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Constructive Incremental Learning from Only Local Information

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We introduce a constructive, incremental learning system for regression problems that models data by means of spatially localized linear models. In contrast to other approaches, the size and shape of the receptive field of each locally linear model, as well as the parameters of the locally linear model itself, are learned independently, that is, without the need for competition or any other kind of communication. Independent learning is accomplished by incrementally minimizing a weighted local cross-validation error. As a result, we obtain a learning system that can allocate resources as needed while dealing with the bias-variance dilemma in a principled way. The spatial localization of the linear models increases robustness toward negative interference. Our learning system can be interpreted as a nonparametric adaptive bandwidth smoother, as a mixture of experts where the experts are trained in isolation, and as a learning system that profits from combining independent expert knowledge on the same problem. This article illustrates the potential learning capabilities of purely local learning and offers an interesting and powerful approach to learning with receptive fields.

1 Introduction

Learning with spatially localized basis functions has become a popular paradigm in machine learning and neurobiological modeling. In the context of radial basis function networks (Moody & Darken, 1988; Poggio & Girosi, 1990), it was demonstrated that such local learning offers an alternative to learning with global basis functions, such as sigmoidal neural networks, and that its theoretical foundation can be solidly grounded in approximation theory (Powell, 1987). In neurophysiological studies, the concept of localized information processing in the form of receptive fields has been known since at least the work of Mountcastle (1957) and Hubel and Wiesel

(1959). Since then, a wealth of experimental evidence has accumulated that suggests that information processing based on local receptive fields is a ubiquitous organizational principle in neurobiology that offers interesting computational opportunities (Lee, Rohrer, & Sparks, 1988; Georgopoulos, 1991; Field, 1994; Olshausen & Field, 1996; Daugman & Downing, 1995).

In this article we explore the computational power of local, receptive field–based incremental learning with the goal of approximating unknown functional relationships between incoming streams of input and output data. By incremental learning we do not mean just that the parameters of the learning system are updated incrementally. We want to address a learning scenario in which limited memory is available such that after a new data point is incorporated in the learning system, it is discarded and cannot be reused, in which input and output distributions of the data are unknown, and in which these distributions may change over time. This situation resembles the learning of sensory and sensorimotor transformations in biology, and it also applies to a variety of artificial domains, ranging from autonomous robotic systems to process control.

Given these constraints on incremental learning, two major problems need to be addressed. The first is how to allocate the appropriate number of resources, such as receptive fields, in order to deal with the trade-off between overfitting and oversmoothing, called the *bias-variance dilemma* (Geman, Bienenstock, & Doursat, 1992). The second problem of incremental learning comes from negative interference: the forgetting of useful knowledge while learning from new data. Methods to prevent negative interference require validation data sets, memorizing of all training data, or strong prior knowledge about the learning problem. However, none of these alternatives is available in the setting we have described as we want to avoid storing data and do not have much knowledge about the structure of the learning task.

In order to address the problems of incremental learning, we resort to techniques from nonparametric statistics (Scott, 1992; Hastie & Tibshirani, 1990). Nearest-neighbor algorithms for pattern recognition and Parzen windows for density estimation are among the best-known methods out of this field (Duda & Hart, 1973). It is interesting to note that many nonparametric methods are essentially receptive field based: predictions are made using data from a restricted local neighborhood around the query point. The size of the neighborhood can be irregular, as typically is the case in nearest-neighbor approaches, or it can be a symmetric smooth weighting function as in Parzen windows. Receptive fields in nonparametric regression are mostly built on the fly and are discarded right after the prediction, a paradigm that has been termed *lazy learning* (Aha, 1997). Necessarily, such nonparametric methods need to store training data. Another characteristic is that predictions are usually based on a single receptive field. This property inspired the field of nonparametric regression to pursue more complex models in a receptive field, for instance, low-order polynomials (Cleveland, 1979; Cleveland & Loader, 1995). In contrast, many neural network algo-

rithms, such as radial basis function systems, focused on combining the activation strengths of many receptive fields to optimize predictions.

In this article, we demonstrate how a nonparametric regression approach can be used to build a receptive field–based learning system for incremental function approximation without the need to store the training data and without discarding receptive fields after using them. A locally linear model will be fitted incrementally within each receptive field such that local function approximation is accomplished in the spirit of a Taylor series expansion. A new property of this learning approach is that each receptive field is trained independent of all other receptive fields, thereby adjusting the parameters of its locally linear model, the size and shape of its receptive field, and the bias on the relevance on its individual input dimensions. New receptive fields are allocated as needed. The resulting algorithm, receptive field–weighted regression (RFWR), achieves robust incremental learning. It also has some interesting relations to previously suggested learning methods. It can be interpreted as a mixture of experts system (Jacobs, Jordan, Nowlan, & Hinton, 1991; Jordan & Jacobs, 1994) where the experts are trained in isolation. It can also be interpreted as a system where a set of experts is trained independently on the same problem and that profits from combining these experts for making predictions (Perrone & Cooper, 1993). Finally, RFWR can be interpreted as a nonparametric memory-based learner (Atkeson, Moore, & Schaal, 1997a) which stores only data that are surprising.

In the next section, we give some motivation for our approach to incremental learning. Section 3 describes the details of our nonparametric incremental learning system and outlines some of its statistical characteristics. Section 4 discusses a variety of empirical evaluations. Section 5 outlines related work, and Section 6 concludes this article.

2 Incremental Learning

2.1 Statistical Assumptions. The assumed statistical model of our problems is the standard regression model,

$$\mathbf{y} = f(\mathbf{x}) + \varepsilon, \tag{2.1}$$

where $\mathbf{x} \in \Re^n$ denotes the n-dimensional vector of input variables, $\mathbf{y} \in \Re^m$ the m-dimensional vector of output variables, and $f(\cdot)$ a deterministic vector-valued function mapping the input $\mathbf{x}$ to the output $\mathbf{y}$. The additive random noise ε is assumed to be independently distributed, $E\{\varepsilon_i \varepsilon_j\} = 0$ for $i \neq j$, and mean zero, $E\{\varepsilon \mid \mathbf{x}\} = 0$, but otherwise of unknown distribution ($E\{\cdot\}$ denotes the expectation operator). The input data are distributed according to the density $p(\mathbf{x})$.

2.2 Localizing Interference. Interference in learning is a natural side effect of the ability to generalize, that is, to interpolate or extrapolate an

output for an unseen input from previously learned data. Generalization is accomplished by allowing changes to the parameters of the learning system to have nonlocal effects. If these effects reduce the overall correctness of predictions to a larger extent than they improve them, interference is called negative or even catastrophic. Incremental learning is particularly endangered by negative interference because there is no direct way to balance the amount of positive interference (that is, generalization) with the amount of negative interference. Any parameter update is usually greedy; its only concern is with the reduction of the error of the current piece of training data. To see the statistical causes of interference, consider using the mean squared error criterion J to select a model $\hat{f}(\cdot)$ to approximate the true function $f(\cdot)$:

$$J = E\left\{\left\|\mathbf{y} - \hat{f}(\mathbf{x})\right\|^2\right\} = \int_{-\infty}^{+\infty} \left\|\mathbf{y} - \hat{f}(\mathbf{x})\right\|^2 p(\mathbf{x}, \mathbf{y}) d\mathbf{x}\, d\mathbf{y}$$
$$= \int_{-\infty}^{+\infty} \left\|\mathbf{y} - \hat{f}(\mathbf{x})\right\|^2 p(\mathbf{y} \mid \mathbf{x}) p(\mathbf{x})\, d\mathbf{x}\, d\mathbf{y}. \tag{2.2}$$

This equation states that, in general, the approximation result for $\hat{f}(\cdot)$ depends on both the conditional distribution $p(\mathbf{y} \mid \mathbf{x})$ and the input distribution $p(\mathbf{x})$ of the data (Fan & Gijbels, 1996). Only for an infinite amount of training data, $\hat{f}(\cdot)$ will asymptotically depend solely on $p(\mathbf{y} \mid \mathbf{x})$ (Papoulis, 1991):

$$\hat{f}(\mathbf{x}) = E\{\mathbf{y} \mid \mathbf{x}\} = \int_{-\infty}^{+\infty} \mathbf{y}\, p(\mathbf{y} \mid \mathbf{x})\, d\mathbf{y}. \tag{2.3}$$

Thus, for a finite amount of data, a stable model $\hat{f}(\cdot)$ can be obtained only if neither of these distributions changes during learning.

These considerations point toward the two major causes for negative interference. If $p(\mathbf{y} \mid \mathbf{x})$ changes, that is, the functional relationship between $\mathbf{x}$ and $\mathbf{y}$ is nonstationary, the parameters in a learning system may have to change. Analogously, if the data for learning are not sampled from a fixed input distribution $p(\mathbf{x})$, the parameters of the learning system may also change. It is particularly a change of the input distribution $p(\mathbf{x})$ that is likely to happen in incremental learning. Imagine a robot learning an inverse dynamics model of its arm, a model that maps joint positions, joint velocities, and joint accelerations to corresponding joint torques. Whenever the robot moves, it receives valid data about this functional relationship. However, since the robot is fulfilling different tasks at different times, the sampled data will come from quite different input distributions; for example, consider the difference between movements for cooking and movements for playing tennis.

One of the interesting properties of learning with localized receptive fields lies in their potential robustness toward interference. If learning is spatially localized—training data at one location have a negligible effect

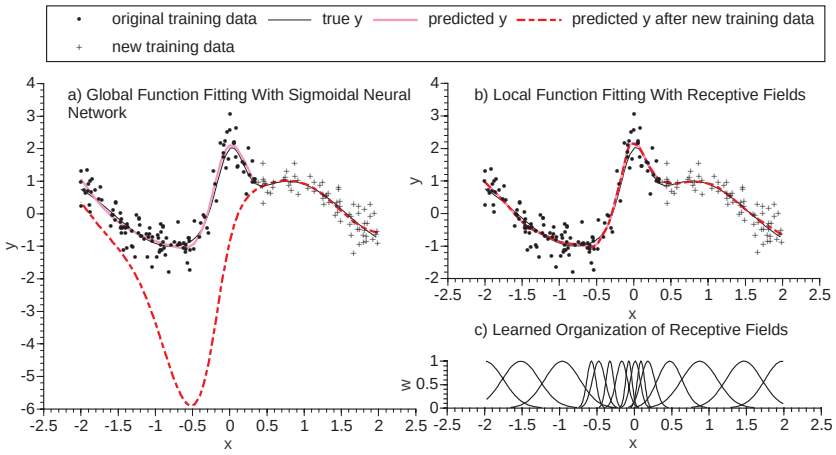

Figure 1: (a) Results of function approximation of the function $y = \sin(2x) + 2\exp(-16x^2) + N(0, 0.16)$ with a sigmoidal neural network. (b) Results of function approximation by a local receptive field–based algorithm, fitting locally linear models in each receptive field. Note that the data trace "true y," "predicted y," and "predicted y after new training data" largely coincide. (c) The organization of the (gaussian) receptive fields of (b) after training.

on the parameters of distant receptive fields—interference will be spatially localized as well. Figure 1 gives an illustration of this effect. Using a synthetic data set suggested by Fan and Gijbels (1995), we trained a three-layer sigmoidal feedforward neural network (six hidden units, using backpropagation with momentum) on 130 noisy data points uniformly distributed in $x \in [-2.0, 0.5]$. The excellent function fit obtained is shown by the "predicted y" trace in Figure 1a. Then we continued training the network on 70 new data points drawn from the same function but with a changed input distribution $x \in [0.5, 2.0]$. The network learned to accommodate these new data points, but in doing so, it also significantly changed its predictions for the previously learned data, although these data are largely separate from the new training data. This effect is due to the nonlocal nature of sigmoidal basis functions and is prone to lead to catastrophic interference, as shown in Figure 1a.

We repeated the same experiment with our receptive field–based learning system, RFWR, which generates locally linear models in each receptive field and blends them for predictions (see Figures 1b and 1c). On the original training data, RFWR achieves comparable results to that of the sigmoidal neural network. After training on the new data, however, no interference

is apparent. The original fit in the left part of the graph was not visibly altered, in contrast to the neural network. Robustness toward negative interference, is accomplished by localizing interference, the best we can do since interference cannot be eliminated for finite data samples.

2.3 Avoiding the Problem of Resource Allocation. Due to the bias-variance trade-off (Geman et al., 1992), learning algorithms have to include a model selection phase in order to find an appropriate compromise between oversmoothing and overfitting. Usually this is accomplished by setting certain meta parameters, for instance, the number of hidden units in a neural network, according to some model selection criterion, such as cross-validation (Stone, 1974). A question frequently asked in model selection (Bishop, 1996) thus becomes: "How many free parameters should be allocated in order to achieve (in expectation) a good bias-variance trade-off?" However, another approach can be pursued: "Given a fixed number of free parameters, how should a given data set be spatially limited in order to achieve (in expectation) a good bias-variance trade-off for the remaining data?" Instead of adapting the complexity of the learning system, one can also adapt the complexity of the region the data are drawn from. For general nonlinear function approximators, it is unclear how to answer this question. For spatially localized function fitting, however, this question translates into: "How should the extent of a receptive field be changed in order to make its associated parametric model fit the data appropriately?" Such an approach transforms the global bias-variance problem into a local one.

The advantage of having each receptive field deal with the bias-variance trade-off individually lies in avoiding the resource allocation problem. In the spirit of a Taylor series expansion, let us assume that we know how to adjust the region of validity, that is, the size and shape of a receptive field, of each locally linear model such that its approximation error at the center—its bias θ_k—is determined by an optimal bias-variance trade-off (see Figure 2a). In order to approximate the entire nonlinear function, we have to cover the input space with sufficiently many locally linear models such that every data point is handled by at least one of them. Importantly, it does not matter whether we allocate too many local models. If we restrict extrapolation of the linear models to the θ_k bound (which actually corresponds to a minimal activation strength of a receptive field), an average of the outputs of all k linear models at a query point x_q cannot have a larger error than $\max(\theta_k)$, as illustrated in Figure 2b. Indeed, allocating too many local models actually has a positive effect. Due to averaging, more overlapping linear models will tend to improve function estimates in the spirit of and with the same limitations as in ensemble methods (Perrone & Cooper, 1993).

Although deriving an optimal local bias-variance trade-off remains hard (Friedman, 1984; Fan & Gijbels, 1996), the local nature of the problem allows new ways to find at least satisfying and computationally affordable solutions. Section 3 will demonstrate how a stochastic approximation of

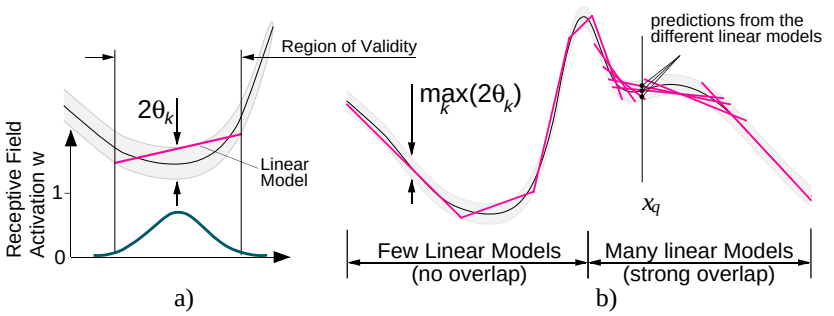

Figure 2: (a) Region of validity of a linear model and its approximation bias θ_k. (b) Function approximation with piecewise linear models.

local leave-one-out cross-validation in conjunction with a regularization approach can be used to realize the local bias-variance trade-off, and even to control approximately the expected bias θ_k in each local model.

2.4 Summary. Given the discussion of the last two sections, a promising route to robust incremental learning seems to be a local receptive field–based system that can also adjust the extent of its receptive fields. However, care must be taken how one goes about accomplishing this goal. Learning methods based on competitive learning usually do not achieve the properties described in the previous section. In competitive learning, the size of a receptive field results from a global competition process of all local models to account for the training data. Therefore, changing the number of local models causes a change of the extent of all receptive fields such that the number of local models becomes a critical choice for the bias-variance trade-off—exactly what we would wish to avoid. The next section explains how an alternative approach based on nonparametric statistics offers a route to achieve our goals without resorting to competitive learning.

3 Receptive Field–Weighted Regression

RFWR constructs a system of receptive fields for incremental function approximation. A prediction $\hat{\mathbf{y}}$ for a query point $\mathbf{x}$ is built from the normalized weighted sum of the individual predictions $\hat{\mathbf{y}}_k$ of all receptive fields:

$$\hat{\mathbf{y}} = \frac{\sum_{k=1}^{K} w_k \hat{\mathbf{y}}_k}{\sum_{k=1}^{K} w_k}. \tag{3.1}$$

The weights w_k correspond to the activation strengths of the corresponding receptive fields. They are determined from the size and shape of each receptive field, characterized by a kernel function. A variety of possible kernels have been suggested (Atkeson, Moore, and Schaal, 1997a). For analytical convenience, we use a gaussian kernel,

$$w_k = \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{c}_k)^T\mathbf{D}_k(\mathbf{x}-\mathbf{c}_k)\right), \qquad \text{where} \quad \mathbf{D}_k = \mathbf{M}_k^T\mathbf{M}_k, \quad (3.2)$$

which parameterizes the receptive field by its location in input space, $\mathbf{c}_k \in \Re^n$, and a positive definite distance metric $\mathbf{D}_k$, determining the size and shape of the receptive field. For algorithmic reasons, it is convenient to generate $\mathbf{D}_k$ from an upper triangular matrix $\mathbf{M}_k$ in order to ensure that $\mathbf{D}_k$ is positive definite.

Within each receptive field, a simple parametric function models the relationship between input and output data. Local polynomials of low order have found widespread use in nonparametric statistics (Nadaraya, 1964; Watson, 1964; Wahba & Wold, 1975; Cleveland, 1979; Cleveland & Devlin, 1988). We will focus on locally linear models because they accomplish a favorable compromise between computational complexity and quality of result (Hastie & Loader, 1993):

$$\hat{\mathbf{y}}_k = (\mathbf{x}-\mathbf{c}_k)^T\mathbf{b}_\mathbf{k} + b_{0,k} = \tilde{\mathbf{x}}^T\beta_k, \qquad \tilde{\mathbf{x}} = ((\mathbf{x}-\mathbf{c}_k)^T, 1)^T, \quad (3.3)$$

where β_k denotes the parameters of the locally linear model and $\tilde{\mathbf{x}}$ is a compact form of the center-subtracted, augmented input vector to simplify the notation.

To clarify the elements and parameters of RFWR, Figure 3 gives a network-like illustration for a single output system. The inputs are routed to all receptive fields, each of which consists of a linear and a gaussian unit. The learning algorithm of RFWR determines the parameters $\mathbf{c}_k$, $\mathbf{M}_k$, and β_k for each receptive field independently—without any information about the other receptive fields, in contrast to competitive learning. RFWR adds and prunes receptive fields as needed, such that the number of receptive fields, K, will automatically adjust to the learning problem at hand. A one-dimensional example of function fitting with RFWR was shown in Figures 1b and 1c. It should be noted that the size of each receptive field adapted according to the local curvature of the function, that there is a certain amount of overlap between the receptive fields, and that the center locations have not been chosen with respect to any explicit optimization criterion.

3.1 Learning with RFWR. Three ingredients of the algorithm need to be discussed: the update of the linear model parameters β_k, the decomposed distance metric $\mathbf{M}_k$, and when and where to add and prune receptive fields. The centers $\mathbf{c}_k$ are not changed after they are allocated. For the sake of clarity,

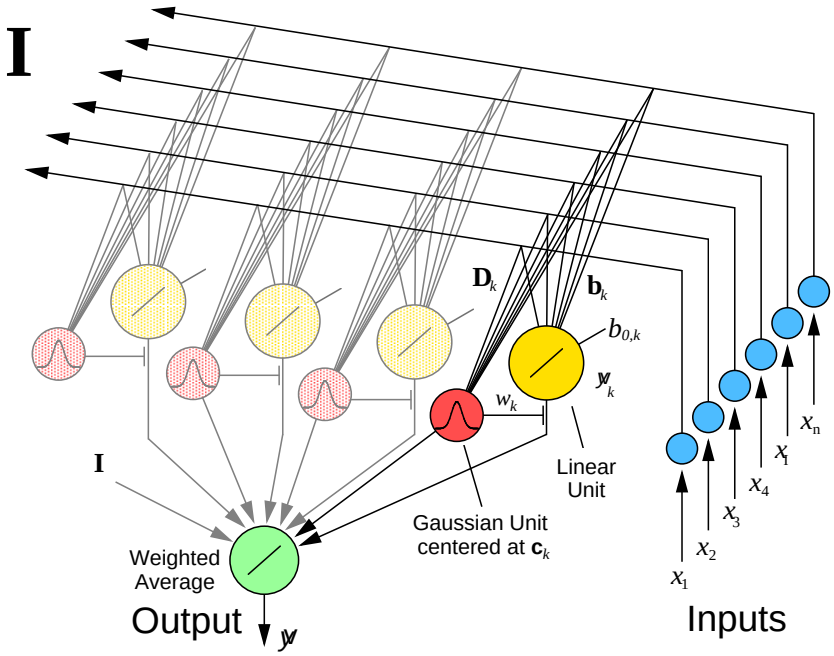

Figure 3: A network illustration of receptive field–weighted regression.

we will drop the subscript k whenever we deal with one receptive field at a time from now on since each receptive field is updated in the same way.

3.1.1 Learning the Linear Model. Learning of β is straightforward since the problem is linear in β. It will be useful to leave the incremental learning framework for a moment and think in terms of a batch update. If we summarize the input part of all p training data points in the rows of the matrix $\mathbf{X} = (\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots, \tilde{\mathbf{x}}_p)^T$, the corresponding output part in the rows of the matrix $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p)^T$, and the corresponding weights in the diagonal matrix $\mathbf{W} = \mathrm{diag}(w_1, w_2, \ldots, w_p)$, the parameter vector β can be calculated from a weighted regression:

$$\beta = (\mathbf{X}^T\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{X}\mathbf{Y} = \mathbf{P}\mathbf{X}^T\mathbf{W}\mathbf{Y}. \tag{3.4}$$

This kind of locally weighted regression has found extensive application in nonparametric statistics (Cleveland, 1979; Cleveland & Loader, 1995), in time-series prediction (Farmer & Sidorowich, 1987, 1988), and in regression

learning problems (Atkeson, 1989b; Moore, 1991; Schaal & Atkeson, 1994a; Atkeson et al., 1997a). The result for β in Equation 3.4 is exactly the same when β is calculated by recursive least squares from one sequential sweep through the training data (Ljung & Söderström, 1986). Given a training point $(\mathbf{x}, \mathbf{y})$, the incremental update of β yields:

$$\begin{aligned} &\beta^{n+1} = \beta^n + w\mathbf{P}^{n+1}\tilde{\mathbf{x}}\mathbf{e}_{cv}^T \\ &\text{where} \quad \mathbf{P}^{n+1} = \frac{1}{\lambda}\left(\mathbf{P}^n - \frac{\mathbf{P}^n\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T\mathbf{P}^n}{\frac{\lambda}{w} + \tilde{\mathbf{x}}^T\mathbf{P}^n\tilde{\mathbf{x}}}\right) \quad \text{and} \quad \mathbf{e}_{cv} = (\mathbf{y} - \beta^{n^T}\tilde{\mathbf{x}}). \end{aligned} \tag{3.5}$$

This update is employed by RFWR. It is useful to note that recursive least squares corresponds to a Newton training method with guaranteed convergence to the global minimum of, in our case, a weighted squared error criterion (Atkeson et al., 1997a). Furthermore, the recursive update avoids an explicit matrix inversion. Differing from the batch update in equation 3.4, equation 3.5 also includes a forgetting factor λ. Changes to the decomposed distance metric $\mathbf{M}$ during learning (see below) will change the weights w. For this reason, it is necessary to include λ in equation 3.5 in order to cancel the contributions gradually from previous data points where $\mathbf{M}$ was not yet learned properly (Ljung & Söderström, 1986).

3.1.2 Learning the Shape and Size of the Receptive Field. Adjusting the shape and size of the receptive field is accomplished by adjusting the decomposed distance metric $\mathbf{M}$. At first glance, one might hope that this could be done by gradient descent in the weighted mean squared error criterion,

$$J = \frac{1}{W}\sum_{i=1}^{p} w_i \left\|\mathbf{y}_i - \hat{\mathbf{y}}_i\right\|^2 \quad \text{where} \quad W = \sum_{i=1}^{p} w_i, \tag{3.6}$$

which is the basis of the solution of locally weighted regression in equation 3.4 (Atkeson et al., 1997a). Unfortunately, minimizing equation 3.6 may result in a quite inappropriate solution. If for each training point one receptive field is centered right on this point, and the corresponding $\mathbf{M}$ is chosen such that the receptive field is so narrow that it is activated only by this data point, the corresponding linear model can fit this one data point with zero error. The function approximation result would strongly tend toward overfitting. It is this property that has made learning algorithms resort to competitive learning with a fixed number of local receptive fields. The global competitive process will prevent receptive fields from modeling just one data point (assuming there are more data points than receptive fields) (Moody & Darken, 1988; Jordan & Jacobs, 1994). But allowing for such a global competitive process takes away the property of being a local learner, even if the receptive fields are actually spatially localized.

An alternative way to address this overfitting effect is to use leave-one-out cross-validation. The cost function to be minimized changes from equation 3.6 to

$$J = \frac{1}{W} \sum_{i=1}^{p} w_i \left\| \mathbf{y}_i - \hat{\mathbf{y}}_{i,-i} \right\|^2 . \tag{3.7}$$

The notation $\hat{\mathbf{y}}_{i,-i}$ denotes that the prediction of the ith data point is calculated from training the learning system with the ith data point excluded from the training set. Thus, it becomes inappropriate for a receptive field to focus on just one training point since the error measure is calculated from data that did not exist in the training set. Leave-one-out cross-validation is usually computationally very expensive since a p-fold training of the learning system is required, for p data points in the training set. Furthermore, for example, for a sigmoidal neural network, it might be unclear how to combine the resultant p different learned parameters into a single solution. However, for linear regression problems, there is a result rendering these concerns irrelevant. Due to the Sherman-Morrison-Woodbury theorem (Belsley, Kuh, & Welsh, 1980), equation 3.7 can be rewritten as:

$$J = \frac{1}{W} \sum_{i=1}^{p} w_i \left\| \mathbf{y} - \hat{\mathbf{y}}_{i,-i} \right\|^2 = \frac{1}{W} \sum_{i=1}^{p} \frac{w_i \left\| \mathbf{y}_i - \hat{\mathbf{y}}_i \right\|^2}{(1 - w_i \tilde{\mathbf{x}}_i^T \mathbf{P} \tilde{\mathbf{x}}_i)^2}. \tag{3.8}$$

This equation states that the leave-one-out cross-validation error can be obtained without p-fold training of the learning system, instead by an adjustment of the weighted mean squared error with the help of the inverted covariance matrix $\mathbf{P}$ (cf. equation 3.4). Equation 3.8 corresponds to a weighted version of the PRESS residual error in standard linear regression techniques (Myers, 1990). Neglecting for a moment how this cost function can be minimized incrementally, we have obtained a criterion that can be used to adjust $\mathbf{M}$ (Schaal & Atkeson, 1994b).

Unfortunately, there is still a point of concern with equation 3.8. Minimizing the locally weighted leave-one-out cross-validation error results in a consistent learning system; with an increasing number of training data, the receptive fields will shrink to a very small size. The advantage of this behavior is that function approximation becomes asymptotically unbiased (i.e., consistent), but as a disadvantage, an ever-increasing number of receptive fields will be required to represent the approximated function. This property can be avoided by introducing a penalty term in equation 3.8:

$$J = \frac{1}{W} \sum_{i=1}^{p} \frac{w_i \left\| \mathbf{y}_i - \hat{\mathbf{y}}_i \right\|^2}{(1 - w_i \tilde{\mathbf{x}}_i^T \mathbf{P} \tilde{\mathbf{x}}_i)^2} + \gamma \sum_{i,j=1}^{n} D_{ij}^2, \tag{3.9}$$

where the scalar γ determines the strength of the penalty. By penalizing the sum of squared coefficients of the distance metric $\mathbf{D}$, we are essentially pe-

nalizing the second derivatives of the function at the site of a receptive field. This is similar to approaches taken in spline fitting (deBoor, 1978; Wahba, 1990) and acts like a low-pass filter: the higher the second derivatives, the more smoothing (and thus bias) will be introduced locally. Another positive effect of the penalty term is that the introduction of bias reduces the variance of the function estimate, a problem usually associated with local function fitting methods (Friedman, 1984). Section 3.3 will outline the properties of equation 3.9 in more detail.

What remains is how to minimize equation 3.9 incrementally by adjusting $\mathbf{M}$ by gradient descent with learning rate α:

$$\mathbf{M}^{n+1} = \mathbf{M}^n - \alpha \frac{\partial J}{\partial \mathbf{M}}. \tag{3.10}$$

Applying the chain rule, the derivative of equation 3.10 can be written as:

$$\begin{aligned} \frac{\partial J}{\partial \mathbf{M}} &= \frac{\partial}{\partial \mathbf{M}} \left(\sum_{i=1}^{p} \frac{w_i \left\| \mathbf{y}_i - \hat{\mathbf{y}}_i \right\|^2}{W(1 - w_i \tilde{\mathbf{x}}_i^T \mathbf{P} \tilde{\mathbf{x}}_i)^2} + \gamma \sum_{i,j=1}^{n} D_{ij}^2 \right) \\ &= \frac{\partial}{\partial \mathbf{M}} \left(\sum_{i=1}^{p} J_{1,i} + J_2 \right) = \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial J_{1,i}}{\partial w_j} \frac{\partial w_j}{\partial \mathbf{M}} + \frac{\partial J_2}{\partial \mathbf{M}}. \end{aligned} \tag{3.11}$$

Without storing data in incremental learning, we cannot use cross-validation and thus cannot obtain the true gradient in equation 3.11. The usual approach to deriving a stochastic gradient would be to drop the two sums in equation 3.11. However, this approximate gradient would be quite inaccurate since the first term of that equation would always be positive: shrinking the receptive field reduces the weight of a data point and thus its contribution to the weighted error. It turns out that we are able to derive a much better stochastic approximation. Given one training point $(\mathbf{x}, \mathbf{y})$ and its associated weight w from equation 3.2, the derivative for this point can be approximated as:

$$\frac{\partial J}{\partial \mathbf{M}} \approx \sum_{i=1}^{p} \frac{\partial J_{1,i}}{\partial w} \frac{\partial w}{\partial \mathbf{M}} + \frac{w}{W} \frac{\partial J_2}{\partial \mathbf{M}} = \frac{\partial w}{\partial \mathbf{M}} \sum_{i=1}^{p} \frac{\partial J_{1,i}}{\partial w} + \frac{w}{W} \frac{\partial J_2}{\partial \mathbf{M}}. \tag{3.12}$$

Summing equation 3.12 over all data points and recalling that W stands for the sum of weights (cf. equation 3.6), equation 3.12 can be verified to result in equation 3.11. Despite the term $J_{1,i}$, it is now possible to obtain an incremental version of the stochastic derivative in equation 3.12 by introducing the "memory traces" W, E, $\mathbf{H}$, and $\mathbf{R}$ (see the notation in equation 3.15):

$$\begin{aligned} W^{n+1} &= \lambda W^n + w \\ E^{n+1} &= \lambda E^n + w \mathbf{e}_{cv}^T \mathbf{e}_{cv} \end{aligned}$$

$$\mathbf{H}^{n+1} = \lambda \mathbf{H}^n + \frac{w\tilde{\mathbf{x}}\mathbf{e}_{cv}^T}{1-h}, \text{ where } h = w\tilde{\mathbf{x}}^T\mathbf{P}^{n+1}\tilde{\mathbf{x}}$$
$$\mathbf{R}^{n+1} = \lambda \mathbf{R}^n + \frac{w^2\mathbf{e}_{cv}^T\mathbf{e}_{cv}\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T}{1-h}. \tag{3.13}$$

The resulting incremental version of the derivative (see equation 3.12) becomes:

$$\frac{\partial J}{\partial \mathbf{M}} \approx \frac{\partial w}{\partial \mathbf{M}} \sum_{i=1}^{p} \frac{\partial J_{1,i}}{\partial w} + \frac{w}{W^{n+1}} \frac{\partial J_2}{\partial \mathbf{M}} \tag{3.14}$$

where:

$$\frac{\partial w}{\partial M_{rl}} = -\frac{1}{2} w(\mathbf{x}-\mathbf{c})^T \frac{\partial \mathbf{D}}{\partial M_{rl}}(\mathbf{x}-\mathbf{c}), \qquad \frac{\partial J_2}{\partial M_{rl}} = 2\gamma \sum_{i,j=1}^{n} D_{ij} \frac{\partial D_{ij}}{\partial M_{rl}}$$

$$\frac{\partial D_{ij}}{\partial M_{rl}} = \delta_{rj} M_{il} + \delta_{ir} M_{jl} \qquad (\delta \text{ is the Kronecker operator})$$

$$\sum_{i=1}^{p} \frac{\partial J_{1,i}}{\partial w} \approx -\frac{E^{n+1}}{\left(W^{n+1}\right)^2} + \frac{1}{W^{n+1}} \left(\mathbf{e}_{cv}^T\mathbf{e}_{cv} - \left(2\mathbf{P}^{n+1}\tilde{\mathbf{x}} \left(\mathbf{y} - \tilde{\mathbf{x}}^T\beta^{n+1} \right)^T \right) \otimes \mathbf{H}^n - \left(2\mathbf{P}^{n+1}\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T\mathbf{P}^{n+1} \right) \otimes \mathbf{R}^n \right).$$

Deriving this derivative is possible due to the fact that an application of the Sherman-Morrison-Woodbury theorem allows us to take derivatives through the inverted covariance matrix $\mathbf{P}$ (Belsley et al., 1980; Atkeson & Schaal, 1995), and that a sum of the form $\Sigma \mathbf{v}_i^T \mathbf{Q} \mathbf{v}_i$ can be written as $\Sigma \mathbf{v}_i^T \mathbf{Q} \mathbf{v}_i = \mathbf{Q} \otimes \Sigma \mathbf{v}_i \mathbf{v}_i^T$, where the operator $\otimes$ denotes an element-wise multiplication of two homomorphic matrices or vectors with a subsequent summation of all coefficients, $\mathbf{Q} \otimes \mathbf{V} = \Sigma Q_{ij} V_{ij}$. It is interesting to note that the stochastic derivative (see equation 3.14) is not just concerned with reducing the error of the current training point, as in many other learning algorithms, but rather that it takes into account the previously encountered training data, too, through the memory traces (see equation 3.13). Thus, both the β and $\mathbf{M}$ update in RFWR are not greedy with respect to the current training sample, a characteristic that will contribute favorably to speed and robustness of incremental learning.

3.1.3 Adding Receptive Fields and Automatic Bias Adjustment. A new receptive field is created if a training sample $(\mathbf{x}, \mathbf{y})$ does not activate any existing receptive field by more than a threshold w_{gen}. The center of the new

receptive field becomes $\mathbf{c} = \mathbf{x}$, $\mathbf{M}$ is set to a manually chosen default value, $\mathbf{M}_{def}$, and all other parameters are initialized to zero, except the matrix $\mathbf{P}$. $\mathbf{P}$ corresponds to an inverted covariance matrix of the weighted inputs (treating the constant input 1 as the $(n+1)$th input). A suitable initialization of $\mathbf{P}$ is as a diagonal matrix, the diagonal elements set to $P_{ii} = 1/r_i^2$, where the coefficients r_i are usually small quantities, such as, 0.001 (Ljung & Söderström, 1986). We summarize all r_i in the $(n+1)$-dimensional vector $\mathbf{r} = (r_1, r_2, \ldots, r_{n+1})^T$.

The parameters $\mathbf{r}$ have an interesting statistical interpretation: they introduce bias in the regression coefficients β and correspond to one of the common forms of biased regression, ridge regression (Belsley et al., 1980). From a probabilistic point of view, they are Bayesian priors that the coefficients of β are zero. From an algorithmic perspective, they are fake data points of the form $[\mathbf{x}_r = (0 \ldots, r_i^2, 0, \ldots)^T, \mathbf{y}_r = 0]$ (Atkeson et al., 1997a). Under normal circumstances, the sizes of the coefficients of $\mathbf{r}$ are too small to introduce noticeable bias. However, ridge regression parameters are important if the input data are locally rank deficient, that is, the matrix inversion in equation 3.4 is close to singular. For high-dimensional input spaces, it is quite common to have locally rank deficient input data. Although RFWR does not explicitly require matrix inversions, the rank deficiency affects the incremental update in equation 3.5 by generating estimates of β with very large variances, causing unreliable predictions. Nonzero ridge regression parameters reduce this variance, though at the cost of introducing bias. An appropriate compromise can be found by including the ridge parameters as adjustable terms in RFWR using gradient descent in the cost (see equation 3.9):

$$\mathbf{r}^{n+1} = \mathbf{r}^n - \alpha_r \frac{\partial J}{\partial \mathbf{r}}. \tag{3.15}$$

After each update of $\mathbf{P}$, the change in $\mathbf{r}$ is added to $\mathbf{P}$. Additionally, it is necessary to add back the fraction of $\mathbf{r}$ that was lost due to the forgetting factor λ; bias should not be forgotten over time. These two computations can be performed together and are surprisingly simple. Section A.1 details this update and the stochastic approximation of $\partial J/\partial \mathbf{r}$, which is analogous to the derivation of equation 3.14.

3.1.4 Pruning Receptive Fields. The last element in RFWR is a pruning facility. A receptive field is pruned if it overlaps too much with another receptive field. This effect is detected by a training sample activating two receptive fields simultaneously more than w_{prune}. The receptive field with the larger determinant of the distance metric $\mathbf{D}$ is pruned. For computational convenience, $\det(\mathbf{D})$ can be approximated by ΣD_{ii}^2 (Deco & Obradovic, 1996). It should be noted that pruning due to overlap aims primarily at computational efficiency, since, as discussed in section 2.3, overlap does not

degrade the approximation quality. The second cause for pruning is if the bias-adjusted weighted mean squared error,

$$wMSE = \frac{E^n}{W^n} - \gamma \sum_{i,j=1}^{n} D_{ij}^2, \tag{3.16}$$

of the linear model of a unit is excessively large in comparison to other units. The bias adjustment term can be derived from the asymptotic behavior of RFWR, outlined in section 3.3 and detailed in Schaal and Atkeson (1997). Empirically, there are usually two ways to adjust $\mathbf{M}$ in order to minimize equation 3.9. The one we normally want to avoid is $\mathbf{M} = 0$, the zero matrix. It indicates that the receptive field performs global regression instead of locally weighted regression. Global linear regression for a nonlinear function has a large *wMSE*. A simple outlier detection test among the *wMSE* of all receptive fields suffices to deal with such behavior. The receptive field is then reinitialized with randomized values. Normally pruning takes place rarely, and if it happens, it is mostly due to an inappropriate initialization of RFWR.

3.1.5 Summary of RFWR. In summary, each receptive field in RFWR has three sets of adjustable parameters: β for the locally linear model, $\mathbf{M}$ for the size and shape of the receptive field, and $\mathbf{r}$ for the bias. The linear model parameters are updated by a Newton method, and the other parameters are updated by gradient descent. A compact pseudo-code overview of RFWR is shown below.

```
Initialize the RFWR with no receptive field (RF);
For every new training sample (x, y):
   a) For k = 1 to #RF:
         Calculate the activation from equation 3.2
         Update the receptive field parameters
         according to equations 3.10 and 3.15
      end;
   b) If no subnet was activated by more than w_gen;
         Create a new RF with c = x, M = M_def
      end;
   c) If two RFs are activated more than w_prune:
         Erase the RF with the larger det(D)
      end;
   d) Calculate the m = E{wMSE} and std = E{(wMSE - m)^2}^0.5
      of all RFs;
   e) For k = 1 to #RF:
         If |wMSE - m| > φstd,
            Reinitialize receptive field with M = ε M_def
         end;
      end;
```

The scalar φ is a (positive) outlier removal threshold, such as $\varphi = 2.576$ or $\varphi = 3.291$ (corresponding to a 99.0% or 99.9% confidence value with respect to a normal distribution), and the scalar ε is a random value $\varepsilon = 1+|N(0, 1)|$. This choice of ensures that the new distance metric will result in a smaller receptive field, which is less likely to converge to an $\mathbf{M} = 0$ solution. It useful to note that the parameters w_{prune}, w_{gen}, and φ can be chosen independently of a particular learning problem and should thus be considered more like constants of the algorithm rather than open parameters.

3.2 Second-Order Gradient Descent. With a little extra computation, it is possible to replace the gradient descent update of $\mathbf{M}$ in equation 3.10 by second-order gradient descent to gain learning speed. For this purpose, we adopted Sutton's (1992a, 1992b) Incremental delta-bar-delta (IDBD) algorithm. The derivation of the algorithm remains as demonstrated in Sutton (1992a, 1992b), only his standard least squares criterion is replaced by our cost function (see equation 3.9), and we apply IDBD to updating a distance metric. Section A.2 provides the details of the algorithm. It is also possible to apply second-order learning to the ridge regression update (see equation 3.15). Empirically, however, we did not find any significant improvements of doing so and, hence, incorporated second-order updates only for the distance metric in RFWR.

3.3 Asymptotic Properties of RFWR. In Schaal and Atkeson (1996, 1997) we derived the asymptotic properties of RFWR's cost function (see equation 3.9). Here we will just mention some of these results that are directly relevant to this article. Assuming that (1) the number of training data points p goes to infinity, (2) within the range of a receptive field a second-order Taylor series expansion fits the training function sufficiently accurately (3) the variance of the noise σ^2 is locally constant, and (4) the input distribution is locally uniform, the following statements can be made:

- The penalty term in the cost function (see equation 3.9) introduces nonvanishing bias like a low-pass filter: the higher the second derivatives (Hessian) of the function, the more bias is incurred.
- The estimated locally linear model $\mathbf{b}$ is asymptotically unbiased.
- The distance metric $\mathbf{D}$ will be a scaled approximation of the Hessian.
- An appropriate penalty term γ for a learning problem can be computed from an estimate of the maximal eigenvalues of the Hessian. This corresponds to a smoothness bias.
- A bias-adjusted weighted mean squared error, *wMSE*, can be formulated in order to compare the approximation quality of receptive fields. This measure was employed in equation 3.16.

These asymptotic results confirm that the penalty term in the cost function (see equation 3.9) has the desired characteristics as mentioned in sections 2.3 and 3.1.2: receptive fields cannot shrink to zero size, and a controlled amount of bias was introduced. It is interesting that the estimated locally linear model **b** tends to become unbiased (under the assumption that $O(2)$ errors of the Taylor series are negligible). This implies that applications requiring a gradient estimate from the function approximator can expect reliable results. The calculation of the gradient estimate is a natural by-product of every lookup in RFWR.

4 Simulation Results

4.1 Basic Function Approximation with RFWR. First, we will establish that RFWR is capable of competing with state-of-the-art supervised learning techniques on a fixed training set. A sufficiently complex learning task that can still be illustrated nicely is to approximate the function

$$z = \max\left\{e^{-10x^2}, e^{-50y^2}, 1.25e^{-5(x^2+y^2)}\right\} + N(0, 0.01), \tag{4.1}$$

from a sample of 500 points, drawn uniformly from the unit square. This function consists of a narrow and a wide ridge that are perpendicular to each other and a gaussian bump at the origin (see Figure 4a). Training data are drawn uniformly from the training set without replacement; training time is measured in epochs—multiples of 500 training samples. The test set consists of 1681 data points corresponding to the vertices of a 41×41 grid over the unit square; the corresponding output values are the exact function values. The approximation error is measured as a normalized mean squared error, *nMSE*—the MSE on the test set normalized by the variance of the outputs of the test set. RFWR's initial parameters are set to $\mathbf{M}_{def} = 5\mathbf{I}$ ($\mathbf{I}$ is the identity matrix), $\gamma = 10^{-7}$, $w_{gen} = 0.1$, and $w_{prime} = 0.9$. The pruning and generation thresholds are of minor importance, determining the overlap of the receptive fields. The choice for the penalty term was calculated to tolerate a maximal bias of 0.1 (Schaal & Atkeson, 1997). The default value for the decomposed distance metric was determined manually such that an initial receptive field covered a significant portion of the input space. Ridge regression parameters did not play any role in this example and were omitted.

A first qualitative evaluation of Figure 4 confirms that RFWR fulfills our expectations. The initially large receptive fields (see Figure 4c) adjust during learning according to the local curvature of the function. They become narrow and elongated in the region of the ridges, and they remain large in the flat parts of the function (See Figure 4d). The number of the receptive fields increased from 16 after one training epoch to 48, and the final approximation result was $nMSE = 0.02$.

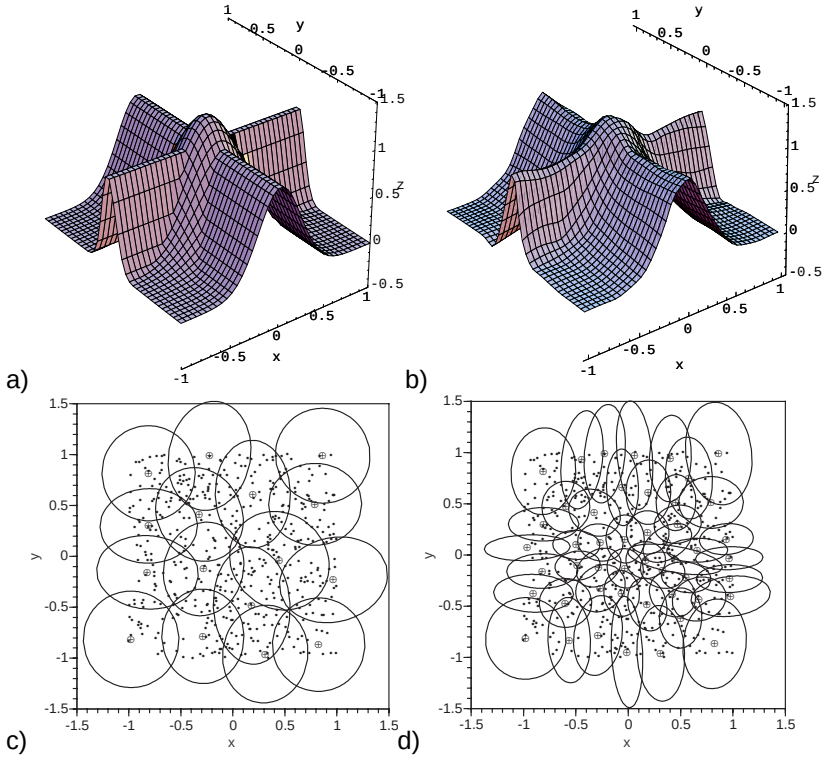

Figure 4: (a) Target function to be approximated. (b) Approximated function after 50 epochs of training. (c) Receptive fields in input space after one epoch, given by contour lines of 0.1 isoactivation and a $\oplus$ mark for the centers (the training data are displayed by small dots). (d) Receptive fields after 50 epochs of training.

We compared the learning results of RFWR with three other algorithms: standard global linear regression and a sigmoidal three-layer backpropagation neural network as baseline comparisons, and the mixture of experts algorithm as a state-of-the-art comparison (Jacobs et al, 1991; Jordan & Jacobs, 1994; Xu, Jordan, & Hinton, 1995). Standard linear regression cannot accomplish a better result than nMSE=1.0 on this example; the function has no linear trend in the chosen region of input space. The sigmoidal network was trained by backpropagation with momentum in a variety of configurations using 20 to 100 units in the hidden layer (the output layer had one linear unit). These networks did not accomplish results better than $nMSE = 0.1$

within 20,000 training epochs. Doubling the number of training samples and reducing the noise level to $N(0, 0.0001)$ finally resulted in $nMSE = 0.02$ for a 100-hidden-unit net after about 15,000 epochs. By using the cascade correlation algorithm (Fahlman & Lebiere, 1990) to fit our original 500-datapoint training set we confirmed that the function in equation 4.1 seems to be a difficult learning task for sigmoidal networks: cascade correlation did not converge when confined to using only sigmoidal hidden units, while it achieved good function fitting ($nMSE = 0.02$) when it was allowed to use gaussian hidden units.

A more natural and interesting comparison is with the mixture of experts (ME) system, particularly as suggested in Xu et al. (1995). In Xu et al. (1995), in contrast to the softmax gating network of Jordan and Jacobs (1994), the experts use a mixture of gaussians as the gating network, and both the gating net and the locally linear models in each leaf of the gating net can be updated by an analytical version of the expectation-maximation (EM) algorithm (Dempster, Laird, & Rubin, 1977). Thus, the basic elements of this form of ME are the same as in RFWR—locally linear models and gaussian receptive fields—while the training methods of the two systems differ significantly—competitive parametric likelihood maximization versus local nonparametric learning. Because ME does not add resources, the performance-determining parameters are how many experts are allocated and how the system is initialized. The algorithm was tested with 25, 50, 75, and 100 experts. Initially, the experts were uniformly distributed in the input space with an initial covariance matrix of the gaussians comparable to the initialization of RFWR's distance metric. We conducted a similar test with RFWR, setting its determining parameter, the penalty γ, to 10^{-6}, 10^{-7}, 10^{-8}, and 10^{-10}.

Figure 5 summarizes the results. Each learning curve is the average of 10 learning trials for each condition of the corresponding algorithm; the training data were randomly generated for each trial. Both algorithms achieve a nMSE=0.12 after only one training epoch—a typical signature of the fast recursive least squares updating of the linear models employed by both algorithms—which is about what the sigmoidal neural network had achieved after 10,000 to 20,000 epochs. Both algorithms converge after about 100 epochs. By adding more experts, the mixture of experts improves its performance to a best average value of $nMSE = 0.04$ with a slight trend to overfitting for 75 experts. RFWR consistently accomplishes a result of $nMSE = 0.02$ for all but the $\gamma = 10^{-6}$ runs, with a slight tendency to overfitting for $\gamma = 10^{-10}$. One standard deviation error bars are indicated by the black bars at the beginning and end of each learning curve.

It was surprising that ME did not achieve the same ultimate fit accuracy as RFWR. This behavior was due to the relatively small training set, the relatively low signal-to-noise ratio of the training data, and the way the gating network assigns training samples to each expert. By significantly increasing the amount of training data and / or lowering the noise, the results

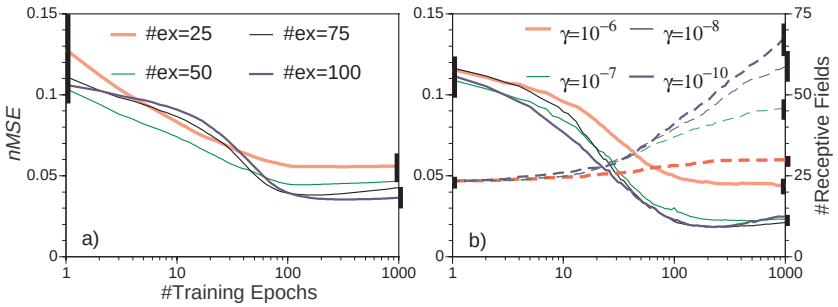

Figure 5: Average learning curves (solid lines) for (a) ME and (b) RFWR. The black bars indicate one standard deviation error bars at the beginning and end of learning; for overlapping traces having approximately the same standard deviation, only one bar is shown. For RFWR (b), the increase of the number of receptive fields over time (dashed lines) is indicated as well.

of both algorithms become indistinguishable. It seems to be the method of credit assignment that makes a significant difference. The expectation step in ME uses normalized weights (i.e., posterior probabilities) to assign training data to the experts. Normalized weights create much sharper decision boundaries between the experts than unnormalized weights as in RFWR. Thus, in the case of noise and not too many training data, the ME algorithm tends to establish decision boundaries between the experts that are too sharp and starts fitting noise. Given the underlying assumption of ME that the world was generated by a mixture of linear models, this behavior may be expected. Since in our test cases the world is actually a continuous function and not a mixture of linear models, the assumptions of ME are only an approximation, which explains why the algorithm does not perform entirely appropriately.

The assumptions of RFWR are quite different: each receptive field tries to find a region of validity that allows it to approximate the tangent plane in this region with some remaining bias. In the spirit of a low-order Taylor series expansion, this is a reasonable way to proceed. Thus, RFWR achieves consistent results with low variance (see Figure 5b). It is also interesting to see how the number of receptive fields of RFWR grows as a function of the penalty factor (see Figure 5b). As expected from the derivation of the cost function (see equation 3.9), a very small penalty parameter causes the receptive fields to keep on shrinking and entails a continuous growth of the number of receptive fields. Nevertheless, the tendency toward overfitting remained low, as can be seen in the $\gamma = 10^{-10}$ traces in Figure 5b. When continuing learning until 10,000 epochs, the *nMSE* saturated close to the

current values for all penalty factors. The local cross-validation term (equation 3.9) is responsible for this desirable behavior. When cross-validation was not used, overfitting was significantly more pronounced and the *nMSE* continued increasing for very small penalty factors.

4.2 Dealing with Irrelevant Inputs. In order to establish the usefulness of the ridge regression parameters, we conducted a further comparison with the mixture of experts. In sensorimotor control, it is unlikely that all variables given to the learning system are equally relevant to the task. Possible kinds of extraneous inputs include constant inputs, changing inputs that are meaningless, and copies and linear combinations of other inputs. Ideally, one would like an autonomous learning system to be robust toward such signals. To explore the behavior of ME and RFWR in such cases, three additional inputs were added to the function (see equation 4.1): a) one almost constant input of $N(0.1, 0.001)$, one input with a Brownian walk in the interval $[-0.1, 0.1]$, and one input that was a copy of x with added gaussian noise $N(0, 0.0025)$. Otherwise training data were generated uniformly by the function (in equation 4.1), but with reduced additive noise of $N(0, 0.0025)$ to improve the signal-to-noise ratio. For these tests, the ridge regression coefficients were initialized to 0.25 for each input.

Figure 6 summarizes the average results of 10 trials for each algorithm. In Figure 6a, we show the mean *nMSE* and its standard deviation on two test sets. In Test1, the predictions were generated by using only the regression coefficients of the relevant inputs, β_0, β_1, β_2, on the same 1681-point test set as in the experiment of section 4.1. This was to establish whether these coefficients adjusted correctly to model the target function. Both algorithms achieved good learning results on this test (see Figure 6a). In Test2, we probed the robustness of the learned model toward the irrelevant inputs. We added the noisy constant, the Brownian, and the noisy x-copy input to the test set, but we also added an offset of 0.1 to each of these signals. If the algorithm learned that these inputs were irrelevant, this change should not matter. If the irrelevant inputs were mistakenly employed as signals to improve the *nMSE* on the training data, the predictions should deteriorate. Figure 6a demonstrates that the results of RFWR remained virtually unaltered by this test, while those of ME became significantly worse. This outcome can be explained by looking at the standard deviations of the regression coefficients of all the locally linear models (see Figure 6b). In contrast to ME, RFWR set the regression coefficients of the irrelevant inputs ($\beta_3, \beta_4, \beta_5$) very close to zero, thus achieving the desired robustness. Such behavior was due to an adjustment of the corresponding ridge regression parameters: they increased for the irrelevant inputs and decreased to zero for the relevant inputs. We should point out that ME was not designed to deal with learning problems with irrelevant inputs and that there are ways to improve its performance in such cases. Nevertheless, this experiment clearly illustrates that it is necessary to deal with the problem of irrelevant

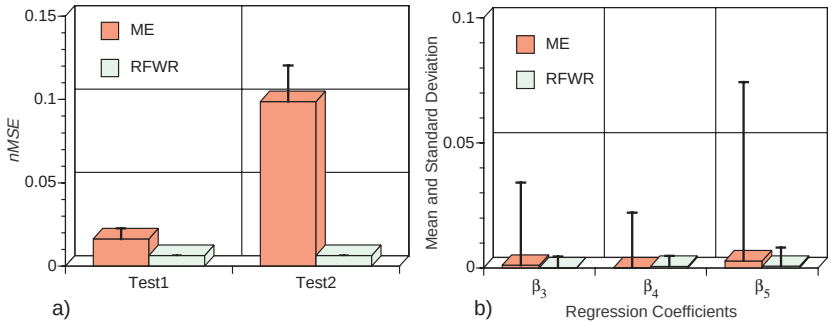

Figure 6: (a) Average *nMSE* of ME and RFWR after 1000 training epochs (see the text for explanation). (b) Mean and standard deviation of the regression coefficients of the irrelevant inputs.

inputs and that local bias adjustment by means of ridge regression is one possible way to do so.

4.3 Shifting Input Distributions. It is easy to conceive of learning tasks where the input distribution of the training data changes over time. To test RFWR's performance on such problems, we designed the following experiment. In three sequential episodes, training data for learning (see equation 4.1) were uniformly drawn from three slightly overlapping input regions in the unit cube: $T_1 = \{(x, y, z) \mid -1.0 < x < -0.2\}$, $T_2 = \{(x, y, z) \mid -0.4 < x < 0.4\}$, and $T_3 = \{(x, y, z) \mid 0.2 < x < 1.0\}$. First, the algorithm was trained on T_1 for 50,000 points and tested on T_1, then trained on T_2 for 50,000 points and tested on T_1 and T_2, and finally trained on T_3 for 50,000 points and tested on test data from all regions. Figure 7 gives an example of how learning proceeded. This test probes how much of the previously learned competence is forgotten when the input distribution shifts. All parameters of RFWR were chosen as in section 4.1.

Because the ME algorithm is not constructive and thus not well suited for learning with strongly shifting input distributions, we chose the resource allocating network (RAN) of Platt (1991) for a comparison, a learning algorithm that is constructive, has no competitive learning component, and has inspired a variety of other algorithms. RAN is a radial basis function (RBF) network that adds RBFs at the site of a training sample according to two criteria: when the approximation of the training sample error is too large, and when no RBF is activated by the training sample more than a ξ value. Both criteria have to be fulfilled simultaneously to create a new RBF. The spherical width of the RBF is chosen according to its distance to the nearest

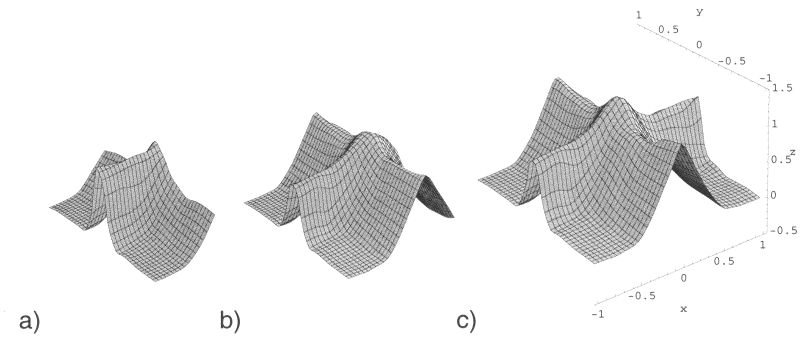

Figure 7: RFWR reconstructed function after training on (a) T_1, (b) then T_2, (c) and finally T_3.

neighboring RBF. By using gradient descent with momentum, the RBF centers are adjusted to reduce the mean squared approximation error, as are the weights of the linear regression network in the second layer of the RBF net. The strategy of RAN is to start initially with very wide RBFs and increase the threshold ξ over time until a prechosen upper limit is reached, causing the creation of ever-smaller RBFs at sites with large error. As in RFWR, we used gaussians (see equation 3.2) as the parametric structure for the RBF.

Figure 8 summarizes the average of 10 learning trials for each algorithm. RFWR shows large robustness toward the shift of input distribution: there is only a minor increase of *nMSE* due to interference in the overlapping parts of the training data (see Figure 7). In contrast, as can be seen in the "original RAN" trace of Figure 8a, RAN significantly increases the *nMSE* during the second and third training episodes. Since RAN starts out with initial RBFs that cover the entire input space, interference is not properly localized, which explains the observed behavior. Note that we already have excluded the constant term in the linear regression layer of RAN (Platt, 1991), a term that is globally active and would decrease the performance in Figure 8 significantly.

From the experience with RFWR, three possible improvements of RAN come to mind. First, instead of starting with very large RBFs initially, we can limit the maximal initial size as in RFWR to $\mathbf{M}_{def}$. Second, we can employ the hyper RBF technique of Poggio and Girosi (1990) to adjust the shape $\mathbf{M}$ of the RBFs by gradient descent as in RFWR (Furlanello, Giuliani, & Trentin, 1995). And third, instead of having the time-varying threshold ξ a global variable, we can define it as an individual variable for each RBF, thus removing the explicit dependency on global training time. By initial-

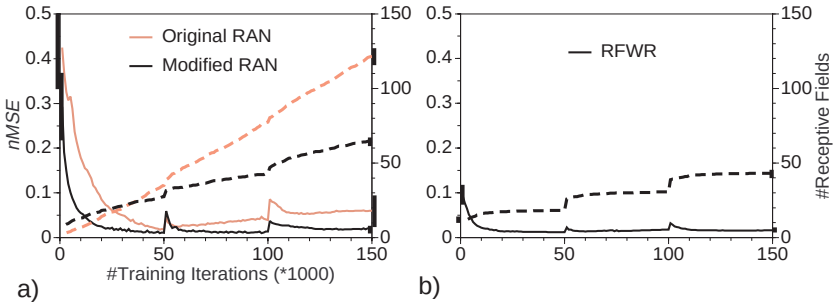

Figure 8: Average learning curves (solid lines) and average number of receptive field / radial basis functions (dashed lines) for (a) RAN and (b) RFWR. The black bars give the one standard deviation at the beginning and the end of learning.

izing RAN with $\mathbf{M}_{def} = 5\mathbf{I}$ as in RFWR, these modifications resulted in a significant improvement of robustness of RAN, as shown in Figure 8a. This version of RAN requires only half as many RBFs, converges more quickly, and achieves very low final approximation errors. As in RFWR, localizing the learning parameters leads to a clear improvement of robustness of incremental learning.

4.4 Sensorimotor Learning. As a last evaluation, we use a traditional example of sensorimotor learning, the approximation of the inverse dynamics of a two-joint arm (Atkeson, 1989a). The configuration of the arm is given by two joint angles, θ_1 and θ_2 (see Figure 9a). The inverse dynamics model is the map from the two joint angles, two joint velocities, and two joint accelerations to the corresponding torques (we assume that the arm controller makes use of a low-gain-feedback proportional integral derivative (PID) controller whose performance is enhanced by feedforward commands from the learned inverse dynamics; An, Atkeson, & Hollerbach, 1988). The torques for the shoulder and elbow joints are learned by separate networks because there is no reason to believe that a receptive field for the elbow torque should have the same shape as for the shoulder torque. For RFWR this would mean that both outputs have the same Hessian, which is definitely not the case. The task goal is to draw a figure 8 in two parts of the work space. Figure 9a shows the desired and the initial performance without the learned commands. Training proceeded in two steps. First, the arm performed sinusoidal movements with varying frequency content in the area of the upper 8. A total of 45,000 training points, sampled at 100 Hz, was used for training. Each training sample was used only once in the sequential order in which it was generated. The learning results are shown in

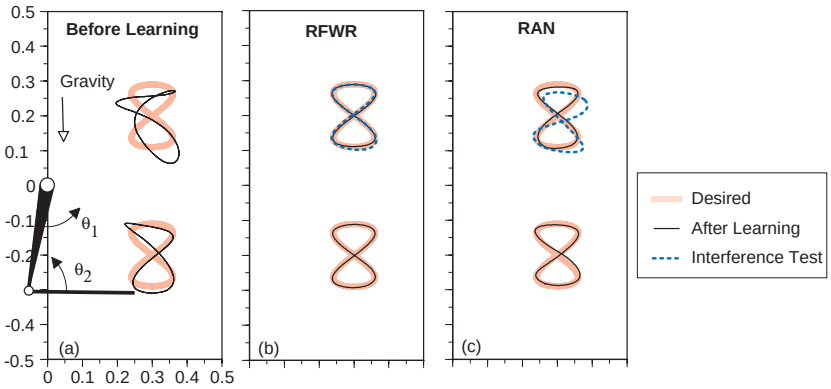

Figure 9: (a) Initial performance of the two-joint arm when drawing the figure 8 without feedforward control signals. (b) Performance of RFWR after learning. (c) Performance of RAN after learning.

the top part of Figure 9b for RFWR and Figure 9c for the modified RAN. Both algorithms were able to track the figure 8 properly.

Next, the algorithms were trained in an analogous fashion on 45,000 samples around the lower figure 8. The bottom parts of Figures 9b and 9c show the corresponding good learning results. However, when returning to performing the upper figure 8, RAN showed significant interference (the dashed line in Figure 9c), although both algorithms were initiated with the same $\mathbf{M}_{def} = 6.0\,\mathbf{I}$. (Note that position, velocity, and acceleration inputs were normalized prior to learning to compensate for the differences in units.) This interference effect highlights the difference between the learning strategy of RBF networks in comparison to the nonparametric statistics approach to modeling with locally linear models. RBF networks need a sufficient overlap of the RBFs to achieve good learning results; one RBF by itself has only limited function approximation capabilities, an effect discussed in the context of hyperacuity (Churchland & Sejnowski, 1992). Gradient descent on the shape parameter $\mathbf{M}$ of the gaussian RBFs quickly decreased $\mathbf{M}$ in our example to achieve an appropriately large overlap. This overlap, however, encourages negative interference, as is evident in Figure 9c. The six-dimensional input space of this example emphasized the need for large overlap, while the two-dimensional example of the previous section did not. Experiments that used a fixed $\mathbf{M}$ as in the original RAN algorithm did not achieve better learning results within a reasonable training time. To avoid interference, there is always the unattractive solution of adding thousands

of quite narrow overlapping RBFs. In the results of Figure 9, both algorithms allocated fewer than 100 receptive fields.

5 Related Work

The field that contributes the most to the development of RFWR is nonparametric statistics. Cleveland (1979) introduced the idea of employing locally linear models for memory-based function approximation, called locally weighted regression (LWR). In a series of articles, he and his colleagues extended the statistical framework of LWR to include multidimensional function approximation and local approximation techniques with higher-order polynomials (Cleveland, Devlin, & Grosse, 1988; Cleveland & Devlin, 1988). Cleveland and Loader (1995) suggested local C_p-tests and local PRESS for choosing the degree of local mixing of different order polynomials as well as local bandwidth adjustment and reviewed a large body of literature on the history of LWR. Hastie and Tibshirani (1990, 1994) give related overviews of nonparametric regression methods. Hastie and Loader (1993) discuss the usefulness of local polynomial regression and show that locally linear and locally quadratic function fitting have appealing properties in terms of the bias-variance trade-off. Friedman (1984) proposed a variable bandwidth smoother for one-dimensional regression problems. Using different statistical techniques, Fan and Gijbels (1992, 1995) suggested several adaptive bandwidth smoothers for LWR and provided detailed analyses of the asymptotic properties of their algorithms.

For the purpose of time-series prediction, LWR was first used by Farmer and Sidorowich (1987, 1988). Atkeson (1989b) introduced the LWR framework for supervised learning in robot control. Moore (1991) employed LWR for learning control based on learning forward models. In the context of learning complex manipulation tasks with a robot, Schaal and Atkeson (1994a, 1994b) demonstrated how LWR can be extended to allow for local bandwidth adaptation by employing local cross-validation and local confidence criteria. Schaal and Atkeson (1996) introduced the first non-memory-based version of LWR. Schaal (1997) applied RFWR for value function approximation in reinforcement learning. Locally weighted learning for classification problems can be found, for example, in Lowe (1995). Aha (1997) compiled a series of articles on nonparametric local classification and regression learning, among which Atkeson, Moore, and Schaal (1997a, 1997b) give an extended survey on locally weighted learning and locally weighted learning applied to control.

Besides nonparametric statistics, RFWR is related to work on constructive learning algorithms, local function approximation based on RBFs, and Kohonen-like self-organizing maps (SOM). An RBF function approximator with a locally linear model in each RBF was suggested by Millington (1991) for reinforcement learning. Platt (1991) suggested a constructive RBF-based learning system. Furlanello et al. (1995) and Furlanello and Giuliani (1995)

extended Platt's method by using Poggio and Girosi's (1990) hyper-RBFs and local principal component analysis. For learning control, Cannon and Slotine (1995) derived a constructive RBF network that used wavelet-like RBFs to adapt to spatial frequency; this is similar to local bandwidth adaptation in nonparametric statistics and the adjustable receptive fields in RFWR. Orr (1995) discussed recursive least squares methods and ridge regression for learning with RBF networks. He also suggests several other methods, including generalized cross-validation, for regularizing ill-conditioned regression.

One of the most established constructive learning systems is cascade correlation (Fahlman & Lebiere, 1990), a system sharing ideas with projection pursuit regression (Friedman & Stützle, 1981). Related to this line of research is the upstart algorithm of Frean (1990), the SOM-based cascading system of Littmann and Ritter (1993), and the work of Jutten and Chentouf (1995). The first usage of locally linear models for regression problems in the context of SOMs was by Ritter and Schulten (1986), who extended Kohonen maps to fit locally linear models (LLM) within each of the units of the SOM. Related to this work is van der Smagt and Groen's (1995) algorithm, which extended LLM to a hierarchical approximation in which each Kohonen unit itself can contain another LLM network. Fritzke (1994, 1995) demonstrated how SOMs can constructively add units, in the context of both RBF and LLM regression problems. Bruske and Sommer (1995) combined Fritzke's ideas with Martinetz and Schulten's (1994) neural gas algorithm to accomplish a more flexible topographic representation as in the original SOM work. A large body of literature on constructive learning stems from fitting high-order global polynomials to data, for instance, as given in Sanger (1991), Sanger, Sutton, and Matheus (1992), and Shin and Ghosh (1995). Due to the global character of these learning methods, the danger of negative interference is quite large. Additional references on constructive learning for regression can be found in the survey by Kwok and Yeung (1995).

The idea of the mixture of experts in Jacobs et al. (1991) and hierarchical mixtures of experts in Jordan and Jacobs (1994) is related to RFWR as the mixture of experts approach looks for similar partitions of the input space, particularly in the version of Xu et al. (1995). Ormoneit and Tresp (1995) suggested methods to improve the generalization of mixture models when fit with the EM algorithm (Dempster et al., 1977) by introducing Bayesian priors. Closely related to the hierarchical mixture of experts are nonparametric decision tree techniques, in which the seminal work of Breiman, Friedman, Olshen, and Stone (1984) introduced classification and regression trees (CART), and Friedman (1991) proposed the MARS algorithm, a CART derivative particularly targeted at smooth function approximation for regression problems.

Finally, adaptive receptive fields and the way receptive fields are created in RFWR resemble in part the classification algorithms of Reilly, Cooper, and Elbaum (1982) and Carpenter and Grossberg (1987).

6 Discussion

This article emphasizes two major points. First, truly local learning—learning without competition, without gating nets, without global regression on top of the local receptive fields—is a feasible approach to learning; moreover, it can compete with state-of-the-art learning systems. Second, truly incremental learning—learning without knowledge about the input and conditional distributions, learning that must cope with continuously incoming data with many partially redundant and / or partially irrelevant inputs—needs to have a variety of mechanisms to make sure that incremental learning is robust. A carefully designed local learning system can accomplish this robustness.

RFWR borrowed in particular from work in nonparametric statistics. Following the definition of Hájek (1969), the term *nonparametric* indicates that the function to be modeled potentially consists of very large families of distributions that cannot be indexed by a finite-dimensional parameter vector in a natural way. This view summarizes the basic assumptions of our learning system, with the addition of prior knowledge about smoothness incorporated in a penalty term. If more prior knowledge is available for a particular problem, it should be incorporated in the learning system. It is unlikely that a nonparametric learner outperforms problem-tailored parametric learning (e.g., fitting sinusoidal data with a sinusoid is the best one can do). The examples given throughout this article highlight when local nonparametric learning can be advantageous, but there is no claim that it is generally superior to other learning systems. On the other hand, when it comes to learning without having strong prior knowledge about the problem, nonparametric methods can be quite beneficial. For instance, Quartz and Sejnowski (1997) claim that constructive nonparametric learning might be one of the key issues in understanding the development of the organization of brains.

RFWR makes use of several new algorithmic features. We introduced a stochastic approximation to leave-one-out local cross-validation—cross-validation that does not need a validation set anymore. This technique can potentially be useful for many other domains because it requires only that the (local) parameters to be estimated are linear in the inputs. By employing a novel penalized local cross-validation criterion, we were able to derive locally adaptive multidimensional distance metrics. These distance metrics can be interpreted as local approximations of the Hessians of the function to be modeled. In order to speed up learning of the distance metric, we derived a second-order gradient descent method. Finally, the penalized local cross-validation criterion could also be employed to achieve automatic local bias adjustment of the relevance of input dimensions, obtained by local ridge regression. Using all these features, the constructive process of RFWR only needs to monitor the activation strength of all receptive fields in order to decide when to create a new receptive field; most constructive learning

systems need to monitor an approximation error criterion as well, which can easily lead to an unfavorable bias-variance trade-off.

Several issues have not been addressed in this article and are left to future research. RFWR makes use of gradient-based learning, which requires a proper choice of learning rates. Although we incorporated second-order learning derived from Sutton (1992a, 1992b), it may still be useful to do some experimentation with the choice of the learning rates in order to achieve close to optimal learning speed without entering unstable domains. It is also necessary to choose a roughly appropriate initial distance metric **D** (cf. equation 3.2), characterizing the initial size of a receptive field. An initial receptive field that is way too large has the danger that the receptive field grows to span the entire input domain: the initial receptive field has to be such that structure in the data cannot be mistaken for high variance noise. As a positive side effect of local learning, however, these open parameters can be explored by allowing just a small number of receptive fields on an initial data set and monitoring their learning behavior; each receptive field learns independently, and there is no need to do parameter exploration with a large number of receptive fields.

A last algorithmic point concerns computational complexity. Recursive least squares is an $O(n^2)$ process (i.e., quadratic in the number of inputs), and the update of a full distance metric is worse than $O(n^2)$. If the dimensionality of the inputs goes beyond about 10, a learning task with many receptive fields will run fairly slowly on a serial computer. Fitting only diagonal distance metrics alleviates this effect and might be necessary anyway since the number of open parameters in the learning system might become too large compared to the number of training data points.

This discussion naturally leads to the longstanding question of how local learning methods can deal with high-dimensional input spaces at all. As nicely described in Scott (1992), the curse of dimensionality has adverse effects on all systems that make use of neighboring points in the Euclidean sense, since the concept of "neighborhood" becomes gradually more counterintuitive when growing beyond 10 input dimensions, and it pretty much vanishes beyond 20 dimensions: every point is about the same distance from every other point. In such domains, the parametric model chosen for learning, be it local or global, becomes the key to success, essentially meaning that any learning system requires strong bias in high-dimensional worlds. However, it remains unclear whether high-dimensional input spaces have *locally* high-dimensional distributions. Our experience in sensorimotor learning is that this may not be true for many interesting problems, as physical systems do not realize arbitrary distributions. For instance, a seven-degree-of-freedom anthropomorphic robot arm, whose inverse dynamics model requires learning in a 21-dimensional input space, seems to realize locally not more than 4- to 8-dimensional input distributions. In Vijayakumar and Schaal (1997) we incorporated local dimensionality reduction as

a preprocessing step in every receptive field, allowing us to approximate high-dimensional data successfully.

As a last point, one might wonder in how far a local learning system like RFWR could have any parallels with neurobiological information processing. Particularly inspired by work on the visual cortex, one of the mainstream assumptions about receptive field–based learning in the brain is that receptive fields are broadly tuned and widely overlapping and that the size of the receptive fields does not seem to be a free parameter in normal learning (as opposed to developmental and reorganizational processes after lesions, e.g., Merzenich, Kaas, Nelson, Sur, & Felleman, 1983). This view emphasizes that accuracy of encoding must be achieved by subsequent postprocessing steps. In contrast, RFWR suggest overlapping but much more finely tuned receptive fields, such that accuracy can be achieved directly by one or several overlapping units. Fine tuning can be achieved not only by a change of the size of the receptive field, but also by "plug-in" approaches, where several receptive fields tuned for different spatial frequencies contribute to learning (Cannon & Slotine, 1995). To distinguish between those two principles, experiments that test for interference and generalization during learning can provide valuable insights into the macroscopic organization of learning. Shadmehr and Mussa-Ivalidi (1994), Imamizu, Uno, and Kawato (1995), and Shadmehr, Brashers-Krug, and Mussa-Ivaldi (1995) provide examples of such investigations in motor control.

Whether the learning principles of RFWR are biologically relevant remains speculative. What we have demonstrated, however, is that there are alternative and powerful methods to accomplish incremental constructive learning based on local receptive fields, and it might be interesting to seek cases where such learning systems might be applied. Receptive field–based local learning is an interesting research topic for neural computation, and truly local learning methods are just starting to demonstrate their potential.

Appendix

A.1 Ridge Regression Derivatives. Each ridge regression parameter can be conceived of as a weighted data point of the form $[\mathbf{x}_r = r_i^2(0, \ldots, 1, 0 \ldots)^T, \mathbf{y}_r = 0]$, which was incorporated in the regression by the recursive least squares update (see equation 3.5). Thus, the derivative of the cost function (see equation 3.9) is a simplified version of the derivative in equation 3.14:

$$\frac{\partial J}{\partial r_i} = \frac{2r_i}{W^{n+1}} \left(- \left(2\mathbf{P}^{n+1}\mathbf{x}_r \left(\mathbf{y}_r - \mathbf{x}_r^T \beta^{n+1} \right)^T \right) \otimes \mathbf{H}^{n+1} - \left(2\mathbf{P}^{n+1}\mathbf{x}_r\mathbf{x}_r^T\mathbf{P}^{n+1} \right) \otimes \mathbf{R}^{n+1} \right). \tag{A.1}$$

By taking advantage of the many zero elements of the ridge "data points," the actual computation of this derivative is greatly speeded up.

There are several ways to incorporate the update of the ridge regression parameters in the matrix **P**, and it should be noted that we also need to add back the fraction of the ridge parameters that was forgotten due to the forgetting factor λ in each update of **P** (see equation 3.5). It turns out that there is a quite efficient way to perform this update. At every update of a receptive field, the forgetting factor effectively reduces the contribution of each ridge parameter by:

$$\Delta_{\lambda,i} = (1-\lambda)\, r_i^2. \tag{A.2}$$

The update due to gradient descent is

$$\Delta_{grad,i} = (r_i + \Delta r_i)^2 - r_i^2, \tag{A.3}$$

and the total increment becomes

$$\begin{aligned}\Delta_i &= \Delta_{\lambda,i} + \Delta_{grad,i} = (1-\lambda)\, r_i^2 + (r_i + \Delta r_i)^2 - r_i^2 \\ &= -\lambda r_i^2 + (r_i + \Delta r_i)^2. \end{aligned} \tag{A.4}$$

Due to the fact that the ridge vectors are all unit vectors, it is possible to update **P** by executing a recursive least squares update for the increment—that is, add a ridge data point of the form $[\mathbf{x}_r = \Delta_i(0, \ldots, 1, 0, \ldots)^T, \mathbf{y}_r = 0]$ for every ridge parameter by using equation 3.5. This update can be accelerated by taking into account the zeros in the ridge points. An additional speed-up can be obtained by not updating **P** at every iteration but accumulating the increments until they exceed a manually chosen threshold.

A.2 Second-Order Learning of the Distance Metric. The idea of the incremental delta-bar-delta (IDBD) algorithm (Sutton, 1992a, 1992b) is to replace the learning rate in the gradient descent update (see equation 3.10) by an individual learning rate for each coefficient of **M** of the following form:

$$\begin{aligned} &M_{ij}^{n+1} = M_{ij}^n - \alpha_{ij}^{n+1} \frac{\partial J}{\partial M_{ij}}, \\ &\text{where} \quad \alpha_{ij}^{n+1} = \exp\left(\beta_{ij}^{n+1}\right) \quad \text{and} \quad \beta_{ij}^{n+1} = \beta_{ij}^n - \theta \frac{\partial J}{\partial M_{ij}} h_{ij}^n. \end{aligned} \tag{A.5}$$

Thus, the learning rates α_{ij} are changed in geometric steps by gradient descent in the meta parameter β_{ij} with meta learning rate θ. The term h_{ij} is updated as

$$h_{ij}^{n+1} = h_{ij}^n \left[1 - \alpha_{ij}^{n+1} \frac{\partial^2 J}{\partial M_{ij}^2}\right]^+ - \alpha_{ij}^{n+1} \frac{\partial J}{M_{ij}},$$

$$\text{where we define} \quad [z]^+ = \begin{cases} z & \text{if } z > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{A.6}$$

h_{ij} is initialized to zero when a receptive field is created. It corresponds to a memory term that stores a decaying trace of the cumulative sum of recent changes to M_{ij}. For more details, see Sutton (1992a, 1992b). In order to apply this second-order update, it is necessary to store the parameters α_{ij}, β_{ij}, and h_{ij} and to compute the second derivative in equation A.6. This second derivative of the cost function (see equation 3.9) with respect to the coefficients of the decomposed distance metric becomes:

$$\frac{\partial J}{\partial \mathbf{M}} \approx \frac{\partial w}{\partial \mathbf{M}} \sum_{i=1}^{p} \frac{\partial J_{1,i}}{\partial w} + \frac{w}{W^{n+1}} \frac{\partial J_2}{\partial \mathbf{M}} \tag{A.7}$$

$$\frac{\partial^2 J}{\partial \mathbf{M}^2} \approx \frac{\partial^2 w}{\partial \mathbf{M}^2} \sum_{i=1}^{p} \frac{\partial J_{1,i}}{\partial w} + \frac{\partial w}{\partial \mathbf{M}} \sum_{i=1}^{p} \frac{\partial^2 J_{1,i}}{\partial w^2} \frac{\partial w}{\partial \mathbf{M}} + \frac{w}{W^{n+1}} \frac{\partial^2 J_2}{\partial \mathbf{M}^2}$$

where:

$$\frac{\partial^2 w}{\partial M_{rl}^2} = \frac{1}{w} \left(\frac{\partial w}{\partial \mathbf{M}} \right)^2 - w(x_l - c_l)^2, \quad \frac{\partial^2 J_2}{\partial M_{rl}^2} = 2\gamma \left(2D_{ll} + \sum_{i,j=1}^{n} \left(\frac{\partial D_{ij}}{\partial M_{rl}} \right)^2 \right)$$

$$\begin{aligned} \sum_{i=1}^{p} \frac{\partial^2 J_{1,i}}{\partial w^2} \approx & -\frac{\mathbf{e}_{cv}^T \mathbf{e}_{cv}}{(W^{n+1})^2} \\ & - \frac{2}{W^{n+1}} \left(\left(-\frac{\mathbf{I}}{W^{n+1}} - 2\mathbf{P}^{n+1} \tilde{\mathbf{x}} \tilde{\mathbf{x}}^T \right) \mathbf{P}^{n+1} \tilde{\mathbf{x}} (\mathbf{y} - \tilde{\mathbf{x}}^T \beta^{n+1})^T \right) \otimes \mathbf{H}^n \\ & + \frac{2}{W^{n+1}} \frac{\left(\mathbf{y} - \tilde{\mathbf{x}}^T \beta^{n+1} \right)^T \left(\mathbf{y} - \tilde{\mathbf{x}}^T \beta^{n+1} \right) h}{w} \\ & - \frac{1}{(W^{n+1})^2} \left(\mathbf{e}_{cv}^T \mathbf{e}_{cv} - 2 \left(\mathbf{P}^{n+1} \tilde{\mathbf{x}} \left(\mathbf{y} - \tilde{\mathbf{x}}^T \beta^{n+1} \right)^T \right) \otimes \mathbf{H}^n \right) \\ & + 2 \frac{E^{n+1}}{(W^{n+1})^3} \end{aligned}$$

Equation A.7 makes use of notation and results derived in equations 3.13 and 3.14.

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