Categorical characterizations of the natural numbers require primitive recursion

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Abstract

Simpson and Yokoyama [Ann. Pure Appl. Logic 164 (2012), 284–293] asked whether there exists a characterization of the natural numbers by a second-order sentence which is provably categorical in the theory RCA$_0$. We answer in the negative, showing that for any characterization of the natural numbers which is provably true in WKL$_0$, the categoricity theorem implies $\Sigma^0_1$ induction.

On the other hand, we show that RCA$_0$ does make it possible to characterize the natural numbers categorically by means of a set of second-order sentences. We also show that a certain $\Pi^1_2$-conservative extension of RCA$_0$ admits a provably categorical single-sentence characterization of the naturals, but each such characterization has to be inconsistent with WKL$_0$ + superexp.

Inspired by a question of Väänänen (see e.g. [Vää12] for some related work), Simpson and the second author [SY12] studied various second-order characterizations of $\langle\mathbb{N},S,0\rangle$, with the aim of determining the reverse-mathematical strength of their respective categoricity theorems. One of the general conclusions is that the strength of a categoricity theorem depends heavily on the characterization. Strikingly, however, each of the categoricity theorems considered in [SY12] implies RCA$_0$, even over the much weaker base theory RCA$_0^*$, that is, RCA$_0$ with $\Sigma^0_1$ induction replaced by $\Delta^0_1$ induction in the language with exponentiation. (For RCA$_0^*$, see [SS86].)

This leads to the following question.

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Question 1. [SY12, Question 5.3] Does $\text{RCA}_0$ prove the existence of a second-order sentence or set of sentences $T$ such that $\mathbb{N},0,S$ is a second-order model of $T$ and all second-order models of $T$ are isomorphic to $\mathbb{N},0,S$? One may also consider the same question with $\text{RCA}_0$ replaced by systems which are $\Pi^0_2$-equivalent to $\text{RCA}_0$.

The question as stated admits multiple versions depending on whether we focus on $\text{RCA}_0$ or consider other $\Pi^0_2$-equivalent theories and whether we want the characterizations of the natural numbers to be sentences or sets of sentences. The most basic version, restricted to $\text{RCA}_0$ and single-sentence characterizations, would read as follows:

Question 2. Does there exist a second-order sentence $\psi$ in the language with one unary function $f$ and one constant $c$ such that $\text{RCA}_0$ proves: (i) $\langle \mathbb{N},S,0 \rangle \models \psi$, and (ii) for every $\langle A,f,c \rangle$, if $\langle A,f,c \rangle \models \psi$, then there exists an isomorphism between $\langle \mathbb{N},S,0 \rangle$ and $\langle A,f,c \rangle$?

We answer Question 2 in the negative. In fact, characterizing $\langle \mathbb{N},S,0 \rangle$ not only up to isomorphism, but even just up to equicardinality of the universe, requires the full strength of $\text{RCA}_0$. More precisely:

Theorem 1. Let $\psi$ be a second-order sentence in the language with one unary function $f$ and one individual constant $c$. If $\text{WKL}_0$ proves that $\langle \mathbb{N},S,0 \rangle \models \psi$, then over $\text{RCA}_0$ the statement “for every $\langle A,f,c \rangle$, if $\langle A,f,c \rangle \models \psi$, then there exists a bijection between $\mathbb{N}$ and $A$” implies $\text{RCA}_0$.

Since $\text{RCA}_0$ is equivalent over $\text{RCA}_0^\ast$ to a statement expressing the correctness of defining functions by primitive recursion [SS86, Lemma 2.5], Theorem 1 may be intuitively understood as saying that, for provably true single-sentence characterizations at least, “categorical characterizations of the natural numbers require primitive recursion”.

Do less stringent versions of Question 1 give rise to “exceptions” to this general conclusion? As it turns out, they do. Firstly, characterizing the natural numbers by a set of sentences is already possible in $\text{RCA}_0^\ast$:

Theorem 2. There exists a $\Delta_0$-definable (and polynomial-time recognizable) set $\Sigma$ of $\Sigma^1_1 \wedge \Pi^1_1$ sentences such that $\text{RCA}_0$ proves: for every $\langle A,f,c \rangle$, $\langle A,f,c \rangle$ satisfies all $\xi \in \Sigma$ if and only if it is isomorphic to $\langle \mathbb{N},S,0 \rangle$.

Secondly, even a single-sentence characterization is possible in a $\Pi^2_1$-conservative extension of $\text{RCA}_0$, at least if one is willing to consider rather peculiar theories:

Theorem 3. There is a $\Sigma^1_2$ sentence which is a categorical characterization of $\langle \mathbb{N},S,0 \rangle$ provably in $\text{RCA}_0^\ast + \neg \text{WKL}$.
Theorem 3 is not quite satisfactory, as the theory and characterization it speaks of are false in \( \langle \omega, P(\omega) \rangle \). So, another natural question to ask is whether a single-sentence characterization of the natural numbers can be provably categorical in a true \( \Pi^0_2 \)-conservative extension of \( \text{RCA}_0 \). We show that under an assumption just a little stronger than \( \Pi^0_2 \)-conservativity, the characterization from Theorem 3 is actually “as true as possible”:

**Theorem 4.** Let \( T \) be an extension of \( \text{RCA}_0 \) conservative for first-order \( \forall \Delta_0(\Sigma_1) \) sentences. Let \( \eta \) be a second-order sentence consistent with \( \text{WKL}_0 + \text{superexp} \). Then it is not the case that \( \eta \) is a categorical characterization of \( \langle \mathbb{N}, S, 0 \rangle \) provably in \( T \).

The proofs of our theorems make use of a weaker notion of isomorphism to \( \langle \mathbb{N}, S, 0 \rangle \) studied in [SY12], that of “almost isomorphism”. Intuitively speaking, a structure \( \langle A, f, c \rangle \) satisfying some basic axioms is almost isomorphic to \( \langle \mathbb{N}, S, 0 \rangle \) if it is “equal to or shorter than” the natural numbers. The two crucial facts we prove and exploit are that almost isomorphism to \( \langle \mathbb{N}, S, 0 \rangle \) can be characterized by a single sentence provably in \( \text{RCA}_0 \), and that structures almost isomorphic to \( \langle \mathbb{N}, S, 0 \rangle \) correspond to \( \Sigma^0_1 \)-definable cuts.

The paper is structured as follows. After a short preliminary Section 1, we conduct our study of almost isomorphism to \( \langle \mathbb{N}, S, 0 \rangle \) studied in [SY12], that of “almost isomorphism”. Intuitively speaking, a structure \( \langle A, f, c \rangle \) satisfying some basic axioms is almost isomorphic to \( \langle \mathbb{N}, S, 0 \rangle \) if it is “equal to or shorter than” the natural numbers. The two crucial facts we prove and exploit are that almost isomorphism to \( \langle \mathbb{N}, S, 0 \rangle \) can be characterized by a single sentence provably in \( \text{RCA}_0 \), and that structures almost isomorphic to \( \langle \mathbb{N}, S, 0 \rangle \) correspond to \( \Sigma^0_1 \)-definable cuts.

The paper is structured as follows. After a short preliminary Section 1, we conduct our study of almost isomorphism to \( \langle \mathbb{N}, S, 0 \rangle \) in Section 2. We then prove Theorem 1 in Section 3, Theorems 2 and 3 in Section 4, and Theorem 4 in Section 5.

## 1 Preliminaries

We assume familiarity with subtheories of second-order arithmetic, as presented in [Sim09]. Of the “big five” theories featuring prominently in that book, we only need the two weakest: \( \text{RCA}_0 \), axiomatized by \( \Delta^0_1 \) comprehension and \( \Sigma^0_1 \) induction (and a finite list of simple basic axioms), and \( \text{WKL}_0 \), which extends \( \text{RCA}_0 \) by the axiom \( \text{WKL} \) stating that an infinite binary tree has an infinite branch.

We also make use of some well-known fragments of first-order arithmetic, principally \( \text{I} \Delta_0 + \text{exp} \), which extends induction for \( \Delta_0 \) formulas by an axiom \( \text{exp} \) stating the totality of exponentiation; \( \text{BS}_{1} \), which extends \( \text{I} \Delta_0 \) by the \( \Sigma_1 \) collection (bounding) principle; and \( \text{IS}_{1} \). For a comprehensive treatment of these and other subtheories of first-order arithmetic, refer to [HP93]. To distinguish a class of first-order formulas from its second-order analogue, we use notation without the superscript “\( ^{\text{\text{d}}} \)”. Thus, for instance, a \( \Sigma_1 \) formula is a first-order formula containing a single block of existential quantifiers followed by a bounded part, whereas a \( \Sigma^0_1 \) formula has the same quantifier structure, but may additionally contain second-order parameters. Such a formula is \( \Sigma_1(\bar{X}) \) if all its second-order parameters are among \( \bar{X} \).
A formula is $\Delta_0(\Sigma_1)$ if it belongs to the closure of $\Sigma_1$ under boolean operations and bounded first-order quantifiers.

The theory $\text{RCA}_0^\ast$ was introduced in [SS86]. It differs from $\text{RCA}_0$ in that the $\Sigma^0_1$ induction axiom is replaced by $\text{I}^\ast_0 + \exp$. $\text{WKL}_0^\ast$ is $\text{RCA}_0^\ast$ plus the WKL axiom. Both $\text{RCA}_0^\ast$ and $\text{WKL}_0^\ast$ have $\text{BS}_1 + \exp$ as their first-order part, while the first-order part of $\text{RCA}_0$ and $\text{WKL}_0$ is $\Sigma^0_1$.

We let superexp denote both the “tower of exponents” function defined by $\text{superexp}(x) = \exp_x(2)$ (where $\exp^0_x(2) = 1, \exp^{x+1}_x(2) = 2^{\exp^x(2)}$) and the axiom saying that for every $x$, superexp$(x)$ exists. $\Delta_0(\exp)$ stands for the class of bounded formulas in the language extending the language of Peano Arithmetic by a symbol for $x^\omega$. $\text{I}^\ast_0(\exp)$ is a definitional extension of $\text{I}^\ast_0 + \exp$.

In any model $M$ of a first-order arithmetic theory (possibly the first-order part of a second-order structure), a cut is a nonempty subset of $M$ which is downwards closed and closed under successor. For a cut $J$, we sometimes abuse notation and also write $J$ to denote the structure $\langle J, S, 0 \rangle$, or even $\langle J, +, \cdot, \leq, 0, 1 \rangle$ if $J$ happens to be closed under multiplication. A set $A \subseteq M$ is bounded if there exists $a \in M$ such that $A \subseteq \{0, \ldots, a\}$, and it is unbounded otherwise. Assuming that $M \models \exp$ and $M$ satisfies $\Delta_0(A)$ induction, we can refer to a bounded set $A$ as $M$-finite (or simply finite), and to an unbounded set $A$ as $(M)$-infinite. Under the same assumptions, it makes sense to speak of the internal cardinality $|A|_\#$ of $A$, which is defined to be $\sup(\{x \in M : A \text{ contains a finite subset with at least } x \text{ elements}\})$. $|A|_\#$ is an element of $M$ if $A$ is finite, and a cut in $M$ otherwise.

If $\langle M, \mathcal{P} \rangle \models \text{RCA}_0^\ast$ and $J$ is a cut in $M$, then $\mathcal{P}_J$ will denote the family of sets $\{X \cap J : X \in \mathcal{P}\}$. Theorem 4.8 of [SS86] states that if $J$ is a proper cut closed under $\exp$, then $\langle J, \mathcal{P}_J \rangle \models \text{WKL}_0^\ast$.

The letter $\mathcal{A}$ will always stand for a structure $\mathcal{A} = \langle A, f, c \rangle$ for the language with one unary function and one constant. $N$ stands for the set of numbers defined by the formula $x = x$; in other words, $\mathbb{N}_M = M$. To refer to the set of standard natural numbers, we use the symbol $\omega$.

The general notational conventions regarding cuts apply also to $\mathbb{N}$: for instance, if there is no danger of confusion, we sometimes write that some $\mathcal{A}$ is “isomorphic to $\mathbb{N}$” rather than “isomorphic to $\langle \mathbb{N}, S, 0 \rangle$”.

## 2 Almost isomorphism

The structure $\mathcal{A} = \langle A, f, c \rangle$ is a Peano system if $f$ is one-to-one, $c \notin \text{rng}(f)$, and $\mathcal{A}$ satisfies the natural formulation of the second-order induction axiom with $c$ as the least element and $f$ as successor. A Peano system is said to be almost isomorphic to $\langle \mathbb{N}, S, 0 \rangle$ if for every $a \in A$ there is some $x \in \mathbb{N}$ such that $f^x(c) = a$. Since $\text{RCA}_0^\ast$
is too weak to prove that any function can be iterated an arbitrary number of times, 
\( f^x(c) = a \) needs to be expressed in such a way as to imply the existence of the sequence \( \langle c, f(c), f^2(c), \ldots, f^x(c) \rangle \).

Being almost isomorphic to \( \mathbb{N} \) is a definable property:

**Lemma 5.** There exists a \( \Sigma^1_1 \land \Pi^1_1 \) sentence \( \xi \) in the language with one unary function \( f \) and one individual constant \( c \) such that \( \text{RCA}_0 \) proves: for every \( \mathcal{A} \), \( \mathcal{A} \models \xi \) if and only if \( \mathcal{A} \) is a Peano system almost isomorphic to \( \langle \mathbb{N}, S, 0 \rangle \).

**Proof.** By definition, \( \mathcal{A} \) is a Peano system precisely if it satisfies the \( \Pi^1_1 \) sentence \( \xi_{\text{peano}} \):

\[
\begin{align*}
&\exists c \in \mathbb{N} \land f(c) \neq 2 \land \forall x \in \mathbb{N} \left( f(x) = f(x+1) \right) \land \forall a \in \mathbb{N} \left( f(a) = a \right) \\
&\forall b \in \mathbb{N} \left( f(b) = b \right) \\
&\forall x \in \mathbb{N} \left( f(x) = f(x+1) \right)
\end{align*}
\]

The sentence \( \xi \) will be the conjunction of \( \xi_{\text{peano}} \), the \( \Sigma^1_1 \) sentence \( \xi_{\text{\&}^1} \text{\&} S \):

there exists a discrete linear ordering \( \leq \)

for which \( c \) is the least element and \( f \) is the successor function,

and the \( \Pi^1_1 \) sentence \( \xi_{\Pi^1_1} \):

for every linear ordering \( \leq \) with \( c \) as least element and \( f \) as successor

and for every \( a \), the set of elements \( \leq \)-below \( a \) is Dedekind-finite.

We say that a set \( X \) is *Dedekind-finite* if there is no bijection between \( X \) and a proper subset of \( X \). Note that provably in \( \text{RCA}_0 \), a set \( X \subseteq A \) is finite exactly if \( A \models "X \text{ is Dedekind-finite}". \)

We first prove that Peano systems almost isomorphic to \( \mathbb{N} \) satisfy \( \xi_{\text{\&}^1} \text{\&} S \) and \( \xi_{\Pi^1_1} \). Let \( \mathcal{A} \) be almost isomorphic to \( \mathbb{N} \). Every \( a \in A \) is of the form \( f^x(c) \) for some \( x \in \mathbb{N} \). Moreover, \( x \) is unique. To see this, assume that \( a = f^x(c) = f^{x+y}(c) \) and that \( \langle c, f(c), \ldots, f^x(c) = a, f^{x+1}(c), \ldots, f^{x+y}(c) = a \rangle \) is the sequence witnessing that \( f^{x+y}(c) = a \) (by \( \Delta^0_0 \)-induction, this sequence is unique and its first \( x+1 \) elements comprise the unique sequence witnessing \( f^x(c) = a \)). If \( y > 0 \), then we have \( c \neq f^y(c) \) and then \( \Delta^0_0 \)-induction coupled with the injectivity of \( f \) gives \( f^w(c) \neq f^{x+y}(c) \) for all \( w \leq x \). So, \( y = 0 \).

Because of the uniqueness of the \( f^x(c) \) representation for \( a \in A \), we can define \( \preceq \) on \( A \) by \( \Delta^0_1 \)-comprehension in the following way:

\[
a \preceq b := \exists x \exists y \left( a = f^x(c) \land b = f^y(c) \land x \leq y \right).
\]

Clearly, \( \preceq \) is a discrete linear ordering on \( A \) with \( c \) as the least element and \( f \) as the successor function, so \( \mathcal{A} \) satisfies \( \xi_{\text{\&}^1} \).
For each $a \in A$, the set of elements $\preceq$-below $a$ is finite. Moreover, if $\preceq$ is any ordering of $A$ with $c$ as least element and $f$ as successor, then for each $a \in A$ the set

$$ \{ b \in A : b \preceq a \iff b \preceq a \} $$

contains $c$ and is closed under $f$. Since $\mathbb{A}$ is a Peano system, $\preceq$ has to coincide with $\preceq$. Thus, $\mathbb{A}$ satisfies $\xi_{\preceq, \Pi}$.

For a proof in the other direction, let $\mathbb{A}$ be a Peano system satisfying $\xi_{\preceq, \Sigma}$ and $\xi_{\preceq, \Pi}$. Let $\preceq$ be an ordering on $A$ witnessing $\xi_{\preceq, \Sigma}$. Take some $a \in A$. By $\xi_{\preceq, \Pi}$, the set $[c, a]_{\preceq}$ of elements $\preceq$-below $a$ is finite. Let $\ell$ be the cardinality of $[c, a]_{\preceq}$ and let $b$ be the $\preceq$-maximal element of $[c, a]_{\preceq}$. By $\Delta^0_1(\exp)$-induction on $x$ prove that there is an element below $b^{x+1}$ coding a sequence $\langle s_0, \ldots, s_x \rangle$ such that $s_0 = c$ and for all $y < x$, either $s_{y+1} = f(s_y) \preceq a$ or $s_{y+1} = s_y = a$. Take such a sequence for $x = \ell - 1$. If $a$ does not appear in the sequence, then by $\Delta^0_1(\exp)$-induction the sequence has the form $\langle c, f(c), \ldots, f^{\ell-1}(c) \rangle$ and all its entries are distinct elements of $[c, a]_{\preceq} \setminus \{ a \}$; an impossibility, given that $[c, a]_{\preceq} \setminus \{ a \}$ only has $\ell - 1$ elements. So, $a$ must appear somewhere in the sequence. Taking $w$ to be the least such that $a = s_w$, we easily verify that $a = f^w(c)$.

**Remark.** We do not know whether in $\mathrm{RCA}_0$ it is possible to characterize $\langle \mathbb{N}, S, 0 \rangle$ up to almost isomorphism by a $\Pi^1_1$ sentence. This does become possible in the case of $\langle \mathbb{N}, \leq \rangle$ (given a suitable definition of almost isomorphism, cf. [SY12]), where there is no need for the $\Sigma^1_1$ part of the characterization which guarantees the existence of a suitable ordering.

An important fact about Peano systems almost isomorphic to $\mathbb{N}$ is that their isomorphism types correspond to $\Sigma^0_1$-definable cuts. This correspondence, which will play a major role in the proofs of our main theorems, is formalized in the following definition and lemma.

**Definition 6.** Let $\mathcal{M} = \langle M, \mathcal{X} \rangle$ be a model of $\mathrm{RCA}_0^\ast$. For a Peano system $\mathbb{A}$ in $\mathcal{M}$ which is almost isomorphic to $\langle \mathbb{N}, S, 0 \rangle$, let $J(\mathbb{A})$ be the cut defined in $\mathcal{M}$ by the $\Sigma^0_1$ formula $\varphi(x)$:

$$ \exists a \in A \; f^x(c) = a. $$

For a $\Sigma^0_1$-definable cut $J$ in $\mathcal{M}$, let the structure $\mathbb{A}(J)$ be $\langle A_J, f_J, c_J \rangle$, where the set $A_J$ consists of all the pairs $\langle x, y_x \rangle$ such that $y_x$ is the smallest witness for the formula $x \in J$, the function $f_J$ maps $\langle x, y_x \rangle$ to $\langle x+1, y_{x+1} \rangle$, and $c_J$ equals $\langle 0, y_0 \rangle$.

**Lemma 7.** Let $\mathcal{M} = \langle M, \mathcal{X} \rangle$ be a model of $\mathrm{RCA}_0^\ast$. The following holds:

(a) for a $\Sigma^0_1$-definable cut $J$ in $\mathcal{M}$, the structure $\mathbb{A}(J)$ is a Peano system almost isomorphic to $\langle \mathbb{N}, S, 0 \rangle$, and $J(\mathbb{A}(J)) = J,$

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(b) if $\mathcal{A} \in \mathcal{X}$ is a Peano system almost isomorphic to $\langle \mathbb{N}, S, 0 \rangle$, then there is an isomorphism in $\mathcal{M}$ between $\mathcal{A}(J(\mathcal{A}))$ and $\mathcal{A}$.

(c) if $\mathcal{A} \in \mathcal{X}$ is a Peano system almost isomorphic to $\langle \mathbb{N}, S, 0 \rangle$, then there is an isomorphism in $\mathcal{M}$ between $\mathcal{A}$ and $J(\mathcal{A})$, which also induces an isomorphism between the second-order structures $\langle \mathcal{A}, \mathcal{X} \cap \mathcal{P}(A) \rangle$ and $\langle J(\mathcal{A}), \mathcal{X}_J(\mathcal{A}) \rangle$.

Although all the isomorphisms between first-order structures mentioned in Lemma 7 are elements of $\mathcal{X}$, a cut is not itself an element of $\mathcal{X}$ unless it equals $M$ (because induction fails for the formula $x \in J$ whenever $J$ is a proper cut). Obviously, the isomorphism between second-order structures mentioned in part (c) is also outside $\mathcal{X}$.

Proof. For a $\Sigma^0_1$-definable cut $J$ in $\mathcal{M}$, it is clear that $A_J$ and $f_J$ are elements of $\mathcal{X}$, that $f_J$ is an injection from $A_J$ into $A_J$, and that $c_J$ is outside the range of $f_J$. Furthermore, for every $(x, y, z) \in A_J$, $\Sigma^0_1$ collection in $\mathcal{M}$ guarantees that there is a common upper bound on $y_0, \ldots, y_n$, so $\Delta^0_0$ induction is enough to show that the sequence $(c_J, f_J(c_J), \ldots, f^j_J(c_J)) = \langle x, y_z \rangle$ exists. If $X \subset A_J$, $X \in \mathcal{X}$, is such that $c_J \in X$ but $f^j_J(c_J) \notin X$, then $\Delta^0_0$ induction along the sequence $(c_J, f_J(c_J), \ldots, f^j_J(c_J))$ finds some $w < x$ such that $f^j_J(c_J) \in X$ but $f^j_J(f^j_J(c_J)) \notin X$. Thus, $\mathcal{A}(J)$ is a Peano system almost isomorphic to $\mathbb{N}$, and clearly $J(\mathcal{A}(J))$ equals $J$, so part (a) is proved.

For part (b), if $\mathcal{A}$ is almost isomorphic to $\mathbb{N}$, then each $a \in A$ has the form $a = f^i(c)$ for some $x \in J(\mathcal{A})$, and we know from the proof of Lemma 5 that the element $x$ is unique. Thus, the mapping which takes $f^i(c) \in \mathcal{A}$ to $\langle x, y_z \rangle \in \mathcal{A}(J(\mathcal{A}))$ is guaranteed to exist in $\mathcal{M}$ by $\Delta^0_0$ comprehension. It follows easily from the definitions of $J(\mathcal{A})$ and $\mathcal{A}(J)$ that the mapping $f^i(c) = \langle x, y_z \rangle$ is an isomorphism between $\mathcal{A}$ and $\mathcal{A}(J(\mathcal{A}))$.

For part (c), we assume that $\mathcal{A}$ equals $\mathcal{A}(J(\mathcal{A}))$, which we may do w.l.o.g. by part (b). The isomorphism between $\mathcal{A}$ and $J(\mathcal{A})$ is given by $(x, y_z) \mapsto x$. To prove that this also induces an isomorphism between $\langle \mathcal{A}, \mathcal{X} \cap \mathcal{P}(A) \rangle$ and $\langle J(\mathcal{A}), \mathcal{X}_J(\mathcal{A}) \rangle$, we have to show that for any $X \subseteq A$, it holds that $X \in \mathcal{X}$ exactly if $\{ x : \langle x, y_z \rangle \in X \}$ has the form $Z \cap J(\mathcal{A})$ for some $Z \in \mathcal{X}$.

The “if” direction is immediate: given $Z \in \mathcal{X}$, the set $\{ (x, y_z) : x \in Z \}$ is $\Delta^0_0(Z)$ and thus belongs to $\mathcal{X}$.

To deal with the other direction, we assume that $\mathcal{M}$ is countable. We can do this w.l.o.g. because $J(\mathcal{A})$ is a definable cut, so the existence of a counterexample in some model would imply the existence of a counterexample in a countable model by a downwards Skolem-Löwenheim argument.

By [SS86, Theorem 4.6], the countability of $\mathcal{M}$ means that we can extend $\mathcal{X}$ to a family $\mathcal{X}^+ \supseteq \mathcal{X}$ such that $\langle M, \mathcal{X}^+ \rangle \models \text{WKL}_0$. Note that there are no (M-
finite sets in $\mathcal{X}^+ \setminus \mathcal{X}$. This is because a finite set in $\mathcal{X}^+$ actually has the form 
\[ \{ x \mid \text{bit}(z, x) = 1 \} \] for some $z \in M$, and each such set is $\Delta_0$-definable and thus in $\mathcal{X}$.

Now consider some $X \in \mathcal{X}$, $X \subseteq A$. Let $T$ be the set consisting of the finite binary strings $s$ satisfying:

\[ \forall a, x < \text{lh}(s) \left[ (a = \langle x, y_x \rangle \wedge a \in X \rightarrow (s)_x = 1) \wedge (a = \langle x, y_x \rangle \wedge a \in A \setminus X \rightarrow (s)_x = 0) \right]. \]

$T$ is $\Delta_0(X)$-definable, so it belongs to $\mathcal{X}$, and it is easy to show that it is an infinite tree. Let $B \in \mathcal{X}^+$ be an infinite branch of $T$. Then \( \{ x : (x, y_x) \in X \} = B \cap J(\mathcal{A}). \)

However, $B \cap J(\mathcal{A})$ can also be written as $(B \cap \{0, \ldots, z\}) \cap J(\mathcal{A})$ for an arbitrary $z \in M \setminus J(\mathcal{A})$, and $B \cap \{0, \ldots, z\}$, being a finite set, belongs to $\mathcal{X}$. \( \square \)

**Corollary 8.** Let $\mathcal{M} = (M, \mathcal{X})$ be a model of $\text{RCA}_0^\ast$. Let $\mathcal{A} \in \mathcal{X}$ be a Peano system almost isomorphic to $\langle \mathbb{N}, \cdot, 0 \rangle$. Assume that $J(\mathcal{A})$ is a proper cut under exp, that $\prec$ is a linear ordering on $A$ with least element $c$ and successor function $f$, and that $\oplus, \otimes$ are operations on $A$ which satisfy the usual recursive definitions of addition resp. multiplication with respect to least element $c$ and successor $f$. Then $\langle (\mathcal{A}, \oplus, \otimes, \prec, c, f(c)), \mathcal{X} \cap \mathcal{P}(A) \rangle \models WKL^0_0$.

**Proof.** Write $\mathcal{A}$ for $\langle \mathcal{A}, \oplus, \otimes, \leq, c, f(c) \rangle$. By Lemma 7 part (b), we can assume w.l.o.g. that $\mathcal{A} = \mathcal{A}(J(\mathcal{A}))$. Using the fact that $\mathcal{A}$ is a Peano system, we can prove that for every $x, z \in J(\mathcal{A})$ we have

\[
\begin{align*}
(x, y_x) \oplus (z, y_z) &= (x + z, y_{x+z}), \\
(x, y_x) \otimes (z, y_z) &= (x \cdot z, y_{x\cdot z}), \\
(x, y_x) \preceq (z, y_z) &\iff x \preceq z.
\end{align*}
\]

By the obvious extension of Lemma 7 part (c) to structures with addition, multiplication and ordering, $\langle \mathcal{A}, \mathcal{X} \cap \mathcal{P}(A) \rangle$ is isomorphic to $\langle J(\mathcal{A}), \mathcal{X}(J(\mathcal{A})) \rangle$. Since $J(\mathcal{A})$ is proper and closed under exp, this means that $\langle \mathcal{A}, \mathcal{X} \cap \mathcal{P}(A) \rangle \models WKL^0_0$. \( \square \)

**Remark.** It was shown in [SY12, Lemma 2.2] that in $\text{RCA}_0$ a Peano system almost isomorphic to $\mathbb{N}$ is actually isomorphic to $\mathbb{N}$. In light of Lemma 7, this is a reflection of the fact that in $\text{RCA}_0$ there are no proper $\Sigma^0_1$-definable cuts.

Informally speaking, a Peano system which is not almost isomorphic to $\mathbb{N}$ is “too long”, since it contains elements which cannot be obtained by starting at zero and iterating successor finitely many times. On the other hand, a Peano system which is almost isomorphic but not isomorphic to $\mathbb{N}$ is “too short”. The results of this section, together with our Theorem 1, give precise meaning to the intuitive idea strongly suggested by Table 2 of [SY12], that the problem with characterizing the natural numbers in $\text{RCA}_0^\ast$ is ruling out structures that are “too short” rather than “too long”.

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3 Characterizations: basic case

In this section, we prove Theorem 1.

**Theorem 1 (restated).** Let $\psi$ be a second-order sentence in the language with one unary function $f$ and one individual constant $c$. If $WKL^0_0$ proves that $\langle \mathbb{N}, S, 0 \rangle \models \psi$, then over $RCA^*_0$ the statement “for every $A$, if $A \models \psi$, then there exists a bijection between $\mathbb{N}$ and $A$” implies $RCA^*_0$.

We use a model-theoretic argument based on the work of Section 2 and a lemma about cuts in models of $I\Delta_0 + \exp + I\Sigma_1$.

**Lemma 9.** Let $M \models I\Delta_0 + \exp + I\Sigma_1$. There exists a proper $\Sigma_1$-definable cut $J \subseteq M$ closed under $\exp$.

Note that a proper cut closed under $\exp$ satisfies $B\Sigma_1 + \exp$, the first-order part of $RCA^*_0$ and $WKL^*_0$.

**Proof.** We need to consider a few cases.

- **Case 1.** $M \models \mathbb{L}_{1\Delta_0} + \exp + I\Sigma_1$. Since $M \models \mathbb{L}_{1\Sigma_1}$, there exists a $\Sigma_1$ formula $\varphi(x)$, possibly with parameters, which defines a proper subset of $M$ closed under successor. Replacing $\varphi(x)$ by the formula $\hat{\varphi}(x)$: “there exists a sequence witnessing that for all $y < x$, $\varphi(y)$ holds”, we obtain a proper $\Sigma_1$-definable cut $K \subseteq M$. Define:
  
  $$J := \{ y : \exists x \in K ( y < \text{superexp}(x)) \}.$$ 

  $J$ is a cut closed under $\exp$ because $K$ is a cut, and it is proper because it does not contain $\text{superexp}(b)$ for any $b \notin K$.

  The remaining cases all assume that $M \models \mathbb{L}_{1\Delta_0} + \exp + I\Sigma_1$.

  - **Case 2.** $\text{Log}^*(M)$ is closed under $\exp$. Define $J := \text{Log}^*(M)$.

  - **Case 3.** $\text{Log}^*(M)$ is closed under addition but not under $\exp$. Let $\text{Log}(\text{Log}^*(M))$ be the subset of $M$ defined as $\{ x : \text{exp}(x) \in \text{Log}^*(M) \}$. Since $\text{Log}^*(M)$ is closed under addition, $\text{Log}(\text{Log}^*(M))$ is a cut. Moreover, $\text{Log}(\text{Log}^*(M)) \subseteq \text{Log}^*(M)$, because $\text{Log}^*(M)$ is not closed under $\exp$. Define:
    
    $$J := \{ y : \exists x \in \text{Log}(\text{Log}^*(M)) ( y < \text{superexp}(x)) \}.$$ 

    $J$ is a cut closed under $\exp$ because $\text{Log}(\text{Log}^*(M))$ is a cut, and it is proper because it does not contain $\text{superexp}(b)$ for any $b \in \text{Log}^*(M) \setminus \text{Log}(\text{Log}^*(M))$.

    - **Case 4.** $\text{Log}^*(M)$ is not closed under addition. Let $\frac{1}{2}\text{Log}^*(M)$ be the subset of $M$ defined as $\{ x : 2x \in \text{Log}^*(M) \}$. Since $\text{Log}^*(M)$ is closed under successor,
\( \frac{1}{2} \log^*(M) \) is a cut. Moreover, \( \frac{1}{2} \log^*(M) \subseteq \log^*(M) \), because \( \log^*(M) \) is not closed under addition. Define:

\[
J := \{ y : \exists x \in \frac{1}{2} \log^*(M) \ (y < \supexp(x)) \}.
\]

\( J \) is a cut closed under \( \supexp \) because \( \frac{1}{2} \log^*(M) \) is a cut, and it is proper because it does not contain \( \supexp(b) \) for any \( b \in \log^*(M) \setminus \frac{1}{2} \log^*(M) \).

\( \Box \)

**Remark.** Inspection of the proof reveals immediately that Lemma 9 relativizes, in the sense that in a model of \( I\Delta_0(X) + \exp + \neg \Sigma^0_1(X) \) there is a \( \Sigma^1(X) \)-definable proper cut closed under \( \exp \).

**Remark.** The method used to prove Lemma 9 shows the following result: for any \( n \in \omega \), there is a definable cut in \( I\Delta_0 + \exp \) which is provably closed under \( \exp \) and proper in all models of \( I\Delta_0 + \exp + \neg \Sigma^0_\alpha \). In contrast, there is no definable cut in \( I\Delta_0 + \exp \) provably closed under \( \supexp \); otherwise, \( I\Delta_0 + \exp \) would prove its consistency relativized to a definable cut, which would contradict a result of [Pud85].

We can now complete the proof of Theorem 1. Assume that \( \psi \) is a second-order sentence true of \( \langle \mathbb{N}, S, 0 \rangle \) provably in \( \text{WKL}_0^0 \). Let \( \mathcal{M} = \langle \mathbb{M}, \mathcal{X} \rangle \) be a model of \( \text{RCA}_0^0 + \neg \Sigma^0_\alpha \). Assume for the sake of contradiction that according to \( \mathcal{M} \), the universe of any structure satisfying \( \psi \) can be bijectively mapped onto \( \mathbb{N} \).

Let \( J \) be the cut in \( \mathcal{M} \) guaranteed to exist by the relativized version of Lemma 9. Since \( |A_J|_\mathcal{M} = J \), the model \( \mathcal{M} \) believes that there is no bijection between \( A_J \) and \( \mathbb{N} \), and hence also that \( \mathcal{A}(J) \models \neg \psi \).

By Lemma 7 and its proof, the mapping \( f^\mathcal{A}(c) \mapsto x \) induces an isomorphism between \( \langle \mathcal{A}(J), \mathcal{X} \cap \mathcal{P}(A_J) \rangle \) and \( \langle J, \mathcal{L}J \rangle \). Since \( J \) is closed under addition and multiplication, we can define the operation \( \oplus \) on \( A_J \) by setting \( f^\mathcal{A}(c) \oplus f^\mathcal{A}(c) = f^\mathcal{A}(c) \), and we can define \( \otimes \) and \( \leq \) analogously. By the uniqueness of the \( f^\mathcal{A}(c) \) representation, \( \oplus, \otimes, \leq \) are all elements of \( \mathcal{X} \). Write \( \mathcal{A}(J) \) for \( \langle \mathcal{A}(J), \oplus, \otimes, \leq, c_J, f_J(c_J) \rangle \).

Clearly, \( A_J \) with the structure given by \( \oplus, \otimes, \leq \) satisfies the assumptions of Corollary 8, which means that \( \langle \mathcal{A}(J), \mathcal{X} \cap \mathcal{P}(A_J) \rangle \) is a model of \( \text{WKL}_0^0 \). We also claim that \( \langle \mathcal{A}(J), \mathcal{X} \cap \mathcal{P}(A_J) \rangle \) believes \( \mathbb{N} \models \neg \psi \). This is essentially an immediate consequence of the fact that \( \mathcal{M} \) thinks \( \mathcal{A}(J) \models \neg \psi \), since the subsets of \( A_J \) are exactly the same in \( \langle \mathcal{A}(J), \mathcal{X} \cap \mathcal{P}(A_J) \rangle \) as in \( \mathcal{M} \). There is one minor technical annoyance related to non-unary second-order quantifiers in \( \psi \), as the integer pairing function in \( \mathcal{A}(J) \) does not coincide with that of \( \mathcal{M} \). The reason this matters is that the language of second-order arithmetic officially contains only unary set variables, so e.g. a binary relation is represented by a set of pairs, but a set of \( M \)-pairs of elements of \( A_J \) might not even be a subset of \( A_J \). Clearly, however, since the graph of the \( \mathcal{A}(J) \)-pairing function is \( \Delta^0_\alpha(\text{exp}) \)-definable in \( \mathcal{M} \), a given set of \( M \)-pairs of
elements of $A_j$ belongs to $\mathcal{X}$ exactly if the corresponding set of $\hat{A}$-pairs belongs to $\mathcal{X} \cap \mathcal{P}(A_j)$; and likewise for tuples of greater constant length.

Thus, our claim holds, and we have contradicted the assumption that $\psi$ is true of $\mathbb{N}$ provably in $\text{WKL}_0^\ast$. \hfill \Box (Theorem 1)

We point out the following corollary of the proof.

**Corollary 10.** The following are equivalent over $\text{RCA}_0^\ast$:

1. $\neg \text{RCA}_0$.
2. There exists $\mathcal{M} = (M, \mathcal{X})$ satisfying $\text{WKL}_0^\ast$ such that $|M| \neq |\mathbb{N}|$.

**Proof.** $\text{RCA}_0$ proves that all infinite sets have the same cardinality, which gives (2) $\Rightarrow$ (1). To prove (1) $\Rightarrow$ (2), work in a model of $\text{RCA}_0^\ast + \neg \text{RCA}_0$ and take the inner model of $\text{WKL}_0^\ast$ provided by the proof of Theorem 1. \hfill \Box

**Remark.** The type of argument described above can be employed to strengthen Theorem 1 in two ways.

Firstly, it is clear that $\langle \mathbb{N}, S, 0 \rangle$ could be replaced in the statement of Theorem 1 by, for instance, $\langle \mathbb{N}, \leq, +, 0, 1 \rangle$. In other words, the extra structure provided by addition and multiplication does not help in characterizing the natural numbers without $\Sigma^0_1$.

Secondly, for any fixed $n \in \omega$, the theories $\text{RCA}_0^\ast / \text{WKL}_0^\ast$ appearing in the statement could be extended (both simultaneously) by an axiom expressing the totality of $f_n$, the $n$-th function in the Grzegorczyk-Wainer hierarchy (e.g., the totality of $f_2$ is exp, the totality of $f_3$ is superexp). The proof remains essentially the same, except that the argument used to show Lemma 9 now splits into $n + 2$ cases instead of four.

By compactness, $\text{RCA}_0^\ast / \text{WKL}_0^\ast$ could also be replaced in the statement of the theorem by $\text{RCA}_0^\ast + \text{PRA} / \text{WKL}_0^\ast + \text{PRA}$, where $\text{PRA}$ is primitive recursive arithmetic.

### 4 Characterizations: exceptions

In this section, we prove Theorems 2 and 3.

**Theorem 2 (restated).** There exists a $\Delta_0$-definable (and polynomial-time recognizable) set $\Xi$ of $\Sigma^1_1 \land \Pi^1_1$ sentences such that $\text{RCA}_0^\ast$ proves: for every $\mathcal{A}$, $\mathcal{A}$ satisfies all $\xi \in \Xi$ if and only if it is isomorphic to $\langle \mathbb{N}, S, 0 \rangle$. 

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Proof of Theorem 2. Let the set \( \Xi \) consist of the sentence \( \xi \) from Lemma 5 and the sentences

\[
\exists a_0 \exists a_1 \ldots \exists a_{x-1} \exists a_x [a_0 = c \land a_1 = f(a_0) \land \ldots \land a_x = f(a_{x-1})],
\]

for every \( x \in \mathbb{N} \). (Note that in a nonstandard model of \( \text{RCA}_0^* \), the set \( \Xi \) will contain sentences of nonstandard length.)

Provably in \( \text{RCA}_0^* \), a structure \( \mathbb{A} \) satisfies all sentences in \( \Xi \) exactly if it is a Peano system almost isomorphic to \( \mathbb{N} \) such that for every \( x \in \mathbb{N} \), \( f^x(c) \) exists. Clearly then, \( \mathbb{N} \) satisfies all sentences in \( \Xi \). Conversely, if \( \mathbb{A} \) satisfies all sentences in \( \Xi \), then \( J(\mathbb{A}) = \mathbb{N} \) and so \( \mathbb{A} \) is isomorphic to \( \mathbb{N} \).

\( \square \)

Theorem 3 (restated). There is a \( \Sigma_1^1 \) sentence which is a categorical characterization of \( \langle \mathbb{N}, S, 0 \rangle \) provably in \( \text{RCA}_0^* + \neg \text{WKL} \).

Before proving the theorem, we verify that the theory it mentions is a \( \Pi_1^1 \)-conservative extension of \( \text{RCA}_0^* \).

Proposition 11. The theory \( \text{RCA}_0^* + \neg \text{WKL} \) is a \( \Pi_1^1 \)-conservative extension of \( \text{RCA}_0^* \).

Proof. Let \( \exists X \forall Y \varphi(X,Y) \) be a \( \Sigma_1^1 \) sentence consistent with \( \text{RCA}_0^* \). Take \( (M, X) \) and \( A \in X \) such that \( (M, X) \models \text{RCA}_0^* + \forall Y \varphi(A,Y) \). Let \( \Delta_1(A)\text{-Def} \) stand for the collection of the \( \Delta_1(A) \)-definable subsets of \( M \). \( \Delta_1(A)\text{-Def} \subseteq X \), so obviously \( (M, \Delta_1(A)\text{-Def}) \models \text{RCA}_0^* + \forall Y \varphi(A,Y) \). Moreover, by a standard argument, there is a \( \Delta_1(A) \)-definable infinite binary tree without a \( \Delta_1(A) \)-definable branch, so \( (M, \Delta_1(A)\text{-Def}) \models \neg \text{WKL} \).

\( \square \)

Proof of Theorem 3. Work in \( \text{RCA}_0^* + \neg \text{WKL} \). The sentence \( \psi \), our categorical characterization of \( \mathbb{N} \), is very much like the sentence \( \xi \) described in the proof of Lemma 5, which expressed almost isomorphism to \( \mathbb{N} \). The one difference is that the \( \Sigma_1^1 \) conjunct of \( \xi \):

there exists a discrete linear ordering \( \preceq \)
for which \( c \) is the least element and \( f \) is the successor function,
is strengthened in \( \psi \) to the \( \Sigma_1^1 \) sentence:

there exist binary operations \( \oplus, \otimes \) and a discrete linear ordering \( \preceq \) such that
\( \preceq \) has \( c \) as the least element and \( f \) as the successor function,
\( \oplus \) and \( \otimes \) satisfy the usual recursive definition of addition and multiplication,
and such that \( I\Delta_0 + \exp + \neg \text{WKL} \) holds.
I\Delta_0 + \exp is finitely axiomatizable, so there is no problem with expressing this as a single sentence. Note that \psi is \Sigma^1_2.

Since \neg \text{WKL} holds, the usual +, \cdot and ordering on \mathbb{N} witness that \mathbb{N} satisfies the new \Sigma^1_2 conjunct of \psi. Of course, \mathbb{N} is a Peano system almost isomorphic to \mathbb{N}, and thus it satisfies \psi.

Now let \mathcal{A} be a structure satisfying \psi. Then \mathcal{A} is a Peano system almost isomorphic to \mathbb{N}, so we may consider \nu(\mathcal{A}). The existence of +, \cdot, \leq witnessing the \Sigma^1_2 conjunct of \psi guarantees that \nu(\mathcal{A}) is closed under exp. Moreover, Corollary \text{8} implies that \nu(\mathcal{A}) cannot be a proper cut, because otherwise \mathcal{A} with the additional structure given by +, \cdot, \leq would have to satisfy WKL. So, \nu(\mathcal{A}) = \mathbb{N} and thus \mathcal{A} is isomorphic to \mathbb{N}.

\section{Characterizations: exceptions are exotic}

To conclude the paper, we prove Theorem 4 and some corollaries.

\textbf{Theorem 4 (restated).} Let T be an extension of RCA^0_0 conservative for first-order \forall \Delta_0(\Sigma_1) sentences. Let \eta be a second-order sentence consistent with WKL^0_0 + superexp. Then it is not the case that \eta is a categorical characterization of \langle \mathbb{N}, S, 0 \rangle provably in T.

\textit{Proof.} Let \mathcal{M} = (M, \mathcal{X}) be a countable recursively saturated model of WKL^0_0 + superexp + \eta.

Tanaka’s self-embedding theorem [Tan97] is stated for countable models of WKL_0, but it is part of the folklore that the same proof works for countable recursively saturated models of WKL^0_0. Thus, there is a cut I in M such that \langle M, \mathcal{X} \rangle and \langle I, \mathcal{X}_I \rangle are isomorphic. In particular, \langle I, \mathcal{X}_I \rangle \models \eta.

Let a \in M \setminus I. Define the cut K in M to be

\{ y : \exists x \in I (y < \exp_{a+3}(2)) \}.

(K, \mathcal{X}_K) is a model of WKL^0_0 and I is a \Sigma_1-definable proper cut in K.

T is conservative over RCA^0_0 for first-order \forall \Delta_0(\Sigma_1) sentences, so there is a model \langle L, \mathcal{Y} \rangle \models T such that K \preceq_{\Delta_0(\Sigma_1)} L. We claim that in \langle L, \mathcal{Y} \rangle there is a Peano system \mathcal{A} satisfying \eta but not isomorphic to \mathbb{N}. This will imply that T does not prove \eta to be a categorical characterization of \mathbb{N}. It remains to prove the claim.

We can assume that \eta does not contain a second-order quantifier in the scope of a first-order quantifier. This is because we can always replace first-order quantification by quantification over singleton sets, at the cost of adding some new first-order quantifiers with none of the original quantifiers of \eta in their scope.
Note that \((K, \mathcal{X}_K)\) contains a proper \(\Sigma_1\) definable cut, namely \(I\), which satisfies \(\eta\). Using the universal \(\Sigma_1\) formula, we can express this fact by a first-order \(\exists \Delta_0(\Sigma_1)\) sentence \(\eta^{FO}\). The sentence \(\eta^{FO}\) says the following:

there exists a triple “\(\Sigma_1\) formula \(\varphi(x, w)\), parameter \(p\), bound \(b\)” such that 

\(b\) does not satisfy \(\varphi(x, p)\), the set defined by \(\varphi(x, p)\) below \(b\) is a cut, 

and this cut satisfies \(\eta\).

To state the last part, replace the second-order quantifiers of \(\eta\) by quantifiers over subsets of \([0, \ldots, b - 1]\) (these are bounded first-order quantifiers) and replace the first-order quantifiers by first-order quantifiers relativized to elements below \(b\) satisfying \(\varphi(x, p)\). By our assumptions about the syntactical form of \(\eta\), this ensures that \(\eta^{FO}\) is \(\exists \Delta_0(\Sigma_1)\).

\(L\) is a \(\Delta_0(\Sigma_1)\)-elementary extension of \(K\), so \(L\) also satisfies \(\eta^{FO}\). Therefore, 

\((L, \mathcal{Y})\) also contains a proper \(\Sigma_1\)-definable cut satisfying \(\eta\). By Lemma 7, this means that in \((L, \mathcal{Y})\) there is a Peano system \(\mathcal{A}\) satisfying \(\eta\) but not isomorphic to \(\mathbb{N}\). The claim, and the theorem, is thus proved.

Remark. The assumption that \(\eta\) is consistent with \(\text{WKL}_0^\times\) rather than just \(\text{WKL}_0\) is only needed to ensure that there is a model of \(\text{RCA}_0\) with a proper \(\Sigma_1\)-definable cut satisfying \(\eta\). The assumption can be replaced by consistency with \(\text{WKL}_0\) extended by a much weaker first-order statement, but we were not able to make the proof work assuming only consistency with \(\text{WKL}_0^\times\).

One idea used in the proof of Theorem 4 seems worth stating as a separate corollary.

**Corollary 12.** Let \(\eta\) be a second order sentence. The statement “there exists a Peano system \(\mathcal{A}\) almost isomorphic but not isomorphic to \(\langle \mathbb{N}, S, 0 \rangle\) such that \(\mathcal{A} \models \eta\)” is \(\Sigma_1^1\) over \(\text{RCA}_0^\times\).

**Proof.** By Lemma 7, a Peano system satisfying \(\eta\) and almost isomorphic but not isomorphic to \(\mathbb{N}\) exists exactly if there is a proper \(\Sigma_1^0\)-definable cut satisfying \(\eta\). This can be expressed by a sentence identical to the first-order sentence \(\eta^{FO}\) from the proof of Theorem 4 except for an additional existential second-order quantifier to account for the possible set parameters in the formula defining the cut.

Theorem 4 also has the consequence that if we restrict our attention to \(\Pi^1_1\)-conservative extensions of \(\text{RCA}_0\), then the characterization from Theorem 3 is not only the “truest possible”, but also the “simplest possible” provably categorical characterization of \(\mathbb{N}\).
Corollary 13. Let $T$ be a $\Pi^1_1$-conservative extension of $\text{RCA}_0^*$. Assume that the second-order sentence $\eta$ is a categorical characterization of $\langle \mathbb{N}, S, 0 \rangle$ provably in $T$. Then

(a) $\eta$ is not $\Pi^1_2$,

(b) $T$ is not $\Pi^1_2$-axiomatizable.

Proof. We first prove (b). Assume that $T$ is $\Pi^1_2$-axiomatizable and $\Pi^1_1$-conservative over $\text{RCA}_0^*$. As observed in [Yok09], this means that $T + \text{WKL}_0^*$ is $\Pi^1_1$-conservative over $\text{RCA}_0^*$, so $T$ is consistent with $\text{WKL}_0^* + \text{superexp}$. Hence, Theorem 4 implies that there can be no provably categorical characterization of $\mathbb{N}$ in $T$.

Turning now to part (a), assume that $\eta$ is $\Pi^1_2$. Since $T$ is $\Pi^1_1$-conservative over $\text{RCA}_0^*$ and proves that $\mathbb{N} \models \eta$, then $\text{RCA}_0^* + \eta$ must also be $\Pi^1_1$-conservative over $\text{RCA}_0^*$. But then, by a similar argument as above, $\eta$ is consistent with $\text{WKL}_0^* + \text{superexp}$, which contradicts Theorem 4.

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References


