n) > 1 \) on \( J_1 \) where \( f^m \) denotes the \( m \)-fold composition of \( f \). In this case the map exhibits chaos in \( J_1 \) [15].

Case 2: \( 1 < a \, \text{and} \, -1 < q < \frac{1-a}{1+a} \). This case is included in Case 1 and the map shape is as shown in Fig. 6(c) where \( Y_c \equiv f(Y_d) \) and \( Y_f \equiv f(Y_c) \). \(-1 < q \) guarantees \( Y_d < Y_c \) and \( q < \frac{1-a}{1+a} \) guarantees \( Y_f < Y_d \) and \( Y_c < Y_a \). Hence \( Y_f < Y_d < Y_c \) is satisfied and \( J_2 \equiv [Y_f, Y_c] \) is to be an invariant interval: the map exhibits chaos in \( J_2 \).

Case 3: \( 1 < a \, \text{and} \, q < -1 \). \( f(y_n) < y_n \) is satisfied for \( y_n < Y_a \) and trajectories diverge as shown in Fig. 4(d) and Fig. 6(d).

4. Experiments

we have fabricated a simple test circuit as shown in Fig. 7. The VCCSs are realized by OTAs (LM13600) that is assumed to have signum characteristics. The IFSW is implemented using the comparator COMP (LM339), the monostable multivibrator MM (4538) and an analog switch \( S \) (4066). When \( v_1 \) reaches the threshold \( V_T \), the COMP triggers the MM to close the \( S \) and \( v_1 \) is reset to the base voltage.
Fig. 7 A test circuit and laboratory measurements. $V_T = 0.5\, \text{V}$, $I_1 = I_2 = 0.1\, \text{mA}$, $I_{CTL} = 0.07\, \text{mA}$, $C_1 = 47\, \text{nF}$, $C_2 = 10\, \text{nF}$ ($\alpha = 4.7$). (a) Chaos for $E = -0.1\, \text{V}$ ($q = -0.2$), (b) Chaos for $E = -0.34\, \text{V}$ ($q = -0.68$), horizontal=$v_1$ [0.5 V/div.], vertical=$v_2$ [2.0 V/div.] for phase plane (left figures), horizontal=$t$ [1.0 ms/div.], vertical=$v_1$ [0.5 V/div.] for time-domain (right figures). (a) and (b) correspond to Figs. 4(b) and (c), respectively.

E. Figures 7(a) and (b) show typical chaotic attractors and waveforms confirmed experimentally.

5. Conclusions

We have presented the PWC-CSO and have analyzed its chaotic behavior. Since the vector field is PWC, the trajectory is PWL and the embedded return map is PWL. Using the map, we have clarified parameter conditions for chaotic attractors theoretically. Presenting a simple test circuit, typical phenomena are confirmed experimentally. Future problems include consideration of generalized systems, comparison of ideal circuit model with practical circuits and engineering applications.

References