A Unified Treatment of Models of Thermoelasticity

Von der Fakultät für Mathematik und Informatik der
Technischen Universität Bergakademie Freiberg genehmigte

DISSERTATION

zur Erlangung des akademischen Grades
doctor rerum naturalium
(Dr. rer. nat.),

vorgelegt von

Dipl.-Math. Kay Jachmann
geboren am 09. Januar 1980 in Dresden

Gutachter: Prof. Dr. Michael Reissig (Freiberg)
Prof. Dr. Yaguang Wang (Shanghai)
Prof. Dr. Rainer Picard (Dresden)

Tag der Verleihung: 19. Dezember 2008

Acknowledgment

This thesis was written while I was an employee at the Faculty of Mathematics and Computer Science at the Technical University Bergakademie Freiberg to which staff I would like to express my sincere gratitude for the excellent working conditions. This applies especially to the Institute of Applied Analysis.
I feel deeply indebted to Prof. Michael Reissig, the advisor of this thesis, without whose permanent support and numerous suggestions and hints in many discussions on the subject this work would not have been possible. I am further very grateful to Prof. Yaguang Wang from the Shanghai Jiao Tong University, who invited me to his department for several long-term stays and acted as a second advisor to this thesis.
Special thanks go to Prof. Rainer Picard from the Technical University Dresden for his readiness to act as a referee and to Dr. Jens Wirth from the Imperial College London for always having an open ear for my questions and many fruitful discussions.
For valuable comments and invitations to their departments I would moreover like to thank Prof. Yuming Qin from the Donghua University, Shanghai, Prof. Daoyuan Fang from the Zhejiang University, Hangzhou and Prof. Yoshihiro Shibata from the Waseda University, Tokyo.
Finally, I would like to thank my family and my significant other for their permanent support, their understanding and encouragement.

Freiberg, September 2008 Kay Jachmann

Contents

1. Introduction

1.1. Background

Systems of thermoelasticity describe the elastic and thermal behavior of elastic, heat-conducting media, in particular the reciprocal actions between elastic stresses and temperature differences.

The theory of thermoelasticity was founded in 1838 by Duhamel, [Duh38], who derived the equations for the strain in an elastic body with temperature gradients. Neumann, [Neu41], obtained the same results in 1841. However, the theory was based on independence of the thermal and mechanical effects. The total strain was determined by superimposing the elastic strain and the thermal expansion caused by the temperature distribution only. The theory thus did not describe the motion associated with the thermal state, nor did it include the interaction between the strain and the temperature distributions.

Hence, thermodynamic arguments were needed, and it was Thomson, [Tho57], in 1857 who first used the laws of thermodynamics to determine the stresses and strains in an elastic body in response to varying temperatures.[1]

Using classical thermodynamics methods Landau and Lifshitz, [LL53], in 1953 derived the coupled equations of thermoelasticity. They are in the linear case of a homogeneous and isotropic medium with zero external body forces and zero external heat supply in 1D given by

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0, \\ \theta_t - \kappa \theta_{xx} + \gamma_2 u_{tx} = 0 \end{cases} \tag{1.1a}$$

and in 3D by

$$\begin{cases} U_{tt} - \mu \Delta U - (\mu + \lambda) \nabla \nabla^T U + \gamma_1 \nabla \theta = 0, \\ \theta_t - \kappa \Delta \theta + \gamma_2 \nabla^T U_t = 0. \end{cases} \tag{1.1b}$$

The unknows $u = u(t, x) \in \mathbb{R}$ or $U = U(t, x) \in \mathbb{R}^3$ and $\theta = \theta(t, x) \in \mathbb{R}$ denote the elastic displacement and the temperature difference to the equilibrium state, respectively, t stands for the time-, x for the space-variable and $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$, $\nabla^T = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$. Physical properties of the underlying isotropic medium are described by the thermal

[1] This historical review can be found in [Kov70].

conductivity $\kappa > 0$, the elasticity modules $\alpha > 0$ or μ and λ with $\mu, \lambda + 2\mu > 0$ and the thermoelastic coupling coefficients γ_1 and γ_2 with $\gamma_1\gamma_2 > 0$.
The derivation of the classical thermoelasticity model is based on Fourier's law of heat conduction, i.e., the heat flux q is assumed to be proportional to the temperature gradient

$$q = -\kappa \nabla \theta. \tag{1.2}$$

That implies that the heat equation for the coupled theory is a parabolic one, giving rise to the unphysical property that if a sudden change of temperature is made at some point of the heat-conducting body, it will be felt instantly everywhere, though with exponentially small amplitudes at distant points (cf. [Cat48]). Hence, we observe an infinite propagation speed of thermal disturbances. Moreover, the temperature of a body is the macroscopic consequence of certain kinds of vibratory motions. Heat is transported by near-neighbor excitation in which changes of momentum and energy on a microscopic scale are propagated as waves.
As indicated in the review article of Joseph and Preziosi, [JP89], it is therefore useful to replace (1.2) by the so-called Cattaneo's equation

$$\tau q_t + q = -\kappa \nabla \theta \tag{1.3}$$

or more general even by a heat-flux equation of Jeffreys type

$$\tau q_t + q = -\kappa \nabla \theta - \tau \kappa_1 \nabla \theta_t. \tag{1.4}$$

In the above $\tau > 0$ denotes the (in general very small) relaxation time and $\kappa_1 > 0$ the effective thermal conductivity.
Using Cattaneo's law of heat conduction instead of (1.2) one immediately arrives at the so-called thermoelasticity systems with second sound, given in the linear 1D and 3D cases by

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0, \\ \theta_t + q_x + \gamma_2 u_{tx} = 0, \\ \tau q_t + q + \kappa \theta_x = 0 \end{cases} \tag{1.5a}$$

and

$$\begin{cases} U_{tt} - \mu \Delta U - (\mu + \lambda) \nabla \nabla^T U + \gamma_1 \nabla \theta = 0, \\ \theta_t + \nabla^T q + \gamma_2 \nabla^T U_t = 0, \\ \tau q_t + q + \kappa \nabla \theta = 0. \end{cases} \tag{1.5b}$$

The name is due to the problem of second sound, which arose first in studies of Tisza, [Tis38], and Landau, [Lan41], of heat waves in liquid helium II. Further, we note that as the relaxation parameter τ goes to zero, the second sound models (1.5a) and (1.5b) formally converge to the classical ones in (1.1a) and (1.1b).

In [GN91] the authors Green and Naghdi discussed, using an analogy between the concepts and equations of the purely thermal and the purely mechanical theories, three types of constitutive equations for heat flow in a stationary rigid solid such that when the respective theories are linearized, type I leads to the usual heat conduction by Fourier's law, type II to a telegraph equation (with a possibly vanishing damping term), whose solution is capable of transmitting waves with finite speed, and type III leads to an equation of Jeffreys type as in (1.4). A similar result to the heat flow of type II can be predicted from Cattaneo's law (1.3) and the energy equation for a rigid conductor (cf. [JP89]), and type III describes a heat flow, which is analogous to the flow of one type of 'viscoelastic' fluids. In [GN92] and [GN93] Green and Nagdhi later derived, using the above corresponding constitutive equations, besides the classical model of thermoelasticity (or thermoelasticity of type 1), the thermoelasticity models of type 3 and 2 for isotropic media, respectively.

The models of type 2, also known as thermoelasticity without energy dissipation, are in the linear 1D and 3D cases given by

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0, \\ \theta_{tt} - \kappa \theta_{xx} + \gamma_2 u_{ttx} = 0 \end{cases} \tag{1.6a}$$

and

$$\begin{cases} U_{tt} - \mu \Delta U - (\mu + \lambda) \nabla \nabla^T U + \gamma_1 \nabla \theta = 0, \\ \theta_{tt} - \kappa \Delta \theta + \gamma_2 \nabla^T U_{tt} = 0. \end{cases} \tag{1.6b}$$

The more dissipative models of type 3 are in 1D given by

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0, \\ \theta_{tt} - \kappa \theta_{xx} - \delta \theta_{txx} + \gamma_2 u_{ttx} = 0 \end{cases} \tag{1.7a}$$

and in the 3D case by

$$\begin{cases} U_{tt} - \mu \Delta U - (\mu + \lambda) \nabla \nabla^T U + \gamma_1 \nabla \theta = 0, \\ \theta_{tt} - \kappa \Delta \theta - \delta \Delta \theta_t + \gamma_2 \nabla^T U_{tt} = 0. \end{cases} \tag{1.7b}$$

The thermoelasticity models of type 3, (1.7a) and (1.7b), formally converge to the ones of type 2, (1.6a) and (1.6b), as $\delta \to 0$.

The first equations in all above models (1.1a), (1.5a)-(1.7a), as well as (1.1b) and (1.5b)-(1.7b) describe the elastic behavior of the underlying medium. Including lower order terms, such as dissipation and mass terms, into these equations, i.e., replacing in 1D

$$u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0$$

by

$$u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x + m u_t = 0 \qquad \text{or} \qquad u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x + m^2 u = 0$$

with $m > 0$, is thus not only of a mathematical interest but also worth studying it from an applicants point of view. Another possibility to physically motivate such a dissipation term is to take the external force negatively proportional to the velocity as it was done in [JR00] (cf. Section 9.1, (9.7)).

1.2. Objectives

The aim of this thesis is to clarify a number of questions for all above mentioned problems on thermoelasticity from a unified point of view.
More precisely, we will consider the Cauchy problems for all above mentioned systems, put these into a more general frame allowing us to study them in some sense at once and then discuss questions on

(a) methods for deriving sufficiently nice solution representations for reading off a number of structural properties,

(b) how to derive H^s well-posedness results,

(c) L^p-L^q decay estimates for dual indices p and q,

(d) which ones of these problems display a so-called diffusive structure and

(e) the propagation of singularities.

The methods we will be using for deriving solution representations are based on the Fourier transform and aspects of perturbation theory of matrices related to so-called diagonalization schemes. They go back to an idea of Wang, [Wan03a], [Wan03b], on decoupling hyperbolic-parabolic coupled systems.
Systems of thermoelasticity are often, but not always, hyperbolic-parabolic coupled ones. In such cases it is of particular interest, whether it is the hyperbolic or parabolic part that is determining the behavior of the solution when considering a certain property, e.g., do we obtain hyperbolic or parabolic decay rates in (c), can we find an asymptotic parabolic structure in (d), and do or do we not observe a smoothing effect in (e) should therefore be questions of interest.
It should be pointed out that some of the above tasks have been solved. Nevertheless, the methods that will be introduced in this thesis allow it to easily reproduce these results in an especially nice way. A systematic overview on qualitative properties of solutions to the classical models (1.1a) and (1.1b), apart from considerations on diffusive structures, can be found in the book of Jiang and Racke, [JR00]. Decay estimates for solutions to the Cauchy problems of the thermoelasticity models with second sound in 1D were studied in [YW06a] and in 3D in [WY06]. Well-posedness results, decay estimates and the propagation of singularities for the thermoelasticity models of type 3 were discussed in [RW05] and [YW06b]. In all four just mentioned articles diagonalization methods were applied in order to obtain the desired results. For details we refer to discussions later on in this thesis.
We should further emphasize that the approaches we will use in this thesis are motivated by concrete applications, namely the ones from Section 1.1. We are not interested in generating results which we can not apply and will therefore work in particular with

conditions that imply especially nice results - when satisfied by the applications that we have in mind - rather than trying to cover more general but not applicable situations.
The schedule is as follows. First, we will complete the introductory chapter by discussing some classical results on L^p-L^q decay estimates and some concerning so-called diffusion phenomena. In Chapter 2 we devote our attention to the study of qualitative properties of solutions to Cauchy problems for a class of linear second-order systems in 1D from which we can deduce results for solutions to the previously discussed thermoelasticity models. The latter will be done in the Chapters 3 and 4 for the 1D problems. Chapter 5 will then be dedicated to the discussion of the 3D problems from the previous section.

1.3. Asymptotic properties of solutions for model equations

The study of L^p-L^q decay estimates for linear evolution equations began in 1970 with two articles from Strichartz, [Str70a] and [Str70b]. He considered the Cauchy problem for the free wave equation

$$\begin{cases} u_{tt} - \Delta u = 0, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x) \end{cases} \tag{1.8}$$

in $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$ with suitable initial data, say $\varphi, \psi \in \mathcal{S}$, and found the a priori estimate

$$\|(u_t, \nabla_x u)(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|(\langle D\rangle \varphi, \psi)\|_{W_p^{N_p}} \tag{1.9}$$

for $n \geq 2$, $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$ and $N_p \geq n\left(\frac{1}{p} - \frac{1}{q}\right)$. Here $W_p^{N_p}$ stands for the standard Sobolev space over L^p with regularity N_p.
With the help of the above estimate Strichartz proved global existence of solutions to nonlinear wave equations with small initial data. The method of continuation of local solutions to nonlinear problems with the help of a priori estimates for the corresponding linear problem to obtain global existence and uniqueness results for small data is now standard, a systematic overview with examples may be found in [Rac92].
There are two main approaches for deriving the estimate in (1.9). On the one hand, one may use explicit solution representations, as it was done in [vW71], and on the other hand, one may write the solution as a sum of Fourier integral operators:

$$u(t,x) = \mathcal{F}^{-1}_{\xi\to x}\left(\frac{e^{i|\xi|t} + e^{-i|\xi|t}}{2}\,\hat{\varphi}(\xi) + \frac{e^{i|\xi|t} - e^{-i|\xi|t}}{2|\xi|i}\,\hat{\psi}(\xi)\right). \tag{1.10}$$

The latter was done for example in [Str70a], [Bre75] and [Pec76].
To be a bit more precise with respect to regularity let us state that the classical Sobolev space $W_p^{N_p}$ in (1.9) may be replaced by the generalized Sobolov or Bessel potential space L^{p,r_p} (cf. Appendix B.2) with $r_p = n\left(\frac{1}{p} - \frac{1}{q}\right)$. This can be seen from using the solution

representation (1.10) together with Corollary B.8.
When including a mass term into the wave equation in (1.8), that is, when considering the Cauchy problem for the Klein-Gordon equation

$$\begin{cases} u_{tt} - \Delta u + m^2 u = 0, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x) \end{cases}$$

for some non-vanishing constant m and $\varphi, \psi \in \mathcal{S}$, the estimate in (1.9) changes to

$$\|(u, u_t, \nabla_x u)(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|(\langle D \rangle \varphi, \psi)\|_{L^{p,r_p}} \tag{1.11}$$

for arbitrary space dimensions n, p and q as before, $r_p = N\left(\frac{1}{p} - \frac{1}{q}\right)$ and $N = n$ for $n \geq 2$, $N = 3/2$ for $n = 1$. The estimate was proved in [vW71], [Pec76] and [Hör97] (the special regularity in the case of space dimension $n = 1$ will be apparent from results later on in this thesis, cf. Theorem 2.17, $n_l = 1$).
For solutions to the Cauchy problem for the dissipative wave equation

$$\begin{cases} u_{tt} - \Delta u + m u_t = 0, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), \end{cases} \tag{1.12}$$

$m > 0$, $\varphi, \psi \in \mathcal{S}$, Matsumura proved in [Mat76] with the help of solution representations via Fourier multipliers in particular the estimate

$$\|(u_t, \nabla_x u)(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|(\langle D \rangle \varphi, \psi)\|_{L^{p,r_p}} \tag{1.13}$$

for arbitrary n, dual q, $1 < p \leq 2$ and $r_p = n\left(\frac{1}{p} - \frac{1}{q}\right)$.
When comparing the estimates (1.11) and (1.13) with (1.9) we observe an improvement in the decay rate of $-\frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$ (and in (1.13) even of an additional $-1/2$, yielding a decay for the usual hyperbolic energy). Moreover, while it is evident from d'Alemberts formula that there is no decay of the first order derivatives of solutions to (1.8), measured in L^q, in the case $n = 1$ the inclusion of an additional mass or dissipation term into the wave equation leads to a decay also in the 1D case. It is therefore particularly interesting to ask whether the addition of a mass or dissipation term in the thermoelasticity models (1.1a), (1.5a)-(1.7a) and (1.1b), (1.5b)-(1.7b) will improve decay rates as well.
Another interesting observation for the damped wave equation is that solutions have an underlying parabolic structure. If we consider the two Cauchy problems

$$\begin{cases} u_{tt} - \Delta u + u_t = 0, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x) \end{cases} \quad \text{and} \quad \begin{cases} v_t - \Delta v = 0, \\ v(0,x) = \varphi(x) + \psi(x), \end{cases} \tag{1.14}$$

then the behavior of u is essentially parabolic in the sense that (cf. [YM00], Theorem 2.1)

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(t^{-\frac{n}{2}-1}), \tag{1.15}$$

while $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)}$ and $\|v(t,\cdot)\|_{L^\infty(\mathbb{R}^n)}$ both behave as $\mathcal{O}(t^{-\frac{n}{2}})$ as $t \to \infty$. This observation is referred to as diffusion phenomenon and was first encountered by Hsiao and Liu, [HL92], for a system of hyperbolic conservation laws with damping.

2. Some considerations for a class of linear second-order systems in 1D

2.1. The problem

Systems of thermoelasticity may be of hyperbolic-parabolic, hyperbolic-hyperbolic coupled and even pure hyperbolic type. Let us therefore first recall definitions of hyperbolic and parabolic systems:

Definition 2.1. *For the d-dimensional column vector $U = U(t, x)$ of unknown functions we consider the system*

$$U_t + A_0 U + A_1 U_x = 0 \tag{2.1}$$

with $(t, x) \in \mathbb{R}_{>0} \times \mathbb{R}$ and constant matrices $A_i \in \mathbb{R}^{d \times d}$.
The system (2.1) is said to be of hyperbolic type if the eigenvalues of A_1 are all real and if, in addition, A_1 has a full set of d linearly independent eigenvectors. If A_1 is symmetric, then (2.1) is called symmetric hyperbolic, [Gar64].

Definition 2.2. *The system*

$$U_t + A_0 U + A_1 U_x - A_2 U_{xx} = 0$$

with $(t, x) \in \mathbb{R}_{>0} \times \mathbb{R}$ and constant matrices $A_i \in \mathbb{R}^{d \times d}$ is called parabolic if we have

$$Re\, \lambda > 0$$

for all eigenvalues $\lambda \in spec\,(A_2)$, [KL89].

We want to include both into our considerations and thus study in this chapter linear second-order systems of the form

$$U_t + A_0 U + A_1 U_x - A_2 U_{xx} = 0 \tag{2.2}$$

for d-dimensional unknowns $U = U(t, x)$ with $(t, x) \in \mathbb{R}_{>0} \times \mathbb{R}$ and A_i being (a bit more general) complex and constant $d \times d$ matrices.
The system (2.2) shall be considered together with the initial data

$$U(0, x) = U_0(x). \tag{2.3}$$

A first step is to apply partial Fourier transformation. Doing this and introducing $V = V(t,\xi) = \mathcal{F}_{x\to\xi}\left(U(t,\cdot)\right)(\xi)$, the system (2.2) transfers into

$$V_t + (A_0 + i\xi\, A_1 + \xi^2 A_2)V = 0. \tag{2.4}$$

The schedule is now as follows. In Section 2.2 we will apply a rather new method of frequency analysis in phase space for decoupling hyperbolic-parabolic coupled systems to obtain solution representations for proving H^s well-posedness results in Section 2.3, for concluding L^p-L^q decay estimates in Section 2.4 and to state some results concerning diffusion phenomena in Section 2.5. In Section 2.6 we will make some considerations on the propagation of singularities. The chapter will be closed by discussing some generalizations for the considered initial value problem in Section 2.7.

2.2. Diagonalization in phase space

The phase space will be divided into the three regions

$$\begin{aligned} &Z_{int}(\sigma) = \{|\xi| \le \sigma \ll 1\},\\ &Z_{mid}(\sigma, N) = \{\sigma \le |\xi| \le N\} \qquad \text{and}\\ &Z_{ext}(N) = \{|\xi| \ge N \gg 1\} \end{aligned}$$

of small, bounded and large frequencies, and in each region we will diagonalize the principal part of system (2.4) correspondingly. More specific, we are especially interested in the asymptotic behavior of the eigenvalues of the coefficient matrix $A(\xi) = A_0 + i\xi A_1 + \xi^2 A_2$ for small and large $|\xi|$.

The matrix $A = A(\xi)$ may (for small frequencies ξ) be understood as a perturbed linear operator, i.e., the *unperturbed operator* A_0 is subjected to the *perturbation* $i\xi A_1 + \xi^2 A_2$. Quite a few facts are known about such operator-valued functions and their eigenvalues, and we should thus make some general considerations on the results that can be expected before starting with our diagonalization procedure.

The facts listed in the following section may be found in [Kat80, Kno96].

2.2.1. Some general considerations on the eigenvalues

We are interested in the behavior of the eigenvalues of $A(\xi)$, especially in the asymptotic behavior for small and large frequencies.

We are thus studying the characteristic equation

$$\det(A(\xi) - \mu I) = 0. \tag{2.5}$$

Let us allow ξ to be complex for now. Then (2.5) is an algebraic equation, i.e., an equation of the form $G(\xi,\mu) = 0$, where G denotes a polynomial in ξ and μ. More specific, (2.5) may be written in the form

$$\mu^d + g_{d-1}(\xi)\mu^{d-1} + \ldots + g_1(\xi)\mu + g_0(\xi) = 0, \tag{2.6}$$

where the coefficients $g_i = g_i(\xi)$ are polynomials in ξ of degree not exceeding $2d$.
We can now state:

Lemma 2.1. *The solutions $\mu_i(\xi)$ of (2.5) are branches of algebraic functions.*

More accurate, if the algebraic equation $\det(A(\xi) - \mu I) = 0$ is irreducible, then the eigenvalues $\mu_i(\xi)$ are branches of one d-valued algebraic function (By an algebraic function we simply understand multiple-valued functions $w = F(z)$ which solve algebraic equations $G(z, w) = 0$, G is assumed to be irreducible.). If the characteristic equation is reducible, then the functions $\mu_i(\xi)$ can be classified into several groups, where each group corresponds to one algebraic function.
In the irreducible case it is thus not possible that some of the functions $\mu_i(\xi)$ are identical. If there are identical ones among the algebraic functions in the reducible case, then we speak of permanent degeneracy of the operator $A(\xi)$.
In any case, the number of distinct eigenvalues of $A(\xi)$ is given by $s \leq d$, independent of ξ, apart from some exceptional values, where the $\mu_i(\xi)$ may coincide. These points may, but must not be, branch points. If (2.5) is irreducible, then we have $s = d$ and the exceptional points are all branch points. There can furthermore only be a finite number of these exceptional values in the whole complex plane, due to the fact that the discriminant of (2.5) is a polynomial in ξ of definite degree. In the reducible case, we at least obtain that there is only a finite number of such exceptional values on every compact subset of $\mathbb{C}$. If we have $s = d$, then the operator $A(\xi)$ is diagonalizable at least for all non-exceptional values.
Concerning the regularity of the eigenvalues $\mu_i = \mu_i(\xi)$ we can state:

Lemma 2.2. *The solutions $\mu_i(\xi)$ of (2.5) are continuous for all ξ and holomorphic except for branch points of the algebraic function, whereof $\mu_i(\xi)$ constitutes a branch.*

Remark 2.1. 1. By some calculations one can furthermore prove that the sum of all eigenvalues constituting one of the algebraic functions gives an entire function.
2. For the overall regularity of the roots $\mu_i = \mu_i(\xi)$ we can not expect differentiability (as will be seen in the following considerations). However, if the characteristic equation (2.5) involves a hyperbolic polynomial (i.e., the coefficients $g_i = g_i(\xi)$ from (2.6), ξ is restricted to $\mathbb{R}$ now, are real-valued and (2.6) has only real solutions), then the characteristic roots μ_i are differentiable (with suitable choice of the branches), [Bro79]. (Please note that in our setting hyperbolic polynomials do not connect to hyperbolic systems.)
3. The results of Lemma 2.2 hold as well if $A = A(\xi)$ is given by an arbitrary holomorphic operator-valued function.

If we are studying the very small frequencies, i.e., we consider only a sufficiently small neighborhood of $\xi = 0$, then $\xi = 0$ is the only exceptional point that we might have, and that is the case if and only if the eigenvalues of A_0 are not distinct. If $\xi = 0$ is not a branch point of the algebraic function corresponding to $\mu_i(\xi)$, then we can write down its Taylor expansion. However, if it is a branch point, then $\mu_i(\xi)$ admits in a neighborhood an expansion of the form (Puiseux series)

$$\sum_{n=0}^{\infty} c_n \xi^{\frac{n}{p}}. \tag{2.7}$$

The natural number p may at the most equal the number of branches of the appendant algebraic function.

Example 2.1. Assume the coefficient matrix $A = A(\xi)$ to be given by

$$A(\xi) = \begin{pmatrix} 0 & 1 \\ i\xi & 0 \end{pmatrix}.$$

The eigenvalues are $\pm\frac{(1+i)}{\sqrt{2}}\sqrt{\xi}$, constituting one double-valued function $\frac{(1+i)}{\sqrt{2}}\sqrt{\xi}$. There is one branch point $\xi = 0$.

At $|\xi| = \infty$ the eigenvalues may at the most have a pole of order 2, which is seen by writing

$$A(\xi) = \xi^2(A_2 + i\xi^{-1}A_1 + \xi^{-2}A_0).$$

The eigenvalues of $A_2 + i\xi^{-1}A_1 + \xi^{-2}A_0$ are continuous for $\xi^{-1} \to 0$. More specific, they have an expansion of the form (2.7), when replacing ξ by ξ^{-1}. Hence, the eigenvalues of $A(\xi)$ admit for $|\xi| \to \infty$ expansions of the form

$$\sum_{n=0}^{\infty} \tilde{c}_n \xi^{2-\frac{n}{p}}.$$

2.2.2. Diagonalization for small frequencies

In the region $Z_{int}(\sigma) = \{|\xi| \leq \sigma \ll 1\}$ it is the matrix A_0 that dominates the coefficient matrix $A(\xi)$ of (2.4). Hence, this is the region of the phase space, where we feel the lower order terms of the underlying model the most.

2.2.2.1. The procedure

We start off our calculations by assuming that the matrix A_0 is diagonalizable. If that is not the case, we will have to stop the procedure already at this point.

Example 2.2. In the following considerations we would like to draw the readers attention every once in a while to the special case, when the matrix A_0 is not only diagonalizable, but when all its eigenvalues have the algebraic multiplicity one.

We will now gradually diagonalize the system (2.4), starting off with:

Step 0: Diagonalization modulo $\mathcal{O}(\xi)$-terms

We denote for the matrix $A_0 \in \mathbb{C}^{d\times d}$ by $\lambda_{0,1}, \dots, \lambda_{0,d}$ its eigenvalues, which we order in such a way that we have distinct groups of equal numbers, i.e.,

$$\lambda_{0,1} = \dots = \lambda_{0,k_1}, \quad \dots \quad , \lambda_{0,k_{b_1-1}+1} = \dots = \lambda_{0,k_{b_1}=d}, \quad \lambda_{0,k_m} \neq \lambda_{0,k_n} \text{ for all } m \neq n. \tag{2.8}$$

Next, we denote by $\tilde{l}_j^{(0)}$ and $\tilde{r}_k^{(0)}$ corresponding left and right eigenvectors with the normalization

$$\tilde{l}_j^{(0)} \tilde{r}_k^{(0)} = \delta_{jk}$$

(cf. Lemma B.1) and introduce the notations $\tilde{R}^{(0)} = (\tilde{r}_1^{(0)}, \dots, \tilde{r}_d^{(0)})$ and $\tilde{L}^{(0)} = (\tilde{l}_1^{(0)T}, \dots, \tilde{l}_d^{(0)T})^T$.
The vector function $\tilde{V}^{(0)}(t,\xi) = \tilde{L}^{(0)}\, V(t,\xi)$ then satisfies

$$\tilde{V}_t^{(0)} + (\Lambda_0 + i\xi \tilde{A}_1^{(0)} + \xi^2 \tilde{A}_2^{(0)}) \tilde{V}^{(0)} = 0 \tag{2.9}$$

with matrices $\Lambda_0 = \operatorname{diag}(\lambda_{0,1}, \dots, \lambda_{0,d})$ and $\tilde{A}_i^{(0)} = \tilde{L}^{(0)}\, A_i\, \tilde{R}^{(0)}$.

Notation: In the following the notation $A_k^{(l)}$ hints to the facts that the matrix $A_k^{(l)}$ appears in step l of the procedure and that the corresponding summand is of the order $\mathcal{O}(\xi^k)$. For the matrices $\tilde{L}^{(l)}$ and $\tilde{R}^{(l)}$ and the vector function $V^{(l)}$ the number l also denotes the affiliation to step l. An additional '~' is attached to the terms if the corresponding step involved eigenvalue theory, and Λ_k denotes a diagonal matrix, whose corresponding summand is of the order $\mathcal{O}(\xi^k)$.

Step 1: Diagonalization modulo $\mathcal{O}(\xi^2)$-terms

The next step of the procedure consists of two substeps.
The first substep uses a method developed by Wang and Reissig, [Wan03a, Wan03b, RW05], for decoupling hyperbolic-parabolic coupled systems by frequency analysis in phase space.
We introduce the vector $V^{(1)}(t,\xi) = (I + i\xi K_{(1)}) \tilde{V}^{(0)}(t,\xi)$, where $K_{(1)} = (k_{ij})_{i,j=1}^d \in \mathbb{C}^{d\times d}$ is a constant matrix with $k_{ii} = 0$, $i = 1, \dots, d$, and non-diagonal entries to be determined later.

Notation: For matrices $K_{(l)} \in \mathbb{C}^{d\times d}$, appearing in the procedure, l denotes the affiliation to step l.

The vector $V^{(1)}$ satisfies the system

$$V_t^{(1)} + \left(\Lambda_0 + i\xi A_1^{(1)} + \xi^2 A_2^{(1)} + A_3^{(1)}\right) V^{(1)} = 0 \tag{2.10}$$

with the matrices

$$\begin{aligned} A_1^{(1)} &= [K_{(1)}, \Lambda_0] + \tilde{A}_1^{(0)}, \\ A_2^{(1)} &= [K_{(1)}, \Lambda_0]K_{(1)} - [K_{(1)}, \tilde{A}_1^{(0)}] + \tilde{A}_2^{(0)}, \\ A_3^{(1)} &= i\xi^3 \sum_{j=0}^{l-4} (-i\xi)^j \left(-[K_{(1)}, \Lambda_0]K_{(1)}^2 + [K_{(1)}, \tilde{A}_1^{(0)}]K_{(1)} + [K_{(1)}, \tilde{A}_2^{(0)}] \right) K_{(1)}^j + \mathcal{O}(\xi^l). \end{aligned}$$

In the last formula $\mathcal{O}(\xi^l)$ denotes a $d \times d$ matrix, which entries have at least the behavior $\mathcal{O}(\xi^l)$ as $\xi \to 0$. We have used the facts that the matrix $(I + i\xi K_{(1)})$ is invertible for small frequencies $|\xi| \leq \sigma \ll 1$ and thus that we can write $\tilde{V}^{(0)} = (I + i\xi K_{(1)})^{-1} V^{(1)} = (I - i\xi K_{(1)}(I + i\xi K_{(1)})^{-1}) V^{(1)}$. We further made use of the notation $[K, \Lambda] = K\,\Lambda - \Lambda\,K$, i.e., $[K, \Lambda]$ denotes the commutator of K and Λ.
We obtain

$$[K_{(1)}, \Lambda_0] = \begin{pmatrix} 0 & k_{12}(\lambda_{0,2} - \lambda_{0,1}) & k_{13}(\lambda_{0,3} - \lambda_{0,1}) & \dots & k_{1d}(\lambda_{0,d} - \lambda_{0,1}) \\ k_{21}(\lambda_{0,1} - \lambda_{0,2}) & 0 & k_{23}(\lambda_{0,3} - \lambda_{0,2}) & \dots & k_{2d}(\lambda_{0,d} - \lambda_{0,2}) \\ k_{31}(\lambda_{0,1} - \lambda_{0,3}) & k_{32}(\lambda_{0,2} - \lambda_{0,3}) & 0 & \dots & k_{3d}(\lambda_{0,d} - \lambda_{0,3}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{d1}(\lambda_{0,1} - \lambda_{0,d}) & k_{d2}(\lambda_{0,2} - \lambda_{0,d}) & k_{d3}(\lambda_{0,3} - \lambda_{0,d}) & \dots & 0 \end{pmatrix}. \tag{2.11}$$

The eigenvalues $\lambda_{0,j}$ of A_0 are not necessarily distinct. Therefore, some non-diagonal entries in (2.11) may automatically be zero. In fact, we have block matrices of zeros on the diagonal, due to the ordering (2.8). With an appropriate choice of $K_{(1)}$ we obtain that $A_1^{(1)}$ is a block diagonal matrix, i.e.,

$$A_1^{(1)} = [K_{(1)}, \Lambda_0] + \tilde{A}_1^{(0)} = \operatorname{diag}(B_1^{(1)}, B_2^{(1)}, \dots, B_{b_1}^{(1)}) = \begin{pmatrix} \boxed{B_1^{(1)}} & & & \\ & \boxed{B_2^{(1)}} & & \\ & & \ddots & \\ & & & \boxed{B_{b_1}^{(1)}} \end{pmatrix}, \tag{2.12}$$

where b_1 is the number of distinct eigenvalues of A_0.

Remark 2.2. (on Example 2.2)
Assuming an appropriate choice of $K_{(1)}$, we have that the matrix $A_1^{(1)}$ is diagonal, i.e.,

$$A_1^{(1)} = [K_{(1)}, \Lambda_0] + \tilde{A}_1^{(0)} = \operatorname{diag}(\lambda_{1,1}, \dots, \lambda_{1,d}) =: \Lambda_1$$

if we denote the diagonal entries of $\tilde{A}_1^{(0)}$ by $\lambda_{1,1}, \dots, \lambda_{1,d}$. Hence, $V^{(1)}$ satisfies

$$V_t^{(1)} + \left(\Lambda_0 + i\xi\Lambda_1 + \xi^2 A_2^{(1)} + A_3^{(1)} \right) V^{(1)} = 0. \tag{2.13}$$

We can therefore directly go on with step 2.

Even if not all eigenvalues of A_0 are distinct, it may occur that $A_1^{(1)}$ is already diagonal (if $\tilde{A}_1^{(0)}$ has zeros in the right spots), but in general it is not.
Let us thus continue with the second substep of step 1. Our goal is now to diagonalize $A_1^{(1)}$, and for doing this we have to assume that it is diagonalizable. Otherwise we stop the procedure here.
The eigenvalues of $A_1^{(1)}$ are given by the eigenvalues of $B_1^{(1)}$ to $B_{b_1}^{(1)}$. These we denote by $\lambda_j^{B_k^{(1)}}$ and order them for each block $B_k^{(1)}$ in distinct groups of equal numbers as we have done it in (2.8).
The diagonalizability of $A_1^{(1)}$ immediately implies the diagonalizability of $B_1^{(1)}$ to $B_{b_1}^{(1)}$, and thus we can certainly find matrices $L^{B_k^{(1)}}$ and $R^{B_k^{(1)}}$ of left and right eigenvectors with the normalization $L^{B_k^{(1)}} R^{B_k^{(1)}} = I_m$ and $L^{B_k^{(1)}} B_k^{(1)} R^{B_k^{(1)}} = \operatorname{diag}(\lambda_1^{B_k^{(1)}}, \dots, \lambda_m^{B_k^{(1)}}) =: \Lambda^{B_k^{(1)}}$ for $B_k^{(1)} \in \mathbb{C}^{m\times m}$.
Matrices $\tilde{L}^{(1)}$ and $\tilde{R}^{(1)}$ of left and right eigenvectors of $A_1^{(1)}$ with $\tilde{L}^{(1)}\tilde{R}^{(1)} = I$ are then given by

$$\tilde{L}^{(1)} = \begin{pmatrix} L^{B_1^{(1)}} & & & \\ & L^{B_2^{(1)}} & & \\ & & \ddots & \\ & & & L^{B_{b_1}^{(1)}} \end{pmatrix}, \qquad \tilde{R}^{(1)} = \begin{pmatrix} R^{B_1^{(1)}} & & & \\ & R^{B_2^{(1)}} & & \\ & & \ddots & \\ & & & R^{B_{b_1}^{(1)}} \end{pmatrix}.$$

We conclude $\tilde{L}^{(1)}\Lambda_0\tilde{R}^{(1)} = \Lambda_0\tilde{L}^{(1)}\tilde{R}^{(1)} = \Lambda_0$, due to the fact that the diagonal entries in the block matrices in Λ_0 corresponding to the blocks $B_k^{(1)}$ are all equal. Therefore, we do not destroy the diagonal structure of Λ_0 that we have obtained in step 0 of the procedure when introducing $\tilde{V}^{(1)} = \tilde{L}^{(1)}V^{(1)}$.
The vector $\tilde{V}^{(1)}$ satisfies the system

$$\tilde{V}_t^{(1)} + \left(\Lambda_0 + i\xi\Lambda_1 + \xi^2\tilde{A}_2^{(1)} + \tilde{A}_3^{(1)}\right)\tilde{V}^{(1)} = 0 \tag{2.14}$$

with the matrices $\Lambda_1 = \operatorname{diag}(\Lambda^{B_1^{(1)}}, \dots, \Lambda^{B_{b_1}^{(1)}}) =: \operatorname{diag}(\lambda_{1,1}, \dots, \lambda_{1,d})$ and $\tilde{A}_i^{(1)} = \tilde{L}^{(1)}A_i^{(1)}\tilde{R}^{(1)}$.

Step 2: Diagonalization modulo $\mathcal{O}(\xi^3)$-terms

This step of the procedure involves three substeps.
For the first two we can once more use the method developed by Wang and Reissig.
First, we introduce a vector $V^{(1\frac{1}{2})} = (I + \xi^2 K_{(1\frac{1}{2})})\tilde{V}^{(1)}$, where $K_{(1\frac{1}{2})}$ is a constant matrix having nonzero entries in positions (i,j) only if $\lambda_{0,i} \neq \lambda_{0,j}$ and if the corresponding entry in $\tilde{A}_2^{(1)}$ is not vanishing. These nonzero entries will again be determined later. The vector $V^{(1\frac{1}{2})}$ satisfies

$$V_t^{(1\frac{1}{2})} + \left(\Lambda_0 + i\xi\Lambda_1 + \xi^2 A_2^{(1\frac{1}{2})} + A_3^{(1\frac{1}{2})}\right)V^{(1\frac{1}{2})} = 0 \tag{2.15}$$

with the matrices

$$
\begin{aligned}
A_2^{(1\frac{1}{2})} &= [K_{(1\frac{1}{2})}, \Lambda_0] + \tilde{A}_2^{(1)}, \\
A_3^{(1\frac{1}{2})} &= -\xi^4 \sum_{j=0}^{\lfloor \frac{l-5}{2} \rfloor} (-\xi^2)^j [K_{(1\frac{1}{2})}, \Lambda_0] K_{(1\frac{1}{2})}^{1+j} + i\xi^3 \sum_{j=0}^{\lfloor \frac{l-4}{2} \rfloor} (-\xi^2)^j [K_{(1\frac{1}{2})}, \Lambda_1] K_{(1\frac{1}{2})}^{j} \\
&\quad + \xi^4 \sum_{j=0}^{\lfloor \frac{l-5}{2} \rfloor} (-\xi^2)^j [K_{(1\frac{1}{2})}, \tilde{A}_2^{(1)}] K_{(1\frac{1}{2})}^{j} + \sum_{j=0}^{\lfloor \frac{l-4}{2} \rfloor} (-\xi^2)^j \tilde{A}_3^{(1)} K_{(1\frac{1}{2})}^{j} + \mathcal{O}(\xi^l).
\end{aligned}
$$

Just like in (2.12) we can choose $K_{(1\frac{1}{2})}$ in a way such that

$$
A_2^{(1\frac{1}{2})} = [K_{(1\frac{1}{2})}, \Lambda_0] + \tilde{A}_2^{(1)} = \operatorname{diag}(B_1^{(1\frac{1}{2})}, \dots, B_{b_1}^{(1\frac{1}{2})}), \tag{2.16}
$$

where the blocks on the diagonal have the same size as in (2.12).

Remark 2.3. (on Example 2.2)
If we choose $K_{(1\frac{1}{2})}$ appropriately, then the matrix $A_2^{(1\frac{1}{2})}$ is already diagonal, i.e.,

$$
A_2^{(1\frac{1}{2})} = \operatorname{diag}(\lambda_{2,1}, \dots, \lambda_{2,d}) =: \Lambda_2
$$

if we denote by $\lambda_{2,1}, \dots, \lambda_{2,d}$ the diagonal entries of $\tilde{A}_2^{(1)}$. Hence, $V^{(1\frac{1}{2})}$ satisfies

$$
V_t^{(1\frac{1}{2})} + \left(\Lambda_0 + i\xi\Lambda_1 + \xi^2\Lambda_2 + A_3^{(1\frac{1}{2})}\right) V^{(1\frac{1}{2})} = 0, \tag{2.17}
$$

and we can go on with step 3.

In the second substep we introduce a vector $V^{(2)} = (I + i\xi K_{(2)})V^{(1\frac{1}{2})}$ with $K_{(2)}$ being a constant matrix, again having zeros on the diagonal and non-diagonal nonzero entries in positions (i, j) only if $\lambda_{1,i} \neq \lambda_{1,j}$ and if $A_2^{(1\frac{1}{2})}$ has nonzero entries in these positions. Having the formula (2.11) in mind, we conclude $[K_{(2)}, \Lambda_0] = 0$. Thus, the vector $V^{(2)}$ satisfies

$$
V_t^{(2)} + \left(\Lambda_0 + i\xi\Lambda_1 + \xi^2 A_2^{(2)} + A_3^{(2)}\right) V^{(2)} = 0 \tag{2.18}
$$

with the matrices

$$
\begin{aligned}
A_2^{(2)} &= -[K_{(2)}, \Lambda_1] + A_2^{(1\frac{1}{2})}, \\
A_3^{(2)} &= i\xi^3 \sum_{j=0}^{l-4} (-i\xi)^j \left([K_{(2)}, \Lambda_1]K_{(2)} + [K_{(2)}, A_2^{(1\frac{1}{2})}]\right) K_{(2)}^j + \sum_{j=0}^{l-4} (-i\xi)^j A_3^{(1\frac{1}{2})} K_{(2)}^j + \mathcal{O}(\xi^l).
\end{aligned}
$$

The matrix $A_2^{(2)}$ is in any case a block diagonal matrix of the form

$$
A_2^{(2)} = \operatorname{diag}(\check{B}_1^{(1\frac{1}{2})}, \dots, \check{B}_{b_1}^{(1\frac{1}{2})})
$$

(cf. (2.16)). Each block $\check{B}_k^{(1\frac{1}{2})}$ though is again a block diagonal matrix, assuming that $K_{(2)}$ is chosen in a proper way, where the size of the blocks depend on the multiplicities of the eigenvalues $\lambda_i^{B_k^{(1)}}$ of $B_k^{(1)}$. Hence,

$$
A_2^{(2)} = \operatorname{diag}(B_1^{(2)}, \dots, B_{b_2}^{(2)})
$$

with $b_1 \leq b_2 \leq d$. The case $b_2 = b_1$ occurs if and only if the eigenvalues of $B_k^{(1)}$ are all the same for all $k = 1, \dots, b_1$.
For the third substep we have thus the same situation as we did in the second substep of step 1 for the matrix $A_1^{(1)}$. If we want to go on, we have to assume that $A_2^{(2)}$ is diagonalizable. If that is not the case, the procedure will be stopped here.
If $A_2^{(2)}$ is diagonalizable, then we have diagonalizability for the matrices $B_k^{(2)}$, $k = 1, \dots, b_2$, and we can do just the same calculations as we have done for the diagonalizable $A_1^{(1)}$. The eigenvalues $\lambda_j^{B_k^{(2)}}$ of $B_k^{(2)} \in \mathbb{C}^{m \times m}$ shall be ordered in distinct groups of equal numbers. We can then find left and right eigenvectors with the normalization $L^{B_k^{(2)}} R^{B_k^{(2)}} = I_m$ and $L^{B_k^{(2)}} B_k^{(2)} R^{B_k^{(2)}} = \Lambda^{B_k^{(2)}} = \mathrm{diag}(\lambda_1^{B_k^{(2)}}, \dots, \lambda_m^{B_k^{(2)}})$. These matrices we use to construct matrices $\tilde{L}^{(2)}$ and $\tilde{R}^{(2)}$ of left and right eigenvectors of $A_2^{(2)}$ with $\tilde{L}^{(2)} \tilde{R}^{(2)} = I$ and conclude just like before that $\tilde{L}^{(2)} \Lambda_i \tilde{R}^{(2)} = \Lambda_i \tilde{L}^{(2)} \tilde{R}^{(2)} = \Lambda_i$ for $i = 1, 2$. Hence, the vector $\tilde{V}^{(2)} = \tilde{L}^{(2)} V^{(2)}$ satisfies

$$\tilde{V}_t^{(2)} + \left(\Lambda_0 + i\xi\Lambda_1 + \xi^2 \Lambda_2 + \tilde{A}_3^{(2)} \right) \tilde{V}^{(2)} = 0$$

with $\Lambda_2 = \mathrm{diag}(\Lambda^{B_1^{(2)}}, \dots, \Lambda^{B_{b_2}^{(2)}}) =: \mathrm{diag}(\lambda_{2,1}, \dots, \lambda_{2,d})$ and $\tilde{A}_3^{(2)} = \tilde{L}^{(2)} A_3^{(2)} \tilde{R}^{(2)} = \mathcal{O}(\xi^3)$ for $\xi \to 0$.

Further diagonalization

Reconsidering the foregoing steps we come to the following conclusion:
In the n^{th} step of the procedure, i.e., when diagonalizing modulo $\mathcal{O}(\xi^{n+1})$, we need $n+1$ substeps, n substeps via the procedure introduced by Wang and Reissig and one substep, where we have to use eigenvalue theory and the assumption that the appearing matrix $A_n^{(n)}$ is diagonalizable. If the latter is not the case, then we have to stop the procedure at this point.
Hence, let us assume that the matrices $A_k^{(k)}$ are diagonalizable for $k = 3, \dots, n$. Then the vector

$$\begin{aligned} \tilde{V}^{(n)} = \tilde{L}^{(n)} (I + i\xi K_{(n)}) \cdot \ldots \cdot (I + a_n \xi^n K_{\left((n-1)\frac{1}{n}\right)}) \cdot \ldots \\ \cdot \tilde{L}^{(3)} (I + i\xi K_{(3)}) (I + \xi^2 K_{\left(2\frac{2}{3}\right)}) (I + i\xi^3 K_{\left(2\frac{1}{3}\right)}) \tilde{V}^{(2)} \end{aligned}$$

satisfies, assuming that all matrices are chosen properly,

$$\tilde{V}_t^{(n)} + \left(\Lambda_0 + i\xi\Lambda_1 + \xi^2\Lambda_2 + i\xi^3\Lambda_3 + \ldots + a_n \xi^n \Lambda_n + \tilde{A}_{n+1}^{(n)} \right) \tilde{V}^{(n)} = 0$$

with $\tilde{A}_{n+1}^{(n)} = \mathcal{O}(\xi^{n+1})$. Here we use $a_n = i$ for odd n and $a_n = 1$ for even n.

Remark 2.4. (on Example 2.2)
The matrices $A_k^{(k)}$ are for all $k = 1, \dots, n$ already diagonal, and the vector $\tilde{V}^{(n)}$ takes the much more simple form

$$\tilde{V}^{(n)} = (I + a_n \xi^n K_{\left((n-1)\frac{1}{n}\right)}) \cdot \ldots \cdot (I + \xi^4 K_{\left(3\frac{1}{4}\right)}) (I + i\xi^3 K_{\left(2\frac{1}{3}\right)}) \tilde{V}^{(2)}.$$

Summary of the procedure

Let us for easy reference write down a scheme of the procedure:

Step 0: We diagonalize modulo $\mathcal{O}(\xi)$-terms in $0+1$ substeps.
Assumption: A_0 is diagonalizable.
For $\tilde{V}^{(0)} = \tilde{L}^{(0)}V$ we obtain a coefficient matrix with diagonal part $\Lambda_0 = \operatorname{diag}(\lambda_{0,1},\dots,\lambda_{0,d})$, $\lambda_{0,j}$ ordered in distinct groups of equal numbers.

Step 1: We diagonalize modulo $\mathcal{O}(\xi^2)$-terms in $1+1$ substeps.
- *Substep 1:* For $V^{(1)} = (I+i\xi K_{(1)})\tilde{V}^{(0)}$ we obtain a coefficient matrix with block-diagonal part $\Lambda_0 + i\xi A_1^{(1)}$, $A_1^{(1)} = \operatorname{diag}(B_1^{(1)},\dots,B_{b_1}^{(1)})$, $B_k^{(1)} \in \mathbb{C}^{m_k\times m_k}$.
- *Assumption:* $A_1^{(1)}$ is diagonalizable.
- *Substep 2:* For $\tilde{V}^{(1)} = \tilde{L}^{(1)}V^{(1)}$ we obtain a coefficient matrix with diagonal part $\Lambda_0 + i\xi\Lambda_1$, $\Lambda_1 = \operatorname{diag}(\lambda_{1,1},\dots,\lambda_{1,d}) = \operatorname{diag}(\lambda_1^{B_1^{(1)}},\dots,\lambda_{m_1}^{B_1^{(1)}},\dots,\lambda_1^{B_{b_1}^{(1)}},\dots,\lambda_{m_{b_1}}^{B_{b_1}^{(1)}})$, $\lambda_j^{B_k^{(1)}}$ ordered in distinct groups of equal numbers, $k=1,\dots,b_1$.

⋮

Step n: We diagonalize modulo $\mathcal{O}(\xi^{n+1})$-terms in $n+1$ substeps.
- *Substep* 1: For $V^{\left((n-1)\frac{1}{n}\right)} = (I + a_n\xi^n K_{\left((n-1)\frac{1}{n}\right)})\tilde{V}^{(n-1)}$ we obtain a coefficient matrix with block-diagonal part $\Lambda_0 + \ldots + a_{n-1}\xi^{n-1}\Lambda_{n-1} + a_n\xi^n A_n^{\left((n-1)\frac{1}{n}\right)}$, $A_n^{\left((n-1)\frac{1}{n}\right)} = \operatorname{diag}(B_1^{\left((n-1)\frac{1}{n}\right)},\dots,B_{b_1}^{\left((n-1)\frac{1}{n}\right)})$.
- *Substep* 2: For $V^{\left((n-1)\frac{2}{n}\right)} = (I + a_{n-1}\xi^{n-1}K_{\left((n-1)\frac{2}{n}\right)})V^{\left((n-1)\frac{1}{n}\right)}$ we obtain a coefficient matrix with block-diagonal part $\Lambda_0 + \ldots + a_{n-1}\xi^{n-1}\Lambda_{n-1} + a_n\xi^n A_n^{\left((n-1)\frac{2}{n}\right)}$, $A_n^{\left((n-1)\frac{2}{n}\right)} = \operatorname{diag}(B_1^{\left((n-1)\frac{2}{n}\right)},\dots,B_{b_2}^{\left((n-1)\frac{2}{n}\right)})$, $b_2 \geq b_1$.

  ⋮
- *Substep n:* For $V^{(n)} = (I+i\xi K_{(n)})V^{\left((n-1)\frac{n-1}{n}\right)}$ we obtain a coefficient matrix with block-diagonal part $\Lambda_0 + \ldots + a_{n-1}\xi^{n-1}\Lambda_{n-1} + A_n^{(n)}$, $A_n^{(n)} = \operatorname{diag}(B_1^{(n)},\dots,B_{b_n}^{(n)})$, $B_k^{(n)} \in \mathbb{C}^{m_k\times m_k}$, $b_n \geq b_{n-1} \geq \ldots \geq b_2 \geq b_1$.
- *Assumption:* $A_n^{(n)}$ is diagonalizable.
- *Substep n+1:* For $\tilde{V}^{(n)} = \tilde{L}^{(n)}V^{(n)}$ we obtain a coefficient matrix with diagonal part $\Lambda_0+\ldots+a_n\xi^n\Lambda_n$, $\Lambda_n = \operatorname{diag}(\lambda_{n,1},\dots,\lambda_{n,d}) = \operatorname{diag}(\lambda_1^{B_1^{(n)}},\dots,\lambda_{m_1}^{B_1^{(n)}},\dots,\lambda_1^{B_{b_n}^{(n)}},\dots,\lambda_{m_{b_n}}^{B_{b_n}^{(n)}})$, $\lambda_j^{B_k^{(n)}}$ ordered in distinct groups of equal numbers, $k=1,\dots,b_n$.

It is of significant importance for our procedure that we have under the imposed assumptions:

Lemma 2.3. *In substep k of step n, $n \geq 1$, $1 \leq k \leq n+1$, we do not alter the diagonal part of the coefficient matrix when transforming the system for $V^{\left((n-1)\frac{k-1}{n}\right)}$ ($\tilde{V}^{(n-1)}$ for $k=1$) into the one for $V^{\left((n-1)\frac{k}{n}\right)}$ ($\tilde{V}^{(n)}$ for $k=n+1$).*

Proof. The proof is a simple generalization of our foregoing calculations and follows from the choice of the matrices involved in the transformations and the proposed ordering of the eigenvalues $\lambda_{i,j}$, $0 \leq i \leq n-1$, $1 \leq j \leq d$. □

2.2.2.2. Definitions and results of the procedure

Let us now go on with formulating some basic definitions, proposing assumptions and concluding results from the observations of the last section.

Definition 2.3. *Let us call the system (2.4) diagonalizable for small (large) frequencies up to the order n ($-n+2$) if we can rewrite it in $Z_{int}(\sigma)$ ($Z_{ext}(N)$), using a number of continuous, regular $\mathcal{O}(1)$-transformations for $\xi \to 0$ (for $|\xi| \to \infty$), into a system for a new vector W, where the coefficient matrix is diagonal modulo terms of the order $\mathcal{O}(\xi^{n+1})$ ($\mathcal{O}(\xi^{-n+1})$).*
By an $\mathcal{O}(1)$-transformation we want to understand a transformation of type $V_2 = A\,V_1$, where all entries of A and its inverse are of the order $\mathcal{O}(1)$ uniformly for $\xi \to 0$ (for $|\xi| \to \infty$).

Definition 2.4. *We call the system (2.4) fully diagonalizable for small (large) frequencies if there is a finite number of continuous, regular $\mathcal{O}(1)$-transformations for $\xi \to 0$ (for $|\xi| \to \infty$), so that the coefficient matrix of the system for the new vector W is completely diagonal.*

Remark 2.5. The full diagonalizability of the system (2.4) for small frequencies is equivalent to the existence of a continuous matrix $T_{int}(\xi)$ in $Z_{int}(\sigma)$ that transfers $A(\xi)$ via a similarity transformation to the diagonal matrix of its eigenvalues $\Lambda(\xi) = \mathrm{diag}(\mu_1(\xi), \dots, \mu_d(\xi))$, i.e.,

$$\Lambda(\xi) = T_{int}(\xi) A(\xi) T_{int}^{-1}(\xi). \tag{2.19}$$

The above uniform diagonalizability, i.e., the existence of a continuous matrix $T_{int}(\xi)$ in $Z_{int}(\sigma)$ such that (2.19) holds, certainly implies diagonalizability of $A(\xi)$ for every fixed $\xi \in Z_{int}(\sigma)$, and using the results on analytic perturbation theory in [Kat80] we can moreover state that the converse is true if all eigenvalues of A_0 are distinct. If the latter does not hold true, then pointwise diagonalizability may not imply the uniform one.
Analogous results hold for large frequencies.

With the notation $A_0^{(0)} := A_0$ we now introduce the assumptions:

($\mathbf{A_n}$) The matrices $A_k^{(k)}$ are diagonalizable for $k = 0, 1, \dots, n$

and:

(A'_n) The matrices $A_k^{(k)}$ are symmetrizable for $k = 0, 1, \ldots, n$.

The assumption (A'_n) is introduced due to the facts that it is satisfied for most practical useful cases and leads to especially nice solution representations. Note that an ordering of the eigenvalues of $A_k^{(k)}$ in the relevant blocks may be achieved by arranging them in an ascending way. The assumption (A'_n) certainly implies (A_n).
With the considerations from the last section we can state:

Proposition 2.4. *If the assumption (A_n) holds, then the system (2.4) is diagonalizable for small frequencies up to the order n.*

In the case that all eigenvalues of the diagonalizable matrix A_0 are distinct we have that the system (2.4) is diagonalizable for small frequencies up to the order n for every $n \in \mathbb{N}_0$. We want to generalize this observation. Let us thus assume that (A_n) holds and introduce the assumption

(B_n) $\forall i, j \in \{1, \ldots, d\}$ with $i \neq j$ $\exists k \in \{0, 1, \ldots, n\} : \lambda_{k,i} \neq \lambda_{k,j}$.

We note that the assumption (B_n) is not satisfied for any n if and only if there are two identical eigenvalues of $A(\xi)$ in $Z_{int}(\sigma)$, that is, if $A(\xi)$ is permanently degenerate.

Lemma 2.5. *Suppose that for some n the assumptions (A_n) and (B_n) hold. Then the system (2.4) is diagonalizable for small frequencies up to the order n for every $n \in \mathbb{N}_0$.*

Remark 2.6. The assumptions in Lemma 2.5 are certainly equivalent to

$$\forall n \in \mathbb{N}_0 : (\mathrm{A_n}) \text{ holds} \quad \wedge \quad \exists n \in \mathbb{N}_0 : (\mathrm{B_n}) \text{ is satisfied.} \tag{2.20}$$

Note that the second condition is not necessary for the results of Lemma 2.5. Consider therefore the example $A = A(\xi) = B + i\xi B + \xi^2 B$, where B denotes a $d \times d$ constant and diagonalizable matrix with at least one multiple fold eigenvalue. In step 0 of the procedure we diagonalize the coefficient matrix completely. Nevertheless, we can follow the procedure (noting that all appearing matrices $K_{(l)}$ vanish and that the matrices $A_l^{(l)}$ are already diagonal), in which (B_n) won't be satisfied for any n.

There is quite some information following from our procedure. First, we obtain:

Lemma 2.6. *Suppose (A_n) holds. Then the characteristic roots $\mu_j = \mu_j(\xi)$, $j = 1, \ldots, d$, of the coefficient matrix $A(\xi) = A_0 + i\xi A_1 + \xi^2 A_2$ of (2.4) behave for $|\xi| \leq \sigma \ll 1$ as*

$$\mu_j(\xi) = \lambda_{0,j} + i\lambda_{1,j}\,\xi + \lambda_{2,j}\,\xi^2 + \ldots + a_n \lambda_{n,j}\,\xi^n + \mathcal{O}(\xi^{n+1}),$$

where $a_n = i$ for odd n and $a_n = 1$ for even n. The numbers $\lambda_{k,j}$ are real if even (A'_n) holds, otherwise they may be complex.

We suppose now that for some n the assumptions ($\mathrm{A_n}$) and ($\mathrm{B_n}$) hold and denote $A^{(n)}(\xi) = \Lambda_0 + i\xi\Lambda_1 + \xi^2\Lambda_2 + \ldots + a_n\xi^n\Lambda_n + \tilde{A}^{(n)}_{n+1}(\xi)$ and further by $\{\tilde{l}_j = \tilde{l}_j(\xi)\}_{j=1}^d$ a set of linearly independent left and by $\{\tilde{r}_j = \tilde{r}_j(\xi)\}_{j=1}^d$ a set of linearly independent right eigenvectors corresponding to $\mu_j(\xi)$. We use the notations $\tilde{R} = \tilde{R}(\xi) = (\tilde{r}_{kj}(\xi))_{k,j=1}^d$ and $\tilde{L} = \tilde{L}(\xi) = (\tilde{l}_{jk}(\xi))_{j,k=1}^d$ and define the number c by

$$c = \max_{i<j} c_{ij} \qquad \text{and} \qquad c_{ij} = \min\{k \in \{0,1,\ldots,n\} : \lambda_{k,i} \neq \lambda_{k,j}\}. \tag{2.21}$$

Note that c equals the minimal number n for which the assumption ($\mathrm{B_n}$) holds.
We can now state:

Lemma 2.7. *The matrices $\tilde{R}$ and $\tilde{L}$ may be chosen as $\tilde{r}_{kk} = \tilde{l}_{kk} = 1$ and $\tilde{r}_{kj}(\xi)$ as well as $\tilde{l}_{jk}(\xi)$ have at least the asymptotic behavior $\tilde{r}_{kj}(\xi) = \mathcal{O}(\xi^{n+1-c})$, $\tilde{l}_{jk}(\xi) = \mathcal{O}(\xi^{n+1-c})$ for $j \neq k$ and $\xi \to 0$.*

Proof. Let us fix a value i with $c = c_{ij}$ for some $j = 1,\ldots,n$.
For the eigenvalue $\mu_i = \mu_i(\xi)$ of $A^{(n)}(\xi)$ we know that

$$(\mu_i I - A^{(n)}(\xi))\tilde{r}_i = (\mu_i I - \Lambda_0 - i\xi\Lambda_1 - \xi^2\Lambda_2 - \ldots - a_n\xi^n\Lambda_n - \tilde{A}^{(n)}_{n+1}(\xi))\tilde{r}_i = 0.$$

That implies that the components of $\tilde{r}_i$ satisfy

$$\left[\begin{pmatrix} k_{11} & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & k_{22} & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & k_{(i-1)(i-1)} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & k_{(i+1)(i+1)} & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & k_{dd} \end{pmatrix} + \mathcal{O}(\xi^{n+1})\right] \begin{pmatrix} \tilde{r}_{1i} \\ \tilde{r}_{2i} \\ \vdots \\ \tilde{r}_{(i-1)i} \\ \tilde{r}_{(i+1)i} \\ \vdots \\ \tilde{r}_{di} \end{pmatrix} = \mathcal{O}(\xi^{n+1})\tilde{r}_{ii}.$$

Here $k_{jj} = \lambda_{0,i} - \lambda_{0,j} + i(\lambda_{1,i} - \lambda_{1,j})\xi + \ldots + a_n(\lambda_{n,i} - \lambda_{n,j})\xi^n$. The term $\mathcal{O}(\xi^{n+1})$ on the left denotes a $(d-1)\times(d-1)$ matrix with elements being of the order $\mathcal{O}(\xi^{n+1})$ as $\xi \to 0$, while $\mathcal{O}(\xi^{n+1})$ on the right side denotes a $(d-1)\times 1$ vector with elements of the same asymptotic behavior. If we set $\tilde{r}_{ii} = 1$, then we conclude that $\tilde{r}_{ki} = \mathcal{O}(\xi^{n+1-c})$ for all $k \neq i$ (and that this is optimal for $k = j$ with $c = c_{ij}$). The results for the other right and the left eigenvectors can be obtained in the same way and may of course even look better. □

In correspondence to Proposition 2.4 we can thus state:

Proposition 2.8. *The system (2.4) is fully diagonalizable for small frequencies if the assumptions (A_n) and (B_n) hold for some n.*

Remark 2.7. The assumptions in Proposition 2.8 are equivalent to (2.20). As in Remark 2.6 we see that the second condition in (2.20) is not necessary for the full diagonalizability of system (2.4) for small frequencies. The first condition however is necessary, as will be discussed in Section 2.2.2.3.

Moreover, we have:

Proposition 2.9. *We assume that the assumptions (A_n) and (B_n) hold for some n (and thus that the system is fully diagonalizable). Then we obtain:*
The solution of (2.4) has in $Z_{int}(\sigma) = \{|\xi| \leq \sigma \ll 1\}$ the following representation

$$V(t,\xi) = T_{int}^{-1}(\xi)\,\mathrm{diag}(e^{-\mu_1(\xi)t},\dots,e^{-\mu_d(\xi)t})T_{int}(\xi)\hat{U}_0(\xi), \tag{2.22}$$

where

$$T_{int}(\xi) = \tilde{L}(\xi)\,\tilde{L}^{(n)}(I + i\xi K_{(n)}) \cdot \ldots \cdot (I + a_n \xi^n K_{((n-1)\frac{1}{n})}) \cdot \ldots \cdot \tilde{L}^{(1)}(I + i\xi K_{(1)})\tilde{L}^{(0)}.$$

The matrices $K_{(j)}$ and $\tilde{L}^{(j)}$ are matrices used in the diagonalization procedure in Section 2.2.2.1, $\tilde{L} = \tilde{L}(\xi)$ is from Lemma 2.7 and the $\mu_j(\xi)$ are from Lemma 2.6.

Proof. We choose the transformation $\tilde{W} = \tilde{L}\tilde{V}^{(n)}$. The vector $\tilde{W}$ satisfies $\tilde{W}_t + D\tilde{W} = 0$ with $D = \mathrm{diag}(\mu_1(\xi),\dots,\mu_d(\xi))$. Solving this diagonal system together with the backward transformation from $\tilde{W}$ to V gives (2.22). □

Remark 2.8. 1. Noting that we could write down the full asymptotic expansion of the diagonalizer $\tilde{L}$ with the help of our procedure, we observe that $T_{int}(\xi)$ and its inverse are in fact analytic in $Z_{int}(\sigma)$. This is due to the analytic dependence of $A(\xi)$ on ξ (cf. [Kat80]).
2. The assumption (B_n) is motivated by the applications. In all physically reasonable models, that we have studied, it was always satisfied for some n. For the case that the assumption (B_n) is not satisfied for any n, solution representations can be derived as well. For such considerations please refer to Section 2.2.2.4.

2.2.2.3. On the necessity of the assumptions (A_n)

A question of interest certainly is whether the condition

$$\forall n \in \mathbb{N}_0 : (A_n) \text{ holds} \tag{2.23}$$

is necessary for the full diagonalizability of system (2.4) for small frequencies (compare to (2.20) and Remark 2.7), and thus whether a uniform diagonalizability (in the sense of Remark 2.5) implies (2.23).
We can give the positive answer in:

Lemma 2.10. *If system (2.4) is fully diagonalizable for small frequencies, then (A_n) holds for all numbers $n \in \mathbb{N}_0$.*

Proof. The full diagonalizability of system (2.4) implies the existence of a continuous matrix $T_{int}(\xi)$ in $Z_{int}(\sigma)$ so that

$$\Lambda(\xi) = T_{int}(\xi)A(\xi)T_{int}^{-1}(\xi)$$

is diagonal. The above equality gives for $\xi = 0$ the diagonalizability of A_0 and thus that ($\mathrm{A_0}$) is satisfied.
Let us assume that ($\mathrm{A_n}$) is satisfied but ($\mathrm{A_{n+1}}$) is not, i.e., $A_{n+1}^{(n+1)}$ is not diagonalizable. Reconsidering our diagonalization procedure we know that there exists a matrix of the form

$$T^{(n+1)}(\xi) = (I + i\xi K_{(n+1)}) \cdot \ldots \cdot (I + a_{n+1}\xi^{n+1}K_{(n\frac{1}{n+1})}) \cdot \ldots \cdot \tilde{L}^{(1)}(I + i\xi K_{(1)})\tilde{L}^{(0)}$$

such that

$$A(\xi) = (T^{(n+1)}(\xi))^{-1}\left(\Lambda_0 + \ldots + a_n\xi^n\Lambda_n + a_{n+1}\xi^{n+1}A_{n+1}^{(n+1)} + A_{n+2}^{(n+1)}\right)T^{(n+1)}(\xi),$$

where the matrices Λ_i are diagonal and $A_{n+1}^{(n+1)}$ is block-diagonal, i.e., $A_{n+1}^{(n+1)} = \mathrm{diag}(B_1^{(n+1)}, \ldots, B_{b_{n+1}}^{(n+1)})$ with $b_{n+1} < d$. The corresponding blocks to $B_k^{(n+1)}$ in the matrices Λ_i, $i = 0, \ldots, n$, have in any case the same entries on the diagonal. The matrix $A_{n+2}^{(n+1)}$ finally has the asymptotic behavior $A_{n+2}^{(n+1)} = \mathcal{O}(\xi^{n+2})$ for $\xi \to 0$.
With the notation $T(\xi) := T_{int}(\xi)(T^{(n+1)}(\xi))^{-1}$ the above implies

$$T^{-1}(\xi)\Lambda(\xi)T(\xi) = \Lambda_0 + \ldots + a_n\xi^n\Lambda_n + a_{n+1}\xi^{n+1}A_{n+1}^{(n+1)} + A_{n+2}^{(n+1)}.$$

The matrix on the left is uniformly diagonalizable for $\xi \in Z_{int}(\sigma)$. For the matrix on the right we can make the following considerations: The non-diagonalizability of $A_{n+1}^{(n+1)}$ implies the non-diagonalizability of at least one of the blocks $B_k^{(n+1)}$, that is, the non-existence of a constant matrix diagonalizing $B_k^{(n+1)}$. Hence, there does not exist a constant matrix diagonalizing the matrix $\Lambda_0 + \ldots + a_n\xi^n\Lambda_n + a_{n+1}\xi^{n+1}A_{n+1}^{(n+1)}$ for $\xi \neq 0$. This together with the smallness of $A_{n+2}^{(n+1)}$ (and the continuity of T) gives the contradiction. □

It immediately follows (see Remark 2.5):

Corollary 2.11. *Let the coefficient matrix $A = A(\xi)$ of system (2.4) be diagonalizable for every fixed $|\xi| \leq \sigma \ll 1$ and suppose that all eigenvalues of A_0 are distinct. Then (A_n) holds for all numbers $n \in \mathbb{N}_0$.*

If we consider only models with a non-permanently degenerate coefficient matrix $A(\xi)$ of (2.4), then we know that our conditions, i.e., there exists a n so that ($\mathrm{A_n}$) and ($\mathrm{B_n}$) are satisfied, are sufficient and necessary for the full (uniform) diagonalizability for small frequencies.

2.2.2.4. Some considerations for non-fully diagonalizable systems

Let us first consider models for which ($\mathrm{A_n}$) is not satisfied for a certain n.
Hence, let us assume that the matrix $A_n^{(n)}$ is not diagonalizable for some n and thus that the diagonalization procedure stops in the n^{th} step, i.e., when diagonalizing modulo $\mathcal{O}(\xi^{n+1})$. Starting point of our considerations is therefore the system

$$V_t^{(n)} + \left(\Lambda_0 + i\xi\Lambda_1 + \xi^2\Lambda_2 + \ldots + a_{n-1}\xi^{n-1}\Lambda_{n-1} + a_n\xi^n A_n^{(n)} + A_{n+1}^{(n)}\right) V^{(n)} = 0$$

with $A_n^{(n)} = \operatorname{diag}(B_1^{(n)}, \ldots, B_{b_n}^{(n)})$, $b_n < d$, and $A_{n+1}^{(n)}(\xi) = \mathcal{O}(\xi^{n+1})$ being a matrix for which we can write down explicit formulas apart from some remainder $\mathcal{O}(\xi^l)$ for an arbitrary $l \geq n+1$.
At least one $B_k^{(n)}$ is not diagonalizable, but there exists a matrix $S^{B_k^{(n)}}$ transforming it to its Jordan canonical form $J^{B_k^{(n)}}$, i.e., $\left(S^{B_k^{(n)}}\right)^{-1} B_k^{(n)} S^{B_k^{(n)}} = J^{B_k^{(n)}}$. The matrices $J_n = \operatorname{diag}(J^{B_1^{(n)}}, \ldots, J^{B_{b_n}^{(n)}})$ (if $B_k^{(n)}$ is diagonalizable, then $J^{B_k^{(n)}}$ is of course diagonal with equal entries grouped together) and $\tilde{S}^{(n)} = \operatorname{diag}(S^{B_1^{(n)}}, \ldots, S^{B_{b_n}^{(n)}})$ satisfy $\left(\tilde{S}^{(n)}\right)^{-1} A_n^{(n)} \tilde{S}^{(n)} = J_n$.
Noting that $\left(\tilde{S}^{(n)}\right)^{-1} \Lambda_i \tilde{S}^{(n)} = \Lambda_i \left(\tilde{S}^{(n)}\right)^{-1} \tilde{S}^{(n)} = \Lambda_i$ for $i = 0, 1, \ldots, n-1$, due to our construction method, we obtain for the vector $\tilde{V}^{(n)} = \left(\tilde{S}^{(n)}\right)^{-1} V^{(n)}$ the system

$$\tilde{V}_t^{(n)} + \left(\Lambda_0 + i\xi\Lambda_1 + \xi^2\Lambda_2 + \ldots + a_{n-1}\xi^{n-1}\Lambda_{n-1} + a_n\xi^n J_n + \left(\tilde{S}^{(n)}\right)^{-1} A_{n+1}^{(n)} \tilde{S}^{(n)}\right) \tilde{V}^{(n)} = 0.$$

If we need to go on with the diagonalization, then we can block-diagonalize modulo terms of the order $\mathcal{O}(\xi^l)$ with arbitrary $l \geq n+1$ and blocks that are determined by the maximal ones in each $J^{B_k^{(n)}}$ with equal numbers on the diagonal. If one of the latter blocks is diagonal, then we can follow the procedure and try to refine our steps for this part. If it is not (completely) diagonal, then there may occur difficulties even when we have diagonalizability of the corresponding block in the next step (i.e., we can not guarantee that we do not lose the almost diagonal structure of the Jordan block, when diagonalizing).
Assume now that we have obtained an initial value problem of the form

$$\begin{cases} W_t + M\,W + E\,W = 0, \\ W(0,\xi) = W_0(\xi), \end{cases} \tag{2.24}$$

where $E(\xi) = \mathcal{O}(\xi^p)$ for $\xi \to 0$ and $M = M(\xi)$ is an almost diagonal matrix, i.e., the only nonzero entries appear in the diagonal and (let's say) lower secondary diagonal. This initial value problem is one for weakly coupled ordinary differential equations.
From (2.24) we obtain besides the asymptotic behavior of the eigenvalues of $A(\xi)$ for $|\xi| \leq \sigma \ll 1$ (compare to Lemma 2.6) via Duhamel's principle the relation

$$W(t,\xi) = F(t,\xi)W_0(\xi) - \int_0^t F(t-s,\xi)E(\xi)W(s,\xi)ds, \tag{2.25}$$

where $F = F(t,\xi)$ solves

$$\partial_t F + M\,F = 0, \quad F(0,\xi) = I.$$

If on the other hand the assumption ($\mathrm{A_n}$) is always satisfied, but ($\mathrm{B_n}$) is not for any n, i.e., $A(\xi)$ is permanently degenerate, then we can derive a system as in (2.24) as well, where $M = M(\xi)$ is diagonal and the remainder $E(\xi) = \mathcal{O}(\xi^p)$ is sufficiently small. Hence, we obtain besides the asymptotic behavior of the eigenvalues of $A(\xi)$ as in Lemma 2.6 solution representations as in (2.25) with an even diagonal $F = F(t,\xi)$.

2.2.3. Diagonalization for large frequencies

For large frequencies $\xi \in Z_{ext}(N) = \{|\xi| \geq N \gg 1\}$ we can certainly rewrite the initial value problem for (2.4) in the form

$$\begin{cases} \hat{\partial}_t W + (A_2 + i\eta\,A_1 + \eta^2 A_0)W = 0, \\ W(0,\eta) = W_0(\eta), \end{cases} \tag{2.26}$$

where $\eta = 1/\xi$, $\hat{\partial}_t := \eta^2\partial_t$, $W = W(t,\eta) = \eta^{-2}V(t,1/\eta)$ and $W_0(\eta) := \eta^{-2}\mathcal{F}(U_0)(1/\eta)$. Hence, we can repeat arguments from the considerations for small frequencies, and we will use the same notations as before, apart from an additional '$\hat{\ }$' for the appearing matrices.
Let us assume that the matrices $\hat{A}_2^{(0)} := A_2$ and $\hat{A}_{2-k}^{(k)}$ for $k = 1,\ldots,n$ are diagonalizable. Then the vector

$$\tilde{W}^{(n)} = \hat{\tilde{L}}^{(n)}(I + i\eta\hat{K}_{(n)})\cdot\ldots\cdot(I + a_n\eta^n\hat{K}_{\left((n-1)\frac{1}{n}\right)})\cdot\ldots\cdot \hat{\tilde{L}}^{(2)}(I + i\eta\hat{K}_{(2)})(I + \eta^2\hat{K}_{\left(1\frac{1}{2}\right)})\hat{\tilde{L}}^{(1)}(I + i\eta\hat{K}_{(1)})\hat{\tilde{L}}^{(0)}W$$

satisfies, assuming that all matrices are chosen properly,

$$\hat{\partial}_t\tilde{W}^{(n)} + \left(\hat{\Lambda}_2 + i\eta\hat{\Lambda}_1 + \eta^2\hat{\Lambda}_0 + i\eta^3\hat{\Lambda}_{-1} + \ldots + a_n\eta^n\hat{\Lambda}_{2-n} + \hat{\tilde{B}}_{1-n}^{(n)}\right)\tilde{W}^{(n)} = 0$$

with $\hat{\tilde{B}}_{1-n}^{(n)} = \mathcal{O}(\eta^{n+1})$ for $\eta \to 0$.
Consequently, we have that $\tilde{V}^{(n)}(t,\xi) = \xi^{-2}\tilde{W}^{(n)}(t,1/\xi)$ satisfies

$$\tilde{V}_t^{(n)} + \left(\xi^2\hat{\Lambda}_2 + i\xi\hat{\Lambda}_1 + \hat{\Lambda}_0 + i\xi^{-1}\hat{\Lambda}_{-1} + \ldots + a_{2-n}\xi^{2-n}\hat{\Lambda}_{2-n} + \hat{\tilde{A}}_{1-n}^{(n)}\right)\tilde{V}^{(n)} = 0$$

with $\hat{\tilde{A}}_{1-n}^{(n)} = \mathcal{O}(\xi^{-n+1})$ for $|\xi| \to \infty$.
In the following we will summarize the facts that we immediately obtain from our considerations for small frequencies.
We introduce the assumptions:

($\hat{\mathbf{A}}_{\mathbf{n}}$) The matrices $\hat{A}_{2-k}^{(k)}$ are diagonalizable for $k = 0,1,\ldots,n$,

($\hat{\mathbf{A}}'_{\mathbf{n}}$) the matrices $\hat{A}_{2-k}^{(k)}$ are symmetrizable for $k = 0,1,\ldots,n$

and

($\hat{\mathrm{B}}_\mathbf{n}$) $\forall i,j \in \{1,\dots,d\}$ with $i \neq j$ $\exists k \in \{0,1,\dots,n\} : \hat{\lambda}_{2-k,i} \neq \hat{\lambda}_{2-k,j}$.

In the latter we have used the notation $\hat{\Lambda}_{2-k} = \mathrm{diag}(\hat{\lambda}_{2-k,1},\dots,\hat{\lambda}_{2-k,d})$ and that the assumption ($\hat{\mathrm{A}}_\mathrm{n}$) holds.

Proposition 2.12. *Suppose that the assumption ($\hat{A}_n$) holds. Then we have:*

(i) The system (2.4) is diagonalizable for large frequencies up to the order $-n+2$.

(ii) The characteristic roots $\mu_j = \mu_j(\xi)$, $j = 1,\dots,d$, of the coefficient matrix $A(\xi) = A_0 + i\xi A_1 + \xi^2 A_2$ of (2.4) behave for $|\xi| \geq N \gg 1$ as

$$\mu_j(\xi) = \hat{\lambda}_{2,j}\,\xi^2 + i\hat{\lambda}_{1,j}\,\xi + \hat{\lambda}_{0,j} + \ldots + a_{2-n}\hat{\lambda}_{2-n,j}\,\xi^{2-n} + \mathcal{O}(\xi^{1-n}),$$

where we use $a_k = i$ for odd k and $a_k = 1$ for even k. The numbers $\hat{\lambda}_{k,j}$ are real if even ($\hat{A}'_n$) holds, otherwise they may be complex.

Suppose now that for some n the assumptions ($\hat{\mathrm{A}}_\mathrm{n}$) and ($\hat{\mathrm{B}}_\mathrm{n}$) hold. We define the number $\hat{c}$ by

$$\hat{c} = \max_{i<j} \hat{c}_{ij} \qquad \text{and} \qquad \hat{c}_{ij} = \min\{k \in \{0,1,\dots,n\} : \hat{\lambda}_{2-k,i} \neq \hat{\lambda}_{2-k,j}\} \tag{2.27}$$

and note that this is the minimal number n for which ($\hat{\mathrm{B}}_\mathrm{n}$) holds.
We can now state:

Proposition 2.13. *We assume that ($\hat{A}_n$) and ($\hat{B}_n$) hold for some n. Then we have:*

(i) The system (2.4) is fully diagonalizable for large frequencies.

(ii) The solution of (2.4) has in $Z_{ext}(N) = \{|\xi| \geq N \gg 1\}$ the representation

$$V(t,\xi) = T_{ext}^{-1}(\xi)\,\mathrm{diag}(e^{-\mu_1(\xi)t},\dots,e^{-\mu_d(\xi)t})T_{ext}(\xi)\hat{U}_0(\xi), \tag{2.28}$$

where

$$T_{ext}(\xi) = \hat{\hat{L}}(\xi)\,\hat{\hat{L}}^{(n)}(I+i\xi^{-1}\hat{K}_{(n)})\cdot\ldots\cdot(I+a_n\xi^{-n}\hat{K}_{((n-1)\frac{1}{n})})\cdot\ldots\cdot\hat{\hat{L}}^{(1)}(I+i\xi^{-1}\hat{K}_{(1)})\hat{\hat{L}}^{(0)}.$$

The matrices $\hat{K}_{(j)}$ and $\hat{\hat{L}}^{(j)}$ are matrices used in the diagonalization procedure, $\hat{\hat{L}}(\xi) = I + R(\xi)$ with $R(\xi) = \mathcal{O}(\xi^{-(n+1)+\hat{c}})$ for $|\xi| \to \infty$, and the $\mu_j(\xi)$ are from Proposition 2.12.

Remark 2.9. From Remark 2.8 we know that the entries of $T_{ext}(\xi)$ and its inverse are smooth and bounded in $Z_{ext}(N)$.

Remark 2.10. If the system (2.2) is a parabolic one, then we have $\mathrm{Re}\,\hat{\lambda}_{2,j} > 0$ for all j in Proposition 2.12. Considering energy estimates we can therefore expect an exponential decay in $Z_{ext}(N)$ for solutions of (2.2). If the system (2.2) is hyperbolic, then we have $\hat{\lambda}_{2,j} = 0$ and $\hat{\lambda}_{1,j} \in \mathbb{R}$ for all j, and the decay behavior depends on the lower order terms of the system.

Analogous results to the ones in Sections 2.2.2.3 and 2.2.2.4 can be obtained.

2.2.4. Diagonalization for bounded frequencies away from zero

We are studying the Cauchy problem for (2.4) in $Z_{mid}(\sigma, N) = \{\sigma \le |\xi| \le N\}$.
For most practical problems that we have in mind it is sufficient to work with an assumption:

$$\textbf{(C)} \quad \exists C = C(\sigma, N) > 0 \, \forall \xi \in Z_{mid}(\sigma, N) \, \forall \mu(\xi) \in \operatorname{spec}(A(\xi)) : \operatorname{Re} \mu(\xi) \ge C > 0.$$

Remark 2.11. One may check (C) by assuming that there is a purely imaginary eigenvalue in $Z_{mid}(\sigma, N)$ and constructing a contradiction using the characteristic polynomial of $A(\xi)$. If we have for all eigenvalues $\mu = \mu(\xi)$ that $\operatorname{Re} \mu(\xi) > 0$ for some $\xi \in Z_{mid}(\sigma, N)$, then continuity and the compactness of $Z_{mid}(\sigma, N)$ give (C).

With assumption (C) and the compactness of $Z_{mid}(\sigma, N)$ we can prove:

Proposition 2.14. *The solution V to the Cauchy problem of (2.4) satisfies in $Z_{mid}(\sigma, N)$*

$$|V(t, \xi)| \lesssim e^{-ct} |\hat{U}_0(\xi)|,$$

where c is a positive constant, provided (C) holds.

Proof. The proof is a simple generalization of the one given in (and we would therefore like the reader to refer to) [RW05], Proposition 3.3. □

Remark 2.12. Considering energy estimates we can expect an exponential decay from $Z_{mid}(\sigma, N)$ for solutions to (2.2), provided (C) holds.

Remark 2.13. The assumption (C) is of course not satisfied for all practical useful cases. We should assume at least $\operatorname{Re} \mu(\xi) \ge 0$ for all eigenvalues $\mu = \mu(\xi)$ and $\xi \in Z_{mid}(\sigma, N)$ to guarantee the stability of our solution, but it may happen that there is a region for which one of the eigenvalues is purely imaginary or that there are single points $\xi_0 \in Z_{mid}(\sigma, N)$ for which an eigenvalue $\mu(\xi)$ becomes imaginary. However, we have only one problem in which (C) is not satisfied (see Chapter 4) and will study this one separately.

Let us summarize the previous considerations. We have diagonalized the principal part of system (2.4) in the three regions $Z_{int}(\sigma) = \{|\xi| \le \sigma \ll 1\}$, $Z_{mid}(\sigma, N) = \{\sigma \le |\xi| \le N\}$ and $Z_{ext}(N) = \{|\xi| \ge N \gg 1\}$.
In $Z_{int}(\sigma)$ we used the diagonalization procedure described in Section 2.2.2.1 to derive, under the assumptions that $(\mathrm{A_n})$ and $(\mathrm{B_n})$ hold for some n, a solution representation given in Proposition 2.9 for which we have a clear understanding of the behavior in time from our results on the asymptotic behavior of the eigenvalues of the coefficient matrix of system (2.4) in Lemma 2.6. The same diagonalization procedure was then applied to the region $Z_{ext}(N)$ of large frequencies and, under the assumptions that $(\hat{\mathrm{A}}_\mathrm{n})$ and $(\hat{\mathrm{B}}_\mathrm{n})$ hold for some n, a solution representation was derived in Proposition 2.13 with a clear understanding of the in time behavior from the results of Proposition 2.12. Finally, for

bounded frequencies we have derived an estimate guaranteeing exponential stability for solutions of (2.4) in $Z_{mid}(\sigma, N)$, provided (C) holds.
We will now go on and use our observations to derive well-posedness results in Section 2.3, results on L^p-L^q decay estimates in Section 2.4, some concerning diffusion phenomena in Section 2.5 and some on the propagation of singularities in Section 2.6.

2.3. Well-posedness

Before we start, let us formulate the condition

$$\forall \xi \in Z_{ext}(N)\, \forall \mu(\xi) \in \operatorname{spec}(A(\xi)) : \operatorname{Re}\mu(\xi) \geq 0 \tag{2.29}$$

for the eigenvalues $\mu = \mu(\xi)$ of the coefficient matrix $A = A(\xi)$ of system (2.4).
We can now state:

Theorem 2.15. *(H^s well-posedness of the Cauchy problem)*
We consider the Cauchy problem

$$\begin{cases} U_t + A_0 U + A_1 U_x - A_2 U_{xx} = 0, \\ U(0,x) = U_0(x) \end{cases}$$

in $\mathbb{R}_{\geq 0} \times \mathbb{R}$ with A_i being complex and constant $d \times d$ matrices.
Suppose that for some n the assumptions $(\hat{A}_n)$ and $(\hat{B}_n)$ hold together with (2.29) and that we have $U_0 \in H^s$ for a fixed $s \in \mathbb{R}$.
Then there exists a (in $C([0,\infty), \mathcal{S}')$) unique solution satisfying

$$U \in C([0,\infty), H^s).$$

Further, we get for an arbitrary $T > 0$ the a priori estimate

$$\|U\|_{C([0,T],H^s)} \leq C_T \|U_0\|_{H^s}$$

with a T-dependent constant C_T.

Proof. With the use of a C^∞-function ϕ_{ext} having its support in $Z_{ext}(N)$ with $\phi_{ext}(\xi) = 1$ for $|\xi| \geq 2N$, $0 \leq \phi_{ext}(\xi) \leq 1$, we can write down the solution to the above Cauchy problem in the form

$$U(t,x) = \mathcal{F}^{-1}(V(t,\xi)) = \mathcal{F}^{-1}\left((1-\phi_{ext}(\xi))V(t,\xi)\right) + \mathcal{F}^{-1}(\phi_{ext}(\xi)V(t,\xi)), \tag{2.30}$$

where $V = V(t,\xi)$ is the solution of the Cauchy problem to system (2.4).
We start by studying the first summand. Using the representation

$$V(t,\xi) = V_0(\xi) - \int_0^t A(\xi)V(s,\xi)\,ds,$$

Gronwall's lemma and the compact support of $(1-\phi_{ext})$ we can derive the estimate

$$|(1-\phi_{ext}(\xi))V(t,\xi)| \lesssim e^{C_N t}|1-\phi_{ext}(\xi)||V_0(\xi)|$$

with a constant C_N depending on N. This assures H^s-regularity of the first summand in (2.30) for all times $t \in [0,\infty)$ and moreover continuity of the first summand measured in H^s, i.e., $\mathcal{F}^{-1}\left((1-\phi_{ext}(\xi))V(t,\xi)\right) \in C([0,\infty),H^s)$.
For the second summand in (2.30) we can use the solution representation given by (2.28) in Proposition 2.13. Using this together with (2.29) we instantly obtain $\mathcal{F}^{-1}(\phi_{ext}(\xi)V(t,\xi)) \in C([0,\infty),H^s)$.
This yields $U \in C([0,\infty),H^s)$. The uniqueness of the solution is clear. □

2.4. L^p-L^q decay estimates

In this section we will state and prove decay estimates for solutions to the Cauchy problem for (2.2). More accurate, we will, based on the results given in the Propositions 2.9, 2.13 and 2.14, prove decay estimates for such solutions subjected to operators with smooth symbols supported in the regions of small, bounded and large frequencies.
We therefore introduce functions $\phi_{int}, \phi_{mid}, \phi_{ext} \in C^\infty(\mathbb{R})$ having their support in $Z_{int}(\sigma)$, $Z_{mid}(\sigma/2, 2N)$ and $Z_{ext}(N)$, respectively, so that $\phi_{int}+\phi_{mid}+\phi_{ext} \equiv 1$.
Of significant importance for the derivation of decay estimates for the respective solutions is the behavior of the eigenvalues of the coefficient matrix from (2.4). For each eigenvalue $\mu = \mu(\xi)$ we have to distinguish basically two cases: Is μ purely imaginary? Or does it have a non-vanishing real part (We restrict to that case in $Z_{mid}(\sigma, N)$ completely.)? However, if μ has a non-vanishing real part, then it might still be the imaginary part that is determining the decay.
If the real part of the eigenvalue is the decay-determining one, then calculations are more or less straight forward. If it is the imaginary part, then we will make use of the theory of oscillatory integrals (cf. Section B.3), i.e., primarily of the Lemma B.3 of van der Corput.

2.4.1. L^p-L^q decay estimates for small frequencies

We assume $U_0 \in \mathcal{S}$ and that there exists a number n so that the assumptions $(\mathrm{A_n})$ and $(\mathrm{B_n})$ hold. To guarantee stability of the solution it is moreover necessary to assume

$$\forall \xi \in Z_{int}(\sigma)\, \forall \mu(\xi) \in \operatorname{spec}(A(\xi)) : \operatorname{Re}\mu(\xi) \geq 0. \tag{2.31}$$

Now we fix for each eigenvalue $\mu_j = \mu_j(\xi)$ with the help of its asymptotic expansion from Lemma 2.6 a decay-determining number $n_{s,j}$ by $n_{s,j} = \min(p_j, q_j)$, where p_j and q_j shall be defined by the following:

(i) We set $p_j = \infty$ if μ_j is purely imaginary in $Z_{int}(\sigma)$.
If the latter is not the case, then we choose p_j to be the minimal one of all numbers $m \in 2\mathbb{N}_0$ with $\operatorname{Re}\lambda_{m,j} > 0$.

(ii) We set $q_j = \infty$, unless there exists a natural number $m \geq 2$ for which $\operatorname{Im}(a_m\lambda_{m,j}) \neq 0$. In that case we choose q_j to be the minimal one of all such numbers m.

Remark 2.14. If all matrices A_i from (2.2) are from $\mathbb{R}^{d\times d}$, and if we have (A'_n) instead of (A_n), then (all numbers $\lambda_{k,j}$ in the asymptotic expansion of μ_j are real and) we can replace the definitions of p_j and q_j by:

(i) Set $p_j = \infty$ if μ_j is purely imaginary in $Z_{int}(\sigma)$.
Otherwise choose p_j to be the minimal one of all numbers $m \in 2\mathbb{N}_0$ with $\lambda_{m,j} > 0$.

(ii) Set $q_j = \infty$, unless there exists a number $m \in 2\mathbb{N}+1$ (i.e., $m \geq 3$) for which $\lambda_{m,j} \neq 0$. In that case we choose q_j to be the minimal one of all such numbers m.

Finally, we set $n_s := \max_{j=1,\dots,d} n_{s,j}$.
We can now state:

Theorem 2.16. *We assume $U_0 \in \mathcal{S}$, the existence of a number n so that the assumptions (A_n) and (B_n) hold and (2.31). Let n_s be defined as above. Then the following L^p-L^q decay estimates hold for solutions U to the initial value problem for (2.2):*

$$\|\phi_{int}(D)U\|_{L^q} \lesssim \begin{cases} e^{-ct}\|U_0\|_{L^p}, & n_s = 0, \\ (1+t)^{-\frac{1}{n_s}\left(\frac{1}{p}-\frac{1}{q}\right)}\|U_0\|_{L^p}, & 2 \leq n_s < \infty, \\ \|U_0\|_{L^p}, & n_s = \infty. \end{cases}$$

Here $1 \leq p \leq 2$, $\frac{1}{p}+\frac{1}{q} = 1$, and c is a positive constant.

Remark 2.15. 1. An exponential decay is obtained if we have for all eigenvalues μ_j that $\operatorname{Re}\lambda_{0,j} > 0$.
2. We obtain no decay if there is at least one purely imaginary eigenvalue in $Z_{int}(\sigma)$ taking the form

$$\mu_j(\xi) = i\operatorname{Im}\lambda_{0,j} + i\lambda_{1,j}\xi \quad \text{with a } \lambda_{1,j} \in \mathbb{R}.$$

In that case we can not expect a decay, which is easily understood by noting that the Cauchy problem for the wave equation (transformed to a first-order system for the usual energy) fits into this case.

Proof. We have from Proposition 2.9 the solution representation

$$\begin{aligned} \phi_{int}(D)U &= \mathcal{F}^{-1}\left(T_{int}^{-1}(\xi)\operatorname{diag}(e^{-\mu_1(\xi)t},\dots,e^{-\mu_d(\xi)t})T_{int}(\xi)\phi_{int}(\xi)\hat{U}_0(\xi)\right) \\ &= \left(\sum_{j,r=1}^{d}\mathcal{F}^{-1}\left(c_{jrk}(\xi)e^{-\mu_j(\xi)t}\phi_{int}(\xi)\hat{U}_{0,r}(\xi)\right)\right)_{k=1}^{d} \end{aligned}$$

with $\hat{U}_0(\xi) = (\hat{U}_{0,1}(\xi), \dots, \hat{U}_{0,d}(\xi))^T$ and $c_{jrk} = c_{jrk}(\xi)$ being smooth functions in $Z_{int}(\sigma)$. We will now follow Brenner, [Bre75], to derive the desired L^p-L^q estimates by interpolating L^2-L^2 with L^1-L^∞ estimates via the Riesz-Thorin interpolation theorem (see e.g. [BL76]). We start with the L^2-L^2 estimates. They are derived with the use of Parseval's equation and take the form

$$\|\phi_{int}(D)U\|_{L^2} \lesssim \begin{cases} e^{-ct}\|U_0\|_{L^2}, & n_s = 0, \\ \|U_0\|_{L^2}, & 2 \le n_s \le \infty \end{cases} \tag{2.32}$$

with an (in the following universal) positive constant c.
Now we are concerned with L^1-L^∞ estimates and start off with

$$\|\phi_{int}(D)U\|_{L^\infty} \lesssim \sum_{j=1}^{d} \underbrace{\sum_{k,r=1}^{d} \sup_{x\in\mathbb{R}} \left| \int_{Z_{int}(\sigma)} e^{ix\xi - \mu_j(\xi)t} c_{jrk}(\xi)\phi_{int}(\xi)d\xi \right|}_{=:S_j} \|U_0\|_{L^1}.$$

We want to study each summand S_j separately and therefore fix an arbitrary $j \in \{1, \dots, d\}$. Let us first assume that the number $n_{s,j}$ of the corresponding eigenvalue μ_j is finite and given by p_j. In that case given we obtain

$$\begin{aligned} S_j &\lesssim \int_{Z_{int}(\sigma)} \left|e^{-\mu_j(\xi)t}\right| d\xi \\ &\lesssim \begin{cases} e^{-ct}, & p_j = 0, \\ \displaystyle\int_{Z_{int}(\sigma)} e^{-c\xi^{p_j}\cdot t} d\xi \lesssim (1+t)^{-\frac{1}{p_j}}, & p_j > 0. \end{cases} \end{aligned}$$

Now we consider the case that $n_{s,j}$ is finite and given by $q_j < p_j$. First, we conclude by substituting $y = x/t$

$$S_j = \sum_{k,r=1}^{d} \sup_{y\in\mathbb{R}} \left| \int_{Z_{int}(\sigma)} e^{i\,t\,\phi_{j,y}(\xi)}\, \psi_{jrk}(\xi) d\xi \right|$$

with smooth and real-valued functions

$$\phi_{j,y}(\xi) = -\mathrm{Im}\,\lambda_{0,j} + (y - \mathrm{Re}\,\lambda_{1,j})\xi - \mathrm{Im}\,(a_{q_j}\lambda_{q_j,j})\xi^{q_j} + \mathcal{O}(\xi^{q_j+1})$$

and smooth, compactly supported and complex-valued functions

$$\psi_{jrk}(\xi) = c_{jrk}(\xi)\phi_{int}(\xi)$$

in the case of a purely imaginary eigenvalue $\mu_j = \mu_j(\xi)$ and

$$\psi_{jrk}(\xi) = c_{jrk}(\xi)\phi_{int}(\xi)e^{-\mathrm{Re}\,(a_{p_j}\lambda_{p_j,j})\xi^{p_j}t + \mathcal{O}(\xi^{p_j+1})t}$$

otherwise. Here we have used (2.31) and $q_j < p_j$ to determine the looks of $\phi_{j,y}$ and ψ_{jrk}. Now we apply Corollary B.4 and obtain for times $t \ge 1$

$$S_j \lesssim t^{-\frac{1}{q_j}} \sum_{k,r=1}^{d} \int_{Z_{int}(\sigma)} \left| \frac{d}{d\xi}\, \psi_{jrk}(\xi) \right| d\xi.$$

Taking into account that for all numbers $n \in 2\mathbb{N}$ $(n \geq 2)$ we have

$$\int_0^\sigma \xi^{n-1} t e^{-c\xi^n t} d\xi \lesssim \int_0^\infty y^{n-1} e^{-cy^n} dy \lesssim 1,$$

where we have used the substitution $y^n = \xi^n t$, we finally conclude for arbitrary $t > 0$

$$S_j \lesssim (1+t)^{-\frac{1}{q_j}}.$$

In the case $n_{s,j} = \infty$ we can not get more than

$$S_j \lesssim 1.$$

Summarizing the previous considerations yields

$$\|\phi_{int}(D)U\|_{L^\infty} \lesssim \begin{cases} e^{-ct}\|U_0\|_{L^1}, & n_s = 0, \\ (1+t)^{-\frac{1}{n_s}}\|U_0\|_{L^1}, & 2 \leq n_s < \infty, \\ \|U_0\|_{L^1}, & n_s = \infty. \end{cases}$$

Combining (2.32) with the above L^1-L^∞ estimates and applying the Riesz-Thorin interpolation theorem proves our assertions. □

2.4.2. L^p-L^q decay estimates for large frequencies

As in the case of small frequencies we assume $U_0 \in \mathcal{S}$, the existence of a number n so that $(\hat{\mathrm{A}}_\mathrm{n})$ and $(\hat{\mathrm{B}}_\mathrm{n})$ hold and that

$$\forall \xi \in Z_{ext}(N)\, \forall \mu(\xi) \in \operatorname{spec}(A(\xi)) : \operatorname{Re}\mu(\xi) \geq 0.$$

Then there are three possible cases for the set of all eigenvalues:

(i) All eigenvalues $\mu = \mu(\xi)$ are always purely imaginary in $Z_{ext}(N)$.

(ii) All eigenvalues $\mu = \mu(\xi)$ have a positive real part in the whole zone $Z_{ext}(N)$.

(iii) There are some purely imaginary eigenvalues in $Z_{ext}(N)$ and some with a positive real part for all $\xi \in Z_{ext}(N)$.

We will treat (in order to observe some special phenomena for the case *(ii)* and for the sake of simplicity) each case separately, starting off with *(i)*.

2.4.2.1. The case of only purely imaginary eigenvalues

We want to make use of the asymptotic expansions of the purely imaginary eigenvalues $\mu_j = \mu_j(\xi)$ from Proposition 2.12 and first note that all coefficients $a_k\hat{\lambda}_{k,j}$ are imaginary. Now we define a decay- and regularity-determining number n_l by:

Set $n_l = \infty$, unless there exists a number $m \in \{-2\} \cup \mathbb{N} = \{-2, 1, 2, \ldots\}$ with

$$\forall j \in \{1, \ldots, d\} \exists k \in \{-2\} \cup \mathbb{N} \text{ with } k \leq m : \hat{\lambda}_{-k,j} \neq 0. \tag{2.33}$$

Then we denote by n_l the minimal one of all above numbers.

To be precise with respect to regularity we will make use of the Bessel potential spaces $L^{p,r} = \langle D \rangle^{-r} L^p$ (see Section B.2). We can now state:

Theorem 2.17. *Assume $U_0 \in \mathcal{S}$, the existence of a number n so that $(\hat{A}_n)$ and $(\hat{B}_n)$ hold and that all eigenvalues $\mu = \mu(\xi)$ are purely imaginary in $Z_{ext}(N)$. Define the number n_l as above. Then the following L^p-L^q decay estimates hold for solutions U to the initial value problem for (2.2):*
For dual indices $q, 1 < p \leq 2$ we obtain the estimates

$$\|\phi_{ext}(D)U\|_{L^q} \lesssim \begin{cases} (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|U_0\|_{L^{p,r_p}}, & n_l = -2, \\ (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|U_0\|_{L^{p,r_p^{n_l}}}, & n_l \in \mathbb{N}, \\ \|U_0\|_{L^{p,r_p}}, & n_l = \infty \end{cases}$$

with $r_p = \frac{1}{p} - \frac{1}{q}$ and $r_p^{n_l} = \left(1 + \frac{n_l}{2}\right)\left(\frac{1}{p} - \frac{1}{q}\right)$.

Remark 2.16. 1. We obtain no decay if there is an eigenvalue of the form

$$\mu_j(\xi) = i\hat{\lambda}_{1,j}\xi + i\mathrm{Im}\,\hat{\lambda}_{0,j} \quad \text{with a } \hat{\lambda}_{1,j} \in \mathbb{R} \text{ in } Z_{ext}(N).$$

Noting once more that the Cauchy problem for the wave equation (transformed to a first-order system for the usual energy) fits into this case, we should not expect a decay.
2. If all matrices A_i from (2.2) are from $\mathbb{R}^{d\times d}$, and if we have $(\hat{\mathrm{A}}_\mathrm{n}^{\prime})$ instead of $(\hat{\mathrm{A}}_\mathrm{n})$, then only the cases $n_l = \infty$ and $n_l \in 2\mathbb{N}_0 + 1$ can occur.

Proof. From Proposition 2.13 we have the solution representation

$$\begin{aligned} \phi_{ext}(D)U &= \mathcal{F}^{-1}\left(T_{ext}^{-1}(\xi)\,\mathrm{diag}(e^{-\mu_1(\xi)t}, \ldots, e^{-\mu_d(\xi)t})T_{ext}(\xi)\phi_{ext}(\xi)\hat{U}_0(\xi)\right) \\ &= \left(\sum_{j,r=1}^{d} \mathcal{F}^{-1}\left(c_{jrk}(\xi)e^{-\mu_j(\xi)t}\phi_{ext}(\xi)\hat{U}_{0,r}(\xi)\right)\right)_{k=1}^{d}, \end{aligned}$$

where $c_{jrk} = c_{jrk}(\xi)$ are smooth functions with

$$|d_\xi^m c_{jrk}(\xi)| \lesssim |\xi|^{-m} \tag{2.34}$$

uniformly for all j, r, k and $m = 0, 1$ in $Z_{ext}(N)$.
We will work with a dyadic decomposition of the phase space and therefore choose a function $\varphi \in C_0^\infty(\mathbb{R})$ such that $\mathrm{supp}\,\varphi = \{2^{-1} \leq |\xi| \leq 2\}$, $\varphi(\xi) > 0$ for $2^{-1} < |\xi| < 2$ and

$$\sum_{m=-\infty}^{\infty} \varphi(2^{-m}\xi) = 1 \qquad \text{for } \xi \neq 0.$$

Such a function exists, a proof may be found in [BL76], Lemma 6.1.7. We use the notations

$$\begin{aligned} \varphi_m(t,\xi) &= \varphi(2^{-m}(1+t)|\xi|) \qquad \text{for } m \in \mathbb{N} \text{ and} \\ \varphi_0(t,\xi) &= 1 - \sum_{m=1}^{\infty} \varphi_m(t,\xi). \end{aligned}$$

We fix k, j and r and will now derive L^p-L^q decay estimates for the multipliers

$$\mathcal{F}^{-1}\left(e^{-\mu_j(\xi)t} c_{jrk}(\xi)\phi_{ext}(\xi)\varphi_m(t,\xi)\hat{\omega}(\xi)\right) \tag{2.35}$$

with $\omega \in \mathcal{S}$. First, we note that $\operatorname{supp}\phi_{ext}(\cdot)\varphi_0(t,\cdot) = \emptyset$. Hence, we have to study the above multipliers for $m \geq 1$ only. They involve the eigenvalue $\mu_j = \mu_j(\xi)$, and we therefore fix a number q_j by setting $q_j = \infty$, unless there exists a $k \in \{-2\} \cup \mathbb{N}$ with $\hat{\lambda}_{-k,j} \neq 0$, in which case we choose q_j to be the minimal one of these numbers. Note that $n_l := \max_{j=1,\dots,d} q_j$.
The L^2-L^2 estimate is given by

$$\left\|\mathcal{F}^{-1}\left(e^{-\mu_j(\xi)t} c_{jrk}(\xi)\phi_{ext}(\xi)\varphi_m(t,\xi)\hat{\omega}(\xi)\right)\right\|_{L^2} \lesssim \|\omega\|_{L^2}, \tag{2.36}$$

and we can thus turn to studying L^1-L^∞ estimates.
We will consider

$$S_m = \left\|\mathcal{F}^{-1}\left(e^{-\mu_j(\xi)t} c_{jrk}(\xi)\phi_{ext}(\xi)\varphi_m(t,\xi)\right)\right\|_{L^\infty}. \tag{2.37}$$

With $y = x/t$ we have

$$S_m \lesssim \sup_{y\in\mathbb{R}} \left|\int_{\mathbb{R}} e^{it\phi_y(\xi)}\psi_m(\xi)d\xi\right|$$

with real-valued and analytic functions

$$\phi_y(\xi) = \begin{cases} -\operatorname{Im}\hat{\lambda}_{2,j}\,\xi^2 + (y - \operatorname{Re}\hat{\lambda}_{1,j})\xi - \operatorname{Im}\hat{\lambda}_{0,j} + \mathcal{O}(\xi^{-1}), & q_j = -2, \\ (y - \operatorname{Re}\hat{\lambda}_{1,j})\xi - \operatorname{Im}\hat{\lambda}_{0,j} - \operatorname{Im}(a_{q_j}\hat{\lambda}_{-q_j,j})\xi^{-q_j} + \mathcal{O}(\xi^{-(q_j+1)}), & q_j \in \mathbb{N}, \\ (y - \operatorname{Re}\hat{\lambda}_{1,j})\xi - \operatorname{Im}\hat{\lambda}_{0,j}, & q_j = \infty \end{cases}$$

for $|\xi| \to \infty$ and smooth and complex-valued functions

$$\psi_m(\xi) = c_{jrk}(\xi)\phi_{ext}(\xi)\varphi_m(t,\xi).$$

We concentrate first on the case $q_j = -2$.
With the transformation $(1+t)\xi = 2^m\eta$ and

$$\tilde{\phi}_{y,m,t}(\eta) = -\operatorname{Im}\hat{\lambda}_{2,j}\,\eta^2 + \left(y - \operatorname{Re}\hat{\lambda}_{1,j}\right)\left(\frac{2^m}{1+t}\right)^{-1}\eta + \left(\frac{2^m}{1+t}\right)^{-2}\mathcal{O}\left(\left(\frac{2^m}{1+t}\eta\right)^{-1}\right),$$

as well as

$$\tilde{\psi}_{m,t}(\eta) = c_{jrk}\left(\frac{2^m}{1+t}\eta\right)\phi_{ext}\left(\frac{2^m}{1+t}\eta\right)\varphi(|\eta|), \tag{2.38}$$

we obtain

$$S_m \lesssim \frac{2^m}{1+t} \sup_{y\in\mathbb{R}} \left| \int_{|\eta|\in\left[\frac{1}{2},2\right]} e^{it\left(\frac{2^m}{1+t}\right)^2 \tilde{\phi}_{y,m,t}(\eta)} \tilde{\psi}_{m,t}(\eta)\, d\eta \right| \lesssim 2^m(1+t)^{-\frac{3}{2}}. \tag{2.39}$$

In the latter we have applied for large times $t \geq 1$ Corollary B.4 together with

$$|d_\eta^2 \tilde{\phi}_{y,m,t}(\eta)| = \left| 2\mathrm{Im}\, \hat{\lambda}_{2,j} + \mathcal{O}\left(\left(\frac{2^m}{1+t}\eta\right)^{-3}\right)\right| \geq c > 0.$$

The above estimate holds independently of y, m, t, and we have used that $\left|\frac{2^m}{1+t}\eta\right| \geq N \gg 1$ on the support of $\tilde{\psi}_{m,t}$. With (2.34) and noting that $|d_\xi \phi_{ext}(\xi)| \lesssim |\xi|^{-1}$ we furthermore have the boundedness of $|d_\eta \tilde{\psi}_{m,t}(\eta)|$ for $|\eta| \in \left[\frac{1}{2}, 2\right]$ independently of m and t.
We proceed similarly for the case $q_j \in \mathbb{N}$. With

$$\begin{aligned}\tilde{\phi}_{y,m,t}(\eta) \quad &= \quad \left(y - \mathrm{Re}\,\hat{\lambda}_{1,j}\right)\left(\frac{2^m}{1+t}\right)^{q_j+1}\eta - \mathrm{Im}\,(a_{q_j}\hat{\lambda}_{-q_j,j})\eta^{-q_j} \\ &\quad + \left(\frac{2^m}{1+t}\right)^{q_j} \mathcal{O}\left(\left(\frac{2^m}{1+t}\eta\right)^{-(q_j+1)}\right)\end{aligned}$$

and

$$|d_\eta^2 \tilde{\phi}_{y,m,t}(\eta)| \quad = \quad |\eta|^{-(q_j+2)} \left| q_j(q_j+1)\mathrm{Im}\,(a_{q_j}\hat{\lambda}_{-q_j,j}) + \mathcal{O}\left(\left(\frac{2^m}{1+t}\eta\right)^{-1}\right)\right| \geq c > 0$$

we obtain

$$S_m \lesssim \frac{2^m}{1+t} \sup_{y\in\mathbb{R}} \left| \int_{|\eta|\in\left[\frac{1}{2},2\right]} e^{it\left(\frac{t+1}{2^m}\right)^{q_j} \tilde{\phi}_{y,m,t}(\eta)} \tilde{\psi}_{m,t}(\eta)\, d\eta \right| \lesssim 2^{m\left(1+\frac{q_j}{2}\right)}(1+t)^{-\left(\frac{3}{2}+\frac{q_j}{2}\right)}. \tag{2.40}$$

For $q_j = \infty$ we have

$$S_m \lesssim 2^m(1+t)^{-1}.$$

The above considerations yield together with the L^2-L^2 estimates (2.36) and the Riesz-Thorin interpolation theorem

$$\begin{aligned}&\left\|\mathcal{F}^{-1}\left(e^{-\mu_j(\xi)t}c_{jrk}(\xi)\phi_{ext}(\xi)\varphi_m(t,\xi)\hat{\omega}(\xi)\right)\right\|_{L^q} \\ &\qquad \lesssim \begin{cases} 2^{m\left(\frac{1}{p}-\frac{1}{q}\right)}(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\omega\|_{L^p}, & q_j = -2, \\ 2^{m\left(1+\frac{q_j}{2}\right)\left(\frac{1}{p}-\frac{1}{q}\right)}(1+t)^{-\left(\frac{3}{2}+\frac{q_j}{2}\right)\left(\frac{1}{p}-\frac{1}{q}\right)}\|\omega\|_{L^p}, & q_j \in \mathbb{N}, \\ 2^{m\left(\frac{1}{p}-\frac{1}{q}\right)}(1+t)^{-\left(\frac{1}{p}-\frac{1}{q}\right)}\|\omega\|_{L^p}, & q_j = \infty \end{cases}\end{aligned}$$

for dual indices q, $1 \leq p \leq 2$.
Taking account of the fact that for any fixed t we have $\varphi_m\varphi_l \equiv 0$ for $|m-l| > 1$, we can replace ω in the above estimates by

$$\omega_m = \mathcal{F}^{-1}\left(\left[\varphi_{m-1}(t,\xi) + \varphi_m(t,\xi) + \varphi_{m+1}(t,\xi)\right]\hat{\omega}(\xi)\right) \qquad (\varphi_{-1} \equiv 0).$$

We restrict again first to the case $q_j = -2$.
Substituting appropriately and summing up, we arrive at

$$(1+t)^{\frac{1}{q}-1}\left\|\mathcal{F}^{-1}\left(e^{-\mu_j\left(\frac{\xi}{1+t}\right)t}c_{jrk}\left(\frac{\xi}{1+t}\right)\phi_{ext}\left(\frac{\xi}{1+t}\right)\hat{\omega}\left(\frac{\xi}{1+t}\right)\right)\right\|_{B^0_{q2}}$$
$$\lesssim (1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}(1+t)^{\frac{1}{p}-1}\left\|\mathcal{F}^{-1}\left(\hat{\omega}\left(\frac{\xi}{1+t}\right)\right)\right\|_{B^r_{p2}}$$

with regularity $r = \frac{1}{p} - \frac{1}{q}$. With the use of the embeddings $B^0_{q2} \hookrightarrow L^q$, $2 \le q < \infty$, and $L^{p,s} \hookrightarrow B^s_{p2}$, $s \in \mathbb{R}$, $1 < p \le 2$, (see e.g. [BL76], Theorem 6.4.4) we finally obtain

$$\left\|\mathcal{F}^{-1}\left(e^{-\mu_j(\xi)t}c_{jrk}(\xi)\phi_{ext}(\xi)\hat{\omega}(\xi)\right)\right\|_{L^q}$$
$$\lesssim (1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}(1+t)^{\frac{1}{p}-1}\left\|\mathcal{F}^{-1}\left(\langle\xi\rangle^r\hat{\omega}\left(\frac{\xi}{1+t}\right)\right)\right\|_{L^p}$$
$$\lesssim (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\omega\|_{L^{p,r}}$$

for dual indices q, $1 < p \le 2$. The cases $q_j \in \mathbb{N}$ and $q_j = \infty$ are treated analogously, and noting that $n_l = \max_{j=1,\ldots,d} q_j$, we have proved our assertions. □

2.4.2.2. The case of no purely imaginary eigenvalues

For the case *(ii)* we define a decay- and regularity-determining number n_l as the minimal one of all even numbers $m \in \{-2\} \cup 2\mathbb{N}_0$ with

$$\forall j \in \{1,\ldots,d\} \exists k \in 2\mathbb{N}_0 \text{ with } k \le m+2 : \operatorname{Re}\hat{\lambda}_{2-k,j} > 0.$$

Theorem 2.18. *Assume $U_0 \in \mathcal{S}$, the existence of a number n so that $(\hat{A}_n)$ and $(\hat{B}_n)$ hold and that all eigenvalues $\mu = \mu(\xi)$ have a positive real part in the whole zone $Z_{ext}(N)$. Define the number n_l as above. Then the following L^p-L^q decay estimates hold for solutions U to the initial value problem for (2.2):*
For dual indices $q, 1 < p \le 2$ we obtain in the cases $n_l \le 0$

$$\|\phi_{ext}(D)U\|_{L^q} \lesssim e^{-ct}\|U_0\|_{L^{p,r_p}} \tag{2.41}$$

with a positive constant c, $r_p = \frac{1}{p} - \frac{1}{q}$, and in the cases $n_l > 0$

$$\|\phi_{ext}(D)U\|_{L^q} \le C_\alpha(1+t)^{-\frac{\alpha}{n_l}}\|U_0\|_{L^{p,\tilde{r}^\alpha_p}}$$

with an arbitrary $\alpha \ge 0$, an α-dependent constant C_α and $\tilde{r}^\alpha_p = \left(\frac{1}{p} - \frac{1}{q}\right) + \alpha$.

Remark 2.17. 1. An exponential stability, i.e., the estimate (2.41), is the natural result for practical useful cases. It is of course always obtained for parabolic systems of type (2.2).
2. For the cases $n_l > 0$ we have the more decay the more regularity we postulate (cf. Example 2.3). Note further that such a phenomenon can be observed only if there is no purely imaginary eigenvalue involved.

Proof. We can follow the lines of the proof of Theorem 2.17. Once again we want to derive L^p-L^q decay estimates for Fourier multipliers of type (2.35). We can, with the same arguments as before, restrict to $m \geq 1$ and define a number $p_j \in \{-2\} \cup 2\mathbb{N}_0$ as the minimal one of all numbers $k \in \{-2\} \cup 2\mathbb{N}_0$ for which $\operatorname{Re} \hat{\lambda}_{-k,j} > 0$. Note that $n_l = \max_{j=1,\dots,d} p_j$.

The L^2-L^2 estimates are in the cases $p_j \leq 0$ given by

$$\left\| \mathcal{F}^{-1}\left(e^{-\mu_j(\xi)t} c_{jrk}(\xi)\phi_{ext}(\xi)\varphi_m(t,\xi)\hat{\omega}(\xi)\right)\right\|_{L^2} \lesssim e^{-ct}\|\omega\|_{L^2}$$

with a positive constant c.

In the cases $p_j > 0$ we arrive with a constant $c_j > 0$, an arbitrary $\alpha \geq 0$ and an α-dependent (universal) positive constant C_α at

$$\begin{aligned}&\left\| \mathcal{F}^{-1}\left(e^{-\mu_j(\xi)t} c_{jrk}(\xi)\phi_{ext}(\xi)\varphi_m(t,\xi)\hat{\omega}(\xi)\right)\right\|_{L^2} \\ &\qquad \leq C_\alpha \left(\frac{2^m}{1+t}\right)^{\alpha} t^{-\frac{\alpha}{p_j}} \left\| \xi^{-\alpha} t^{\frac{\alpha}{p_j}} e^{-c_j \xi^{-p_j} t}\hat{\omega}(\xi)\right\|_{L^2} \leq C_\alpha \left(\frac{2^m}{1+t}\right)^{\alpha} t^{-\frac{\alpha}{p_j}} \|\omega\|_{L^2}\end{aligned}$$

for times $t \geq 1$. Hence, we have for arbitrary $t > 0$

$$\left\| \mathcal{F}^{-1}\left(e^{-\mu_j(\xi)t} c_{jrk}(\xi)\phi_{ext}(\xi)\varphi_m(t,\xi)\hat{\omega}(\xi)\right)\right\|_{L^2} \leq C_\alpha 2^{m\alpha}(1+t)^{-\left(\alpha+\frac{\alpha}{p_j}\right)}\|\omega\|_{L^2}. \qquad (2.42)$$

For the corresponding L^1-L^∞ estimates we proceed as before and consider terms as in (2.37). We obtain

$$S_m \leq C_\alpha \left(\frac{2^m}{1+t}\right)^{\alpha} \int_{|\xi| \in \left[\frac{2^{m-1}}{1+t}, \frac{2^{m+1}}{1+t}\right]} \left|\xi^{-\alpha} e^{-\mu_j(\xi)t}\right| d\xi. \qquad (2.43)$$

In the cases $p_j \leq 0$ we get (choosing $\alpha = 0$)

$$S_m \lesssim 2^m e^{-ct}.$$

In the cases $p_j > 0$ we estimate for large times $t \geq 1$

$$S_m \leq C_\alpha \left(\frac{2^m}{1+t}\right)^{\alpha} \int_{\frac{2^{m-1}}{1+t}}^{\frac{2^{m+1}}{1+t}} \xi^{-\alpha} e^{-c_j \xi^{-p_j} t} d\xi.$$

With the help of the substitution $c_j \xi^{-p_j} t = \nu$ we rewrite the above as

$$\begin{aligned} S_m &\leq C_\alpha \left(\frac{2^m}{1+t}\right)^{\alpha} t^{-\frac{\alpha}{p_j}} t^{\frac{1}{p_j}} \int_{c_j\left(\frac{1+t}{2^{m+1}}\right)^{p_j} t}^{c_j\left(\frac{1+t}{2^{m-1}}\right)^{p_j} t} \nu^{-\frac{1}{p_j}} \nu^{\frac{\alpha}{p_j}-1} e^{-\nu} d\nu \\ &\leq C_\alpha \left(\frac{2^m}{1+t}\right)^{1+\alpha} t^{-\frac{\alpha}{p_j}} \Gamma(\alpha/p_j) \leq C_\alpha \left(\frac{2^m}{1+t}\right)^{1+\alpha} t^{-\frac{\alpha}{p_j}}. \end{aligned}$$

In the latter we need $\alpha > 0$ in order for $\Gamma(\alpha/p_j)$ to be finite. However, we have from (2.43) for any $\alpha \geq 0$ and any $t > 0$

$$S_m \leq C_\alpha 2^{m(1+\alpha)}(1+t)^{-(1+\alpha)}.$$

Hence, we obtain in the cases $p_j \le 0$ the L^1-L^∞ estimates

$$\left\|\mathcal{F}^{-1}\left(e^{-\mu_j(\xi)t}c_{jrk}(\xi)\phi_{ext}(\xi)\varphi_m(t,\xi)\hat{\omega}(\xi)\right)\right\|_{L^\infty} \lesssim 2^m e^{-ct}\|\omega\|_{L^1}$$

and in the cases $p_j > 0$

$$\left\|\mathcal{F}^{-1}\left(e^{-\mu_j(\xi)t}c_{jrk}(\xi)\phi_{ext}(\xi)\varphi_m(t,\xi)\hat{\omega}(\xi)\right)\right\|_{L^\infty} \le C_\alpha 2^{m(1+\alpha)}(1+t)^{-\left(1+\alpha+\frac{\alpha}{p_j}\right)}\|\omega\|_{L^1}.$$

Applying the Riesz-Thorin interpolation theorem yields the L^p-L^q decay estimates

$$\left\|\mathcal{F}^{-1}\left(e^{-\mu_j(\xi)t}c_{jrk}(\xi)\phi_{ext}(\xi)\varphi_m(t,\xi)\hat{\omega}(\xi)\right)\right\|_{L^q} \lesssim 2^{m\left(\frac{1}{p}-\frac{1}{q}\right)}e^{-ct}\|\omega\|_{L^p}$$

in the cases $p_j \le 0$ and

$$\left\|\mathcal{F}^{-1}\left(e^{-\mu_j(\xi)t}c_{jrk}(\xi)\phi_{ext}(\xi)\varphi_m(t,\xi)\hat{\omega}(\xi)\right)\right\|_{L^q} \le C_\alpha 2^{m\left(\left(\frac{1}{p}-\frac{1}{q}\right)+\alpha\right)}(1+t)^{-\left(\left(\frac{1}{p}-\frac{1}{q}\right)+\alpha+\frac{\alpha}{p_j}\right)}\|\omega\|_{L^p}$$

in the cases $p_j > 0$ for dual indices q, $1 \le p \le 2$. Analogous arguments as at the end of the proof of Theorem 2.17 imply the assertions of Theorem 2.18. □

An immediate consequence of the above considerations is:

Corollary 2.19. *Let all assumptions of Theorem 2.18 be satisfied and assume $n_l > 0$. Then we have:*

$$\|\phi_{ext}(D)U\|_{L^q} \le C_\alpha(1+t)^{-\frac{\alpha}{n_l}\left(\frac{1}{p}-\frac{1}{q}\right)}\|U_0\|_{L^{p,r_p^\alpha}}$$

with an arbitrary $\alpha \ge 0$, an α-dependent constant C_α, $1 < p \le 2$, $\frac{1}{p}+\frac{1}{q}=1$ and $r_p^\alpha = (1+\alpha)\left(\frac{1}{p}-\frac{1}{q}\right)$.

Remark 2.18. In contrary to Theorem 2.18 we obtain (for positive α) almost no decay for dual values $q, 1 < p \le 2$, $p, q \approx 2$, but also need almost no regularity.

Proof. The statement of Corollary 2.19 follows from setting $\alpha = 0$ in the L^2-L^2 estimate (2.42) and repeating arguments of the proof of Theorem 2.18. □

Up to now we have neglected that the eigenvalues might have non-vanishing and helpful imaginary parts in the cases $n_l > 0$ (The estimates (2.41) can not be improved.). Let us therefore define a number $\tilde{n}_l$ as $\tilde{n}_l := \max_{j=1,\ldots,d} n_{l,j}$ with $n_{l,j} = \min(p_j, q_j)$, q_j (replace $\hat{\lambda}_{-k,j} \neq 0$ by $\operatorname{Im}(a_{-k}\hat{\lambda}_{-k,j}) \neq 0$) and p_j defined as in the proofs of the Theorems 2.17 and 2.18, respectively.

Corollary 2.20. *Let all assumptions of Theorem 2.18 be satisfied and assume $n_l > 0$. Then we have with $1 < p \le 2$ and $\frac{1}{p}+\frac{1}{q}=1$ for $\tilde{n}_l \in \{-2, 0\}$*

$$\|\phi_{ext}(D)U\|_{L^q} \lesssim (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|U_0\|_{L^{p,r_p}},$$

$r_p = \frac{1}{p}-\frac{1}{q}$, and for $\tilde{n}_l \in \mathbb{N}$

$$\|\phi_{ext}(D)U\|_{L^q} \lesssim (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|U_0\|_{L^{p,r_p^{\tilde{n}_l}}}$$

with $r_p^{\tilde{n}_l} = (1+\frac{\tilde{n}_l}{2})\left(\frac{1}{p}-\frac{1}{q}\right)$.

Remark 2.19. In comparison with Corollary 2.19 we need less regularity for the same decay in the cases $\alpha = \frac{n_l}{2}$ and $\tilde{n}_l < n_l$.

Proof. We can do in the cases $q_j < p_j$ exactly the same calculations as in the proof of Theorem 2.17 for the cases $q_j = -2$ and $q_j \in \mathbb{N}$. In particular, the estimates (2.39) and (2.40) hold in this situation as well. The only change to before is that the functions $\tilde{\psi}_{m,t} = \tilde{\psi}_{m,t}(\eta)$ in (2.38) should be replaced by

$$\tilde{\psi}_{m,t}(\eta) = c_{jrk}\left(\frac{2^m}{1+t}\eta\right)\phi_{ext}\left(\frac{2^m}{1+t}\eta\right)\varphi(|\eta|)e^{-\mathrm{Re}\,\mu_j\left(\frac{2^m}{1+t}\eta\right)t}.$$

Nevertheless, $\displaystyle\int_{|\eta|\in[\frac{1}{2},2]}\left|d_\eta\tilde{\psi}_{m,t}(\eta)\right|d\eta$ is still bounded independently of m and t. □

We finish this section by giving an example for an equation fitting into the case $n_l > 0$:

Example 2.3. Let us consider the Cauchy problem

$$u_t + a_1 u_x + P(D)u = 0, \qquad u(0,x) = u_0(x)$$

with $a_1 \in \mathbb{R}$ and $P(D)$ being a pseudodifferential operator with the symbol

$$P(\xi) = \begin{cases} 0, & |\xi| \le N/2, \\ ia_{-1}\xi^{-1} + \ldots + ia_{-(n_l-1)}\xi^{-(n_l-1)} + a_{-n_l}\xi^{-n_l} + \mathcal{O}(\xi^{-(n_l+1)}), & |\xi| \ge N, \end{cases}$$

$a_{-j} \in \mathbb{R}$, $a_{-n_l} > 0$.

The above equation (which certainly does not fit into our model, but is representative for large frequencies) involves in some sense a very small mass term, which, if we only assume the usual regularity from the data (i.e., $\alpha = 0$), leads to no decay for large frequencies.

However, if we have additional regularity from the data, then the amplitude is decaying stronger (compare with the solution representation

$$\phi_{ext}(D)u(t,x) = \mathcal{F}^{-1}\left(\langle\xi\rangle^{-\beta}e^{-\mu(\xi)t}\phi_{ext}(\xi)\mathcal{F}\left(\langle D_x\rangle^{\beta}u_0\right)\right)$$

with $\mu(\xi) = ia_1\xi + ia_{-1}\xi^{-1} + \ldots + ia_{-(n_l-1)}\xi^{-(n_l-1)} + a_{-n_l}\xi^{-n_l} + \mathcal{O}(\xi^{-(n_l+1)})$), i.e., we have more decay.

2.4.2.3. The mixed case

Here, i.e., in case *(iii)*, we just need to combine the previous considerations.

Define a number n_1 like the number n_l in Section 2.4.2.1 for the purely imaginary eigenvalues and a number n_2 like $\tilde{n}_l$ before Corollary 2.20 in Section 2.4.2.2 for the eigenvalues with a positive real part for all $\xi \in Z_{ext}(N)$. We set $n_l := \max(n_1, n_2)$.

Corollary 2.21. *We assume $U_0 \in \mathcal{S}$, the existence of a number n so that $(\hat{A}_n)$ and $(\hat{B}_n)$ hold and that there are purely imaginary eigenvalues $\mu = \mu(\xi)$ in $Z_{ext}(N)$ and some having a positive real part in the whole zone $Z_{ext}(N)$. Define the number n_l as above. Then the following L^p-L^q decay estimates hold for solutions U to the initial value problem for (2.2):*
For dual indices $q, 1 < p \leq 2$ we obtain the estimates

$$\|\phi_{ext}(D)U\|_{L^q} \lesssim \begin{cases} (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|U_0\|_{L^{p,r_p}}, & n_l \leq 0, \\ (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|U_0\|_{L^{p,r_p^{n_l}}}, & n_l \in \mathbb{N}, \\ \|U_0\|_{L^{p,r_p}}, & n_l = \infty \end{cases}$$

with $r_p = \frac{1}{p} - \frac{1}{q}$ and $r_p^{n_l} = \left(1 + \frac{n_l}{2}\right)\left(\frac{1}{p} - \frac{1}{q}\right)$.

2.4.3. L^p-L^q decay estimates for bounded frequencies away from zero

We immediately obtain from Proposition 2.14:

Theorem 2.22. *We assume $U_0 \in \mathcal{S}$ and (C). Then we have for solutions U to the initial value problem for (2.2) the estimate*

$$\|\phi_{mid}(D)U\|_{L^q} \lesssim e^{-ct}\|U_0\|_{L^p}$$

with a positive constant c, $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

2.5. Diffusion phenomena

The system (2.2) includes hyperbolic-parabolic coupled systems, like they appear for instance in thermoelasticity. It is certainly of interest whether we can prove, similar to our observations for the Cauchy problem of the damped wave equation in Section 1.3, an asymptotic underlying parabolic structure for the corresponding Cauchy problems of such systems.
First, it is natural to prove diffusion phenomena for the decay-determining region of the phase space. As we have seen in Section 2.4 this region might, under the imposed assumptions, be given by $Z_{int}(\sigma)$, the region of small frequencies, or $Z_{ext}(N)$, the region of large frequencies. The region of small frequencies is the one usually determining the decay of the solution to the Cauchy problem of system (2.2). That will be assumed in Section 2.5.1. However, we will consider the case that the large frequencies determine the decay as well in Section 2.5.2.
We are looking for an underlying parabolic structure for large times t and should therefore restrict onto the cases, where all eigenvalues have a positive real part in $Z_{int}(\sigma)\backslash\{0\}$

or $Z_{ext}(N)$, respectively. It is moreover natural to exclude the cases in which we obtain an exponential decay already.
A diffusion phenomenon is then obtained if we can prove for an appropriate reference system (compare to (1.14)) a result as in (1.15).

2.5.1. Diffusion phenomena for small frequencies

We assume that the decay of the solution to the Cauchy problem of system (2.2) is determined by the small frequencies and thus restrict our view to $Z_{int}(\sigma) = \{|\xi| \leq \sigma \ll 1\}$. We suppose that the assumptions of Theorem 2.16 hold, restrict to the situations, where all eigenvalues $\mu_j = \mu_j(\xi)$ have a positive real part in $Z_{int}(\sigma)\backslash\{0\}$ and neglect the cases in which we obtain an exponential decay, i.e., in which the number n_s from Section 2.4.1 vanishes.
With the help of the finite numbers p_j from Section 2.4.1 and $m_s := \max_{j=1,\dots,d} p_j$, $n_s = \max_{j=1,\dots,d} \min\{p_j, q_j\} \leq m_s$, we can define our reference system:
Let $W = W(t,x)$ be the solution to the following problem:

$$\begin{cases} W_t + \displaystyle\sum_{k=0}^{m_s} a_k M_k \partial_x^k W = 0, \\ W(0,x) = \tilde{L}^{(c)} \cdot \ldots \cdot \tilde{L}^{(0)} U_0(x), \end{cases} \tag{2.44}$$

where $a_k = (-1)^{\lfloor k/2 \rfloor}$, and the matrices M_k are diagonal matrices $M_k = \operatorname{diag}(m_1^{(k)}, \dots, m_d^{(k)})$ with $m_j^{(k)} = 0$ if $k > p_j$ and $m_j^{(k)} = \lambda_{k,j}$ otherwise. The matrices $\tilde{L}^{(i)}$ are taken from the diagonalization procedure in Section 2.2.2.1, c is the number from (2.21), and U_0 is the initial data from (2.3).

Lemma 2.23. *If we assume that $U_0 \in H^s$, $s \in \mathbb{R}$, then there exists a (in $C([0,\infty), \mathcal{S}')$) unique solution to (2.44) with regularity*

$$W \in C([0,\infty), H^s).$$

Proof. The assertion follows directly from the solution representation

$$W(t,x) = \mathcal{F}^{-1}_{\xi \to x}\left(\operatorname{diag}\left(e^{-\tilde{\mu}_1(\xi)t}, \dots, e^{-\tilde{\mu}_d(\xi)t}\right) \tilde{L}^{(c)} \cdot \ldots \cdot \tilde{L}^{(0)} \hat{U}_0(\xi)\right), \tag{2.45}$$

where the $\tilde{\mu}_j$ are given by $\tilde{\mu}_j(\xi) = \lambda_{0,j} + i\lambda_{1,j}\xi + \ldots + \lambda_{p_j,j}\,\xi^{p_j}$, i.e., they take the values of the corresponding $\mu_j(\xi)$ from Lemma 2.6 when neglecting the $\mathcal{O}(\xi^{p_j+1})$-terms. ☐

With ϕ_{int} denoting the same cut-off function as in Section 2.4 we can now state:

Theorem 2.24. *We assume $U_0 \in \mathcal{S}$, the existence of a number n so that (A_n) and (B_n) hold and that all eigenvalues $\mu_j = \mu_j(\xi)$ have a positive real part in $Z_{int}(\sigma)\backslash\{0\}$. Assume n_s to be positive and define the number m_s as above.*
Then we obtain for the solution U to the Cauchy problem of system (2.2) the estimate

$$\left\| \phi_{int}(D) \left(U - \tilde{R}^{(0)} \cdot \ldots \cdot \tilde{R}^{(c)}\, W \right) \right\|_{L^q} \lesssim (1+t)^{-\frac{1}{n_s}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{m_s}} \left\|U_0\right\|_{L^p},$$

where $1 \leq p \leq 2$, $\frac{1}{p}+\frac{1}{q}=1$ *and the matrices* $\tilde{R}^{(i)}$ *(from Section 2.2.2.1) are the inverse ones to* $\tilde{L}^{(i)}$*.*

Proof. For the solution of (2.44) we have the solution representation given by (2.45). Further, Proposition 2.9 tells us that for the solution $V(t,\xi)=\mathcal{F}(U)(t,\xi)$ of the Cauchy problem for (2.4) we have in $Z_{int}(\sigma)$ the representation

$$V(t,\xi)=T_{int}^{-1}(\xi)\,\mathrm{diag}\left(e^{-\mu_1(\xi)t},\dots,e^{-\mu_d(\xi)t}\right)T_{int}(\xi)\hat{U}_0(\xi).$$

We know that

$$T_{int}(\xi)=L+P_1(\xi)\quad\text{and}\quad T_{int}^{-1}(\xi)=R+P_2(\xi),$$

where $L:=\tilde{L}^{(c)}\cdot\ldots\cdot\tilde{L}^{(0)}$, $R:=\tilde{R}^{(0)}\cdot\ldots\cdot\tilde{R}^{(c)}$ and $P_i(\xi)=\mathcal{O}(\xi)$ for $\xi\to 0$.
Hence,

$$\phi_{int}(D)\,(U-RW)=\mathcal{F}^{-1}\left(\phi_{int}(\xi)\left(V(t,\xi)-R\hat{W}(t,\xi)\right)\right)$$

can be written as the sum of:

(i) $\mathcal{F}^{-1}\left(\phi_{int}(\xi)\left(R\ \mathrm{diag}\left(e^{-\mu_1(\xi)t}-e^{-\tilde{\mu}_1(\xi)t},\dots,e^{-\mu_d(\xi)t}-e^{-\tilde{\mu}_d(\xi)t}\right)L\,\hat{U}_0(\xi)\right)\right)$,

(ii) $\mathcal{F}^{-1}\left(\phi_{int}(\xi)\left(R\ \mathrm{diag}\left(e^{-\mu_1(\xi)t},\dots,e^{-\mu_d(\xi)t}\right)P_1(\xi)\hat{U}_0(\xi)\right)\right)$,

(iii) $\mathcal{F}^{-1}\left(\phi_{int}(\xi)\left(P_2(\xi)\,\mathrm{diag}\left(e^{-\mu_1(\xi)t},\dots,e^{-\mu_d(\xi)t}\right)(L+P_1(\xi))\hat{U}_0(\xi)\right)\right)$.

The components of *(i)* are given by

$$I_k=\mathcal{F}^{-1}\left(\sum_{j,r=1}^{d}c_{jrk}\left(e^{-\mu_j(\xi)t}-e^{-\tilde{\mu}_j(\xi)t}\right)\phi_{int}(\xi)\hat{U}_{0,r}(\xi)\right),$$

where c_{jrk} are constants and $\hat{U}_0=(\hat{U}_{0,1},\dots,\hat{U}_{0,d})^T$. To estimate these we make use of the fact that we can write

$$e^{-\mu_j(\xi)t}-e^{-\tilde{\mu}_j(\xi)t}=-r_j(\xi)t\,e^{-\tilde{\mu}_j(\xi)t}\int_0^1 e^{-r_j(\xi)ts}ds,$$

with $r_j=r_j(\xi)$ denoting the $\mathcal{O}(\xi^{p_j+1})$-terms from Lemma 2.6 for the corresponding $\mu_j(\xi)$, and thus obtain as a L^1-L^∞ estimate

$$\|I_k\|_{L^\infty}\lesssim(1+t)^{-\frac{1}{n_s}-\frac{1}{m_s}}\|U_0\|_{L^1}$$

and as a L^2-L^2 estimate

$$\|I_k\|_{L^2}\lesssim(1+t)^{-\frac{1}{m_s}}\|U_0\|_{L^2}.$$

This gives after interpolation

$$\|I_k\|_{L^q}\lesssim(1+t)^{-\frac{1}{n_s}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{m_s}}\|U_0\|_{L^p}$$

for $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.
The components of *(ii)* and *(iii)* are of the form

$$J_k = \mathcal{F}^{-1}\left(\sum_{j,r=1}^{d} c_{jrk}(\xi)e^{-\mu_j(\xi)t}\phi_{int}(\xi)\hat{U}_{0,r}(\xi)\right) \tag{2.46}$$

with $c_{jrk}(\xi) = \mathcal{O}(\xi)$ for all j, r and k. By straightforward calculations we can derive, as before, L^1-L^∞ and L^2-L^2 estimates, which yield after interpolation

$$\|J_k\|_{L^q} \lesssim (1+t)^{-\frac{1}{n_s}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{m_s}}\|U_0\|_{L^p}$$

for dual indices $1 \leq p \leq 2$ and q.
This proves the statement of the theorem. □

Taking the estimates from Theorem 2.16 for $2 \leq n_s < \infty$ and

$$\|\phi_{int}(D)W\|_{L^q} \lesssim (1+t)^{-\frac{1}{n_s}\left(\frac{1}{p}-\frac{1}{q}\right)}\|U_0\|_{L^p}$$

for $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$ into consideration, and thus that the difference, as stated in Theorem 2.24, is decaying faster, we can state that the asymptotic profiles (at least from the point of view of decay estimates) of solutions to the Cauchy problem of (2.2) are given by solutions to (2.44).

Remark 2.20. In the case $n_s = 0$, i.e., if we have an exponential decay for the solution U to the Cauchy problem of system (2.2) from the region of small frequencies, we can not expect a diffusion phenomenon. That conclusion may be obtained by noting that the functions $c_{jrk}(\xi) = \mathcal{O}(\xi)$ in the terms (2.46) have no essential helping character.

2.5.2. Diffusion phenomena for large frequencies

As we have mentioned in the beginning of Section 2.5 it may also be the region $Z_{ext}(N) = \{|\xi| \geq N \gg 1\}$ of large frequencies that is determining the non-exponential decay of the solution to the Cauchy problem of (2.2). We assume now that this is the case and restrict our view to this region of the phase space.
We suppose that the assumptions of Theorem 2.18 hold and neglect the cases $n_l \leq 0$.
Now we define a reference system as in the case of small frequencies by introducing for each eigenvalue $\mu_j = \mu_j(\xi)$ (compare to the proof of Theorem 2.18) a number p_j as the minimal one of all numbers $k \in \{-2\} \cup 2\mathbb{N}_0$ with $\operatorname{Re}\hat{\lambda}_{-k,j} > 0$ and observing that $n_l = \max_{j=1,\dots,d} p_j$:
Let $W = W(t,x)$ be the solution to the problem:

$$\begin{cases} W_t + P(D)W = 0, \\ W(0,x) = \hat{\hat{L}}^{(\hat{c})} \cdot \ldots \cdot \hat{\hat{L}}^{(0)}U_0(x), \end{cases} \tag{2.47}$$

where $P(D)$ is a pseudodifferential operator of at the most order two with the symbol $P(\xi)$ being a diagonal matrix with diagonal entries $d_i(\xi)$ given by

$$d_i(\xi) = \begin{cases} i\mathrm{Im}\,\hat{\lambda}_{2,j}\xi^2 + i\hat{\lambda}_{1,j}\xi + i\mathrm{Im}\,\hat{\lambda}_{0,j}, & |\xi| \le N/2, \\ i\mathrm{Im}\,\hat{\lambda}_{2,j}\xi^2 + i\hat{\lambda}_{1,j}\xi + i\mathrm{Im}\,\hat{\lambda}_{0,j} + i\hat{\lambda}_{-1,j}\xi^{-1} + \dots & \\ \qquad + i\hat{\lambda}_{-(p_j-1),j}\xi^{-(p_j-1)} + \hat{\lambda}_{-p_j,j}\xi^{-p_j}, & |\xi| \ge N \gg 1. \end{cases}$$

The matrices $\hat{\hat{L}}^{(i)}$ are taken from the diagonalization procedure in Section 2.2.3, $\hat{c}$ is the number from (2.27), and U_0 is the initial data from (2.3).
The problem (2.47) is well-posed and with ϕ_{ext} denoting the cut-off function from Section 2.4 we obtain with a similar proof to the one for Theorem 2.24:

Theorem 2.25. *We assume $U_0 \in \mathcal{S}$, the existence of a number n so that $(\hat{A}_n)$ and $(\hat{B}_n)$ hold and that all eigenvalues have a positive real part in $Z_{ext}(N)$. We suppose that the number n_l is positive and obtain for the solution U to the Cauchy problem of system (2.2) the estimate*

$$\left\| \phi_{ext}(D)\left(U - \hat{\hat{R}}^{(0)} \cdot \ldots \cdot \hat{\hat{R}}^{(\hat{c})}\, W\right)\right\|_{L^q} \le C_\alpha (1+t)^{-\frac{\alpha}{n_l} - \frac{1}{n_l}} \|U_0\|_{L^{p,\tilde{r}_p^\alpha}},$$

where $\alpha \ge 0$ is arbitrary, C_α an α-dependent constant, $1 < p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\tilde{r}_p^\alpha = \left(\frac{1}{p} - \frac{1}{q}\right) + \alpha$ and the matrices $\hat{\hat{R}}^{(i)}$ (from Section 2.2.3) are the inverse ones to $\hat{\hat{L}}^{(i)}$.

The asymptotic profiles (from the viewpoint of decay estimates) of solutions to the Cauchy problem of (2.2) are thus given by solutions to (2.47).

2.6. Propagation of singularities

Let us now devote ourselves to the study of propagation of singularities for solutions U to the Cauchy problem of (2.2).
We are first interested in so-called mild singularities, i.e., we propose for (linear combinations of) our initial data H^s-regularity everywhere and H^{s+1}-regularity everywhere except at one point and are then interested in the propagation of this H^{s+1}-singularity. Suppose now that the assumptions of Theorem 2.15 hold and thus that we have well-posedness for the Cauchy problem of (2.2). Let us, keeping the practical applications in view, assume moreover that the values $\hat{\lambda}_{k,j}$ from the asymptotic expansions of the eigenvalues $\mu_j = \mu_j(\xi)$ from Proposition 2.12, *(ii)* are all real. The latter is for instance the case when $(\hat{\mathrm{A}}_{\mathrm{n}}')$ is satisfied instead of $(\hat{\mathrm{A}}_{\mathrm{n}})$ and all matrices A_i from (2.2) are from $\mathbb{R}^{d\times d}$. Now we introduce with

$$L := \hat{\hat{L}}^{(\hat{c})} \cdot \ldots \cdot \hat{\hat{L}}^{(1)} \hat{\hat{L}}^{(0)},$$

where the above constant matrices are taken from the diagonalization procedure for large frequencies in Section 2.2.3 and $\hat{c}$ from (2.27), the vectors

$$W_0 = (w_{0,1}, \ldots, w_{0,d})^T = L\, U_0 \qquad \text{and} \quad W = (w_1, \ldots, w_d)^T = L\, U.$$

It is our goal to prove the following result:

Theorem 2.26. *We consider the Cauchy problem*

$$\begin{cases} U_t + A_0 U + A_1 U_x - A_2 U_{xx} = 0, \\ U(0,x) = U_0(x) \end{cases}$$

in $\mathbb{R}_{\geq 0} \times \mathbb{R}$ *with* A_i *being real and constant* $d \times d$ *matrices.*
Suppose that for some n *the assumptions* $(\hat{A}'_n)$ *and* $(\hat{B}_n)$ *hold together with (2.29). We further assume that for a fixed* $s \in \mathbb{R}$ *we have for all* $j = 1, \ldots, d$

$$w_{0,j} \in \left[H^s(\mathbb{R}) \cap H^{s+1}(\mathbb{R}\backslash\{x_0\})\right] \backslash H^{s+1}(\mathbb{R}). \tag{2.48}$$

Then we have for $j \in \{1, \ldots, d\}$ *with (cf. Proposition 2.12, (ii)):*

$$\begin{aligned} &\underline{\hat{\lambda}_{2,j} = 0:} \quad w_j(t,\cdot) \in \left[H^s(\mathbb{R}) \cap H^{s+1}(\mathbb{R}\backslash\{x_0 + \hat{\lambda}_{1,j}t\})\right] \backslash H^{s+1}(\mathbb{R}), \\ &\underline{\hat{\lambda}_{2,j} > 0:} \quad w_j(t,\cdot) \in H^{s+1}(\mathbb{R}) \cap H^{s+2}(\mathbb{R}\backslash\{x_0 + \hat{\lambda}_{1,k}t : k \in K\}), \\ &\qquad\qquad\quad K := \{k \in \{1, \ldots, d\} : \hat{\lambda}_{2,k} = 0\}, \end{aligned}$$

for all $t > 0$.

Proof. The proof of Theorem 2.15 tells us that we feel singularities of the data only for large frequencies ξ. It is therefore sufficient to study the solution representation from Proposition 2.13.
In fact, from Proposition 2.13 we know that (after reordering) the components of $\tilde{U}_{ext} = (\tilde{u}_{ext,1}, \ldots, \tilde{u}_{ext,d})^T = \mathcal{F}^{-1}\left(\phi_{ext}(\xi) T_{ext}(\xi) V(t,\xi)\right)$ are solutions of

$$\begin{cases} (\partial_t - \hat{\lambda}_{2,j}\partial_x^2)\tilde{u}_{ext,j} + \hat{\lambda}_{1,j}\partial_x \tilde{u}_{ext,j} + a_j(D)\tilde{u}_{ext,j} = 0, & j = 1, \ldots, j_1, \\ (\partial_t + \hat{\lambda}_{1,j}\partial_x)\tilde{u}_{ext,j} + b_j(D)\tilde{u}_{ext,j} = 0, & j = j_1 + 1, \ldots, d, \\ \tilde{U}_{ext}(0,x) = \mathcal{F}^{-1}\left(\phi_{ext}(\xi) T_{ext}(\xi) \mathcal{F}(U_0)(\xi)\right), \end{cases}$$

$j_1 \in \{0, \ldots, d\}$ (We of course reorder the initial data as above.).
For the first j_1 equations we assume $\hat{\lambda}_{2,j}$ to be positive. For the remaining ones $\hat{\lambda}_{1,j}$ may take any real value. By including an appropriate cut-off function ψ with $\psi \equiv 1$ on $\operatorname{supp} \phi_{ext}$ into the symbols, we can assume that $a_j, b_k \in \Psi^0_{1,0}(\mathbb{R})$ for all $j = 1 \ldots, j_1$ and $k = j_1 + 1 \ldots, d$.
Taking into account that

$$\tilde{U}_{ext}(0,x) = \phi_{ext}(D) W_0 + \phi^{(1)}_{ext}(D) W_0$$

with each element of $\phi^{(1)}_{ext}$ belonging to $\Psi^{-1}_{1,0}(\mathbb{R})$, we have exactly the same behavior for the components of $\tilde{U}_{ext}(0,x)$ as assumed for the ones of W_0 in (2.48).
By applying the classical theory of linear parabolic and hyperbolic problems (cf. [CP82, Rau91]) we immediately conclude in the cases $j = 1, \ldots, j_1$

$$\tilde{u}_{ext,j} \in C^\infty((0,\infty) \times \mathbb{R}) \tag{2.49}$$

and in the cases $j = j_1 + 1, \dots, d$

$$\tilde{u}_{ext,j}(t,\cdot) \in \left[H^s(\mathbb{R}) \cap H^{s+1}(\mathbb{R}\backslash\{x_0 + \hat{\lambda}_{1,j}t\})\right] \backslash H^{s+1}(\mathbb{R}).$$

Note for the latter that we can write

$$\tilde{u}_{ext,j}(t,x) = P_t(D)\mathcal{F}^{-1}\left(e^{-i\hat{\lambda}_{1,j}\xi t}\hat{\tilde{u}}_{ext,j}(0,\xi)\right),$$

where $P_t \in \Psi^0_{1,0}(\mathbb{R})$ is an elliptic operator and that it also holds true if $\hat{\lambda}_{1,j} = 0$. The results now follow immediately from the fact that

$$\phi_{ext}(D)W = \tilde{U}_{ext} + \phi^{(2)}_{ext}(D)\tilde{U}_{ext},$$

where each element of $\phi^{(2)}_{ext}$ belongs to $\Psi^{-1}_{1,0}(\mathbb{R})$. □

Remark 2.21. 1. In general, for such j with $\hat{\lambda}_{2,j} > 0$ we can not say that

$$w_j(t,\cdot) \notin H^{s+2}_{\text{loc}}(\{x_0 + \hat{\lambda}_{1,k}t\}) \qquad \text{for a } k \in K = \{k \in \{1,\dots,d\} : \hat{\lambda}_{2,k} = 0\}. \tag{2.50}$$

That actually depends on the looks of T_{-1} in the expansion

$$L\,T^{-1}_{ext}(\xi) = I + i\xi^{-1}T_{-1} + \mathcal{O}(\xi^{-2}). \tag{2.51}$$

Hence, (2.50) holds true only if the entry t_{jk} of $T_{-1} = (t_{lm})^d_{l,m=1}$ is not vanishing.
2. In the case that $\hat{\lambda}_{2,j} > 0$ for all $j = 1, \dots, d$ we even obtain

$$U \in C^\infty((0,\infty) \times \mathbb{R}),$$

which is due to the fact that this condition guarantees that the system given in (2.2) is a parabolic one. If the assumptions of Theorem 2.26 hold together with $\hat{\lambda}_{2,j} = 0$ for all $j = 1, \dots, d$, then the system given in (2.2) is a purely hyperbolic one and thus the obtained results should be expected (cf. [Bea89]).

With the above proof we can moreover immediately conclude the following result:

Corollary 2.27. *We consider the Cauchy problem from Theorem 2.26. Suppose that for some n the assumptions $(\hat{A}'_n)$ and $(\hat{B}_n)$ hold together with (2.29). We further assume that for a fixed $s \in \mathbb{R}$ and an integer $m \geq 1$ we have for all $j = 1, \dots, d$*

$$w_{0,j} \in \left[H^s(\mathbb{R}) \cap H^{s+m}(\mathbb{R}\backslash\{x_0\})\right] \backslash H^{s+1}(\mathbb{R}).$$

Then we have, using $K := \{k \in \{1,\dots,d\} : \hat{\lambda}_{2,k} = 0\}$, for $j \in \{1,\dots,d\}$ with:

$\underline{\hat{\lambda}_{2,j} = 0}$: $w_j(t,\cdot) \in H^s(\mathbb{R}) \cap H^{s+m}(\mathbb{R}\backslash\{x_0 + \hat{\lambda}_{1,k}t : k \in K\})$,

$w_j(t,\cdot) \in H^{s+1}_{loc}(\{x_0 + \hat{\lambda}_{1,k}t\})\backslash H^{s+2}_{loc}(\{x_0 + \hat{\lambda}_{1,k}t\})$, *$k \in K$ with* $\hat{\lambda}_{1,k} \neq \hat{\lambda}_{1,j}$,

$w_j(t,\cdot) \notin H^{s+1}_{loc}(\{x_0 + \hat{\lambda}_{1,j}t\})$,

$\underline{\hat{\lambda}_{2,j} > 0}$: $w_j(t,\cdot) \in H^{s+1}(\mathbb{R}) \cap H^{s+m+1}(\mathbb{R}\backslash\{x_0 + \hat{\lambda}_{1,k}t : k \in K\})$,

$w_j(t,\cdot) \notin H^{s+2}_{loc}(\{x_0 + \hat{\lambda}_{1,k}t\})$, $k \in K$,

for all $t > 0$, assuming that $t_{jk} \neq 0$ for all $k \in K$, $k \neq j$, and $T_{-1} = (t_{lm})^d_{l,m=1}$ from (2.51).

The next corollary can be proved by using induction on m in Corollary 2.27 together with (2.2) to deduce regularity of the solutions in the t-variable.

Corollary 2.28. *We consider once more the Cauchy problem from Theorem 2.26. Suppose that for some n the assumptions ($\hat{A}'_n$) and ($\hat{B}_n$) hold together with (2.29). We further assume that for a fixed $s \in \mathbb{R}$*

$$U_0 \in H^s(\mathbb{R}) \cap C^\infty(\mathbb{R}\backslash\{x_0\}).$$

Then we obtain

$$U \in C^\infty(((0,\infty) \times \mathbb{R})\backslash\{(t, x_0 + \hat{\lambda}_{1,k}t) : t > 0, k \in K\})$$

with $K = \{k \in \{1, \ldots, d\} : \hat{\lambda}_{2,k} = 0\}$.

2.7. Generalizations

More general systems in physical space

Instead of systems as in (2.2) we could more generally study linear m^{th}-order systems

$$U_t + \sum_{k=0}^{m} a_k A_k \partial_x^k U = 0 \tag{2.52}$$

for d-dimensional unknowns $U = U(t,x)$, where $(t,x) \in \mathbb{R}_{>0} \times \mathbb{R}$, $a_k = (-1)^{\lfloor k/2 \rfloor}$ and all matrices A_k are complex and constant.
We can apply our diagonalization procedure to systems of type (2.52), work with the same hierarchy of conditions and should then discuss in what way the results mentioned in this chapter have to be altered to fit the above situation.

More general systems in phase space

We can moreover apply our diagonalization procedure to systems of the form

$$V_t + \sum_{k=0}^{m_s} b_k \xi^k A_k V + R_s(\xi) V = 0 \tag{2.53}$$

or

$$V_t + \sum_{k=0}^{m_l} b_{n-k} \xi^{n-k} A_{n-k} V + R_l(\xi) V = 0 \tag{2.54}$$

in the regions $Z_{int}(\sigma)$ of small frequencies or $Z_{ext}(N)$ of large frequencies, respectively. Here $b_k = 1$ for even and $b_k = i$ for odd k. The matrices A_k are complex and constant, $n \in \mathbb{Z}$, $R_s(\xi) = \mathcal{O}(\xi^{m_s+1})$ for $\xi \to 0$ and $R_l(\xi) = \mathcal{O}(\xi^{n-m_l-1})$ for $|\xi| \to \infty$.

Remark 2.22. Note that we could rewrite systems as in (2.4) (or the equivalent ones to (2.52) in phase space) in a neighborhood of any fixed frequency ξ_0 into one as in (2.53) by shifting ξ_0 to the origin (i.e., we write $\xi = \xi - \xi_0 + \xi_0 =: \zeta + \xi_0$ and obtain a system as in (2.53) in a neighborhood of $\zeta = 0$). Thus, we are able to derive solution representations in neighborhoods of ‘problematic’ frequencies ξ_0.

Non-fully diagonalizable systems

In Section 2.2.2.4 we have given some remarks on non-fully diagonalizable systems (2.4) for small frequencies. Solution representations as in (2.25) (note that we can apply Gronwall's lemma to derive useful estimates) together with the information on the asymptotic behavior of the eigenvalues are in a lot of cases sufficient to derive results on decay estimates and diffusion phenomena. For a detailed example we refer to Section 3.3.
Note that analogous considerations hold true for non-fully diagonalizable systems (2.4) for large frequencies as well, and hence we are able to derive in many cases well-posedness/ regularity results and results on the propagation of singularities. However, details shall - due to the fact that they are not needed for the applications that we have in mind - not be discussed here.

3. Applications to parabolic structured models with and without dissipation or mass

In this chapter we will apply the results of the foregoing one to thermoelasticity models of type 1, i.e., the classical model, type 2 and 3 and to second sound models in one space dimension with and without dissipation or mass terms that are not necessarily of parabolic type but have a parabolic structure from the point of view of decay estimates. The first step is always to transform the Cauchy problem for the considered thermoelasticity model into one for a system of type (2.2) or type (2.4) (or at least into one for a system appearing in Section 2.7) by introducing a proper vector function. There is no general concept for the choice of that vector function. This will have to be discussed for each single model separately.
We start off by running through the results of the foregoing chapter for all above mentioned problems without any lower order terms in detail.

3.1. Thermoelasticity models without dissipation or mass

The results of the following section were obtained by Wang and introduced to the author in a private communication.

3.1.1. Classical thermoelasticity

We are examining the Cauchy problem for the hyperbolic-parabolic coupled system of classical thermoelasticity, i.e.,

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0, \\ \theta_t - \kappa \theta_{xx} + \gamma_2 u_{tx} = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \theta(0,x) = \theta_0(x), \end{cases} \tag{3.1}$$

with the positiveness assumptions $\alpha, \kappa, \gamma_1\gamma_2 > 0$ for the constant coefficients.
With the use of $U = (u_+, u_-, \theta)^T$, $u_\pm = u_t \pm \sqrt{\alpha}\, u_x$, we equivalently transform (3.1) into

the initial value problem

$$\begin{cases} U_t + A_1 U_x - A_2 U_{xx} = 0, \\ U(0,x) = U_0(x) = (u_1 + \sqrt{\alpha}\, u_0', u_1 - \sqrt{\alpha}\, u_0', \theta_0)^T \end{cases} \tag{3.2}$$

with the matrices

$$A_1 = \begin{pmatrix} -\sqrt{\alpha} & 0 & \gamma_1 \\ 0 & \sqrt{\alpha} & \gamma_1 \\ \frac{\gamma_2}{2} & \frac{\gamma_2}{2} & 0 \end{pmatrix} \qquad \text{and} \qquad A_2 = \operatorname{diag}(0,0,\kappa).$$

Setting $d = 3$, (3.2) is a Cauchy problem for a system of type (2.2) with matrices A_1 and A_2 from $\mathbb{R}^{d\times d}$. We apply partial Fourier transformation and obtain for $V(t,\xi) = \mathcal{F}_{x\to\xi}(U(t,\cdot))(\xi)$ the system

$$V_t + (i\xi A_1 + \xi^2 A_2)V = 0. \tag{3.3}$$

In the following section we will follow the lines of our considerations in Section 2.2 closely and calculate all appearing matrices in the diagonalization procedure explicitly.

3.1.1.1. The diagonalization procedure

Diagonalization for small frequencies

We are considering (3.3) in $Z_{int}(\sigma) = \{|\xi| \le \sigma \ll 1\}$.

Step 0: Diagonalization modulo $\mathcal{O}(\xi)$-terms

The matrix A_0 is identically zero. Therefore we have

$$\lambda_{0,1} = \lambda_{0,2} = \lambda_{0,3} = 0, \qquad \tilde{L}^{(0)} = \tilde{R}^{(0)} = I,$$

and we are practically skipping step 0 of the procedure. We are thus starting from system (2.9) with $\tilde{V}^{(0)} = V$,

$$\Lambda_0 \equiv 0 \quad \text{and} \quad \tilde{A}_i^{(0)} = A_i.$$

Step 1: Diagonalization modulo $\mathcal{O}(\xi^2)$-terms

Substep 1: Having formula (2.11) in mind we choose the matrix $K_{(1)}$ to be identically zero,

$$K_{(1)} \equiv 0,$$

or equivalently $V^{(1)} = \tilde{V}^{(0)} = V$. Hence, the matrices in (2.10) are given by

$$A_i^{(1)} = \tilde{A}_i^{(0)} = A_i \quad \text{and} \quad A_3^{(1)} \equiv 0.$$

Substep 2: The matrix $A_1^{(1)} = A_1$ is symmetrizable with the eigenvalues

$$\lambda_{1,1} = -\sqrt{\alpha + \gamma_1\gamma_2} < \lambda_{1,2} = 0 < \lambda_{1,3} = \sqrt{\alpha + \gamma_1\gamma_2} \tag{3.4}$$

and matrices

$$\tilde{L}^{(1)} = \begin{pmatrix} -\frac{\gamma_2}{2}\, a_+ & -\frac{\gamma_2}{2}\, a_- & \gamma_1\gamma_2 \\ \gamma_2 & -\gamma_2 & 2\sqrt{\alpha} \\ \frac{\gamma_2}{2}\, a_- & \frac{\gamma_2}{2}\, a_+ & \gamma_1\gamma_2 \end{pmatrix}, \qquad \tilde{R}^{(1)} = \frac{1}{2(\alpha+\gamma_1\gamma_2)} \begin{pmatrix} -\frac{\gamma_1}{a_-} & \gamma_1 & \frac{\gamma_1}{a_+} \\ -\frac{\gamma_1}{a_+} & -\gamma_1 & \frac{\gamma_1}{a_-} \\ 1 & \sqrt{\alpha} & 1 \end{pmatrix},$$

$a_\pm = \sqrt{\alpha+\gamma_1\gamma_2} \pm \sqrt{\alpha}$, of corresponding left and right eigenvectors (The above matrices are of course just one possible choice.).
The eigenvalues in (3.4) are distinct. Hence, (A_1') and (B_1) are satisfied, and we have full diagonalizability of (3.3) for small frequencies. We will now go on with the procedure long enough to obtain necessary information on the asymptotic behavior of the eigenvalues of the coefficient matrix from (3.3).
The matrices in system (2.14) for $\tilde{V}^{(1)} = \tilde{L}^{(1)} V^{(1)}$ are given by

$$\Lambda_1 = \mathrm{diag}(\lambda_{1,1}, \lambda_{1,2}, \lambda_{1,3}),\ \tilde{A}_2^{(1)} = \frac{\kappa}{2(\alpha+\gamma_1\gamma_2)} \begin{pmatrix} \gamma_1\gamma_2 & \gamma_1\gamma_2\sqrt{\alpha} & \gamma_1\gamma_2 \\ 2\sqrt{\alpha} & 2\alpha & 2\sqrt{\alpha} \\ \gamma_1\gamma_2 & \gamma_1\gamma_2\sqrt{\alpha} & \gamma_1\gamma_2 \end{pmatrix},\ \tilde{A}_3^{(1)} \equiv 0,$$

i.e., $\tilde{V}^{(1)}$ satisfies

$$\tilde{V}_t^{(1)} + (i\xi\Lambda_1 + \xi^2 \tilde{A}_2^{(1)})\tilde{V}^{(1)} = 0. \tag{3.5}$$

Step 2: Diagonalization modulo $\mathcal{O}(\xi^3)$-terms

From the fact that (A_1') and (B_1) hold and looking at the diagonal entries of $\tilde{A}_2^{(1)}$ we know (without calculation) that there exists a matrix $K_{(2)}$ so that the vector $\tilde{V}^{(2)} = (I + i\xi K_{(2)})\tilde{V}^{(1)}$ satisfies the system (3.7). Let us nevertheless carry out the procedure in detail.

Substep 1: We choose, due to the fact that $\Lambda_0 = 0$,

$$K_{(1\frac{1}{2})} \equiv 0.$$

Hence, the matrices in system (2.15) for $V^{(1\frac{1}{2})} = \tilde{V}^{(1)}$ are given by

$$A_2^{(1\frac{1}{2})} = \tilde{A}_2^{(1)} \qquad \text{and} \qquad A_3^{(1\frac{1}{2})} \equiv 0.$$

Substep 2: The matrix $K_{(2)}$, with which we finish the diagonalization modulo $\mathcal{O}(\xi^3)$, is given by

$$K_{(2)} = \frac{\kappa}{4(\alpha+\gamma_1\gamma_2)^{\frac{3}{2}}} \begin{pmatrix} 0 & 2\gamma_1\gamma_2\sqrt{\alpha} & \gamma_1\gamma_2 \\ -4\sqrt{\alpha} & 0 & 4\sqrt{\alpha} \\ -\gamma_1\gamma_2 & -2\gamma_1\gamma_2\sqrt{\alpha} & 0 \end{pmatrix}. \tag{3.6}$$

The matrix

$$A_2^{(2)} = \frac{\kappa}{2(\alpha+\gamma_1\gamma_2)}\,\mathrm{diag}(\gamma_1\gamma_2, 2\alpha, \gamma_1\gamma_2) =: \Lambda_2$$

in system (2.18) for $V^{(2)} = (I + i\xi K_{(2)})V^{(1\frac{1}{2})}$ is thus already diagonal and $A_3^{(2)} = \mathcal{O}(\xi^3)$.

Substep 3: We have

$$\lambda_{2,1} = \frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}, \lambda_{2,2} = \frac{\alpha\kappa}{\alpha+\gamma_1\gamma_2}, \lambda_{2,3} = \frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}, \qquad \tilde{L}^{(2)} = \tilde{R}^{(2)} = I,$$

and $\tilde{V}^{(2)} = V^{(2)}$ satisfies

$$\tilde{V}_t^{(2)} + \left(i\xi\Lambda_1 + \xi^2\Lambda_2 + \tilde{A}_3^{(2)}\right)\tilde{V}^{(2)} = 0 \tag{3.7}$$

with $\tilde{A}_3^{(2)} = \mathcal{O}(\xi^3)$ for $\xi \to 0$. A further diagonalization is not necessary for our purposes.

Proposition 3.1. *(i) The characteristic roots $\mu_j = \mu_j(\xi)$ of the coefficient matrix $A(\xi) = i\xi A_1 + \xi^2 A_2$ from (3.3) behave for $|\xi| \le \sigma \ll 1$ as*

$$\begin{aligned}\mu_{1,3}(\xi) &= \mp i\sqrt{\alpha+\gamma_1\gamma_2}\,\xi + \frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}\,\xi^2 + \mathcal{O}(\xi^3),\\ \mu_2(\xi) &= \frac{\alpha\kappa}{\alpha+\gamma_1\gamma_2}\,\xi^2 + \mathcal{O}(\xi^3).\end{aligned}$$

(ii) The solution to the Cauchy problem of (3.3) has in $Z_{int}(\sigma) = \{|\xi| \le \sigma \ll 1\}$ the representation

$$V(t,\xi) = T_{int}^{-1}(\xi)\,\mathrm{diag}(e^{-\mu_1(\xi)t}, e^{-\mu_2(\xi)t}, e^{-\mu_3(\xi)t})T_{int}(\xi)\hat{U}_0(\xi),$$

where $T_{int}(\xi) = \tilde{L}(\xi)(I + i\xi K_{(2)})\tilde{L}^{(1)}$ with a matrix $\tilde{L}(\xi) = I + \mathcal{O}(\xi^2)$ for $\xi \to 0$.

Diagonalization for large frequencies

We are now considering (3.3) in $Z_{ext}(N) = \{|\xi| \ge \mathbb{N} \gg 1\}$.

Step 0: Diagonalization modulo $\mathcal{O}(\xi)$-terms

Having system (2.26) in mind, we start the procedure by diagonalizing A_2. This matrix is already diagonal with the eigenvalues

$$\hat{\lambda}_{2,1} = \hat{\lambda}_{2,2} = 0 < \hat{\lambda}_{2,3} = \kappa \qquad \text{and} \qquad \hat{\tilde{L}}^{(0)} = \hat{\tilde{R}}^{(0)} = I.$$

The vector $\tilde{V}^{(0)} = \hat{\tilde{L}}^{(0)}V$ thus satisfies

$$\tilde{V}_t^{(0)} + \left(\xi^2\hat{\Lambda}_2 + i\xi\hat{\tilde{A}}_1^{(0)}\right)\tilde{V}^{(0)} = 0$$

with

$$\hat{\Lambda}_2 = \mathrm{diag}(0,0,\kappa) = A_2, \qquad \hat{\tilde{A}}_1^{(0)} = A_1.$$

Step 1: Diagonalization modulo $\mathcal{O}(1)$-terms

Substep 1: We choose the matrix $\hat{K}_{(1)}$ from the transformation $V^{(1)} = (I + i\xi^{-1}\hat{K}_{(1)})\tilde{V}^{(0)}$ to take the form

$$\hat{K}_{(1)} = \frac{1}{2\kappa}\begin{pmatrix} 0 & 0 & -2\gamma_1 \\ 0 & 0 & -2\gamma_1 \\ \gamma_2 & \gamma_2 & 0 \end{pmatrix}.$$

The vector $V^{(1)}$ then satisfies

$$V_t^{(1)} + \left(\xi^2 \hat{\Lambda}_2 + i\xi \hat{A}_1^{(1)} + \hat{A}_0^{(1)} + \hat{A}_{-1}^{(1)}\right) V^{(1)} = 0$$

with

$$\hat{A}_1^{(1)} = \operatorname{diag}(-\sqrt{\alpha}, \sqrt{\alpha}, 0), \qquad \hat{A}_0^{(1)} = \frac{1}{2\kappa} \begin{pmatrix} \gamma_1\gamma_2 & \gamma_1\gamma_2 & 2\gamma_1\sqrt{\alpha} \\ \gamma_1\gamma_2 & \gamma_1\gamma_2 & -2\gamma_1\sqrt{\alpha} \\ \gamma_2\sqrt{\alpha} & -\gamma_2\sqrt{\alpha} & -2\gamma_1\gamma_2 \end{pmatrix}$$

and $\hat{A}_{-1}^{(1)} = \mathcal{O}(\xi^{-1})$ for $|\xi| \to \infty$.

Substep 2: The matrix $\hat{A}_1^{(1)}$ is already diagonal. Hence, we can practically skip this substep, i.e., the vector $\tilde{V}^{(1)} = \hat{\tilde{L}}^{(1)} V^{(1)}$ satisfies with

$$\hat{\lambda}_{1,1} = -\sqrt{\alpha},\ \hat{\lambda}_{1,2} = \sqrt{\alpha},\ \hat{\lambda}_{1,3} = 0 \qquad \text{and} \qquad \hat{\tilde{L}}^{(1)} = \hat{\tilde{R}}^{(1)} = I$$

the system

$$\tilde{V}_t^{(1)} + \left(\xi^2 \hat{\Lambda}_2 + i\xi \hat{\Lambda}_1 + \hat{\tilde{A}}_0^{(1)} + \hat{\tilde{A}}_{-1}^{(1)}\right) \tilde{V}^{(1)} = 0$$

with

$$\hat{\Lambda}_1 = \operatorname{diag}(-\sqrt{\alpha}, \sqrt{\alpha}, 0), \quad \hat{\tilde{A}}_0^{(1)} = \hat{A}_0^{(1)} \qquad \text{and} \qquad \hat{\tilde{A}}_{-1}^{(1)} = \mathcal{O}(\xi^{-1}).$$

Moreover, we see that $(\hat{\mathrm{A}}_1')$ and $(\hat{\mathrm{B}}_1)$ are satisfied. Therefore, we have full diagonalizability of (3.3) for large frequencies.

Step 2: Diagonalization modulo $\mathcal{O}(\xi^{-1})$-terms

Again from the fact that $(\hat{\mathrm{A}}_1')$ and $(\hat{\mathrm{B}}_1)$ hold and considering the diagonal entries of $\hat{A}_0^{(1)}$ we know that there exist matrices $\hat{K}_{(1\frac{1}{2})}$ and $\hat{K}_{(2)}$ so that the vector $\tilde{V}^{(2)} = (I + i\xi^{-1}\hat{K}_{(2)})(I + \xi^{-2}\hat{K}_{(1\frac{1}{2})})\tilde{V}^{(1)}$ satisfies

$$\tilde{V}_t^{(2)} + \left(\xi^2 \hat{\Lambda}_2 + i\xi \hat{\Lambda}_1 + \hat{\Lambda}_0 + \hat{\tilde{A}}_{-1}^{(2)}\right) \tilde{V}^{(2)} = 0$$

with $\hat{\Lambda}_0 = \operatorname{diag}\left(\frac{\gamma_1\gamma_2}{2\kappa}, \frac{\gamma_1\gamma_2}{2\kappa}, -\frac{\gamma_1\gamma_2}{\kappa}\right)$ and $\hat{\tilde{A}}_{-1}^{(2)} = \mathcal{O}(\xi^{-1})$ for $|\xi| \to \infty$. The matrices $\hat{K}_{(1\frac{1}{2})}$ and $\hat{K}_{(2)}$ can be calculated explicitly and take the form

$$\hat{K}_{(1\frac{1}{2})} = \frac{\sqrt{\alpha}}{2\kappa^2} \begin{pmatrix} 0 & 0 & -2\gamma_1 \\ 0 & 0 & 2\gamma_1 \\ \gamma_2 & -\gamma_2 & 0 \end{pmatrix} \qquad \text{and} \qquad \hat{K}_{(2)} = \frac{\gamma_1\gamma_2}{4\kappa\sqrt{\alpha}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We do not need to diagonalize any further for our purposes.
We have obtained:

Proposition 3.2. *(i) The characteristic roots $\mu_j = \mu_j(\xi)$ of the coefficient matrix $A(\xi) = i\xi A_1 + \xi^2 A_2$ from (3.3) behave for $|\xi| \geq N \gg 1$ as*

$$\begin{aligned} \mu_{1,2}(\xi) &= \mp i\sqrt{\alpha}\,\xi + \frac{\gamma_1\gamma_2}{2\kappa} + \mathcal{O}(\xi^{-1}), \\ \mu_3(\xi) &= \kappa\xi^2 - \frac{\gamma_1\gamma_2}{\kappa} + \mathcal{O}(\xi^{-1}). \end{aligned}$$

(ii) The solution to the Cauchy problem of (3.3) has in $Z_{ext}(N) = \{|\xi| \geq N \gg 1\}$ *the representation*

$$V(t,\xi) = T_{ext}^{-1}(\xi)\,\mathrm{diag}(e^{-\mu_1(\xi)t}, e^{-\mu_2(\xi)t}, e^{-\mu_3(\xi)t})T_{ext}(\xi)\hat{U}_0(\xi),$$

where

$$T_{ext}(\xi) = \hat{\hat{L}}(\xi)(I + i\xi^{-1}\hat{K}_{(2)})(I + \xi^{-2}\hat{K}_{(1\frac{1}{2})})(I + i\xi^{-1}\hat{K}_{(1)})$$

with a matrix $\hat{\hat{L}}(\xi) = I + \mathcal{O}(\xi^{-2})$ *for* $|\xi| \to \infty$.

Diagonalization for bounded frequencies away from zero

At last we are analyzing (3.3) in $Z_{mid}(\sigma, N) = \{\sigma \leq |\xi| \leq \mathbb{N}\}$ and start with:

Lemma 3.3. *The assumption (C) holds.*

Proof. We assume that there is a purely imaginary eigenvalue $\mu = i\,a$ with $a \in \mathbb{R}$ of the coefficient matrix $A(\xi) = i\xi A_1 + \xi^2 A_2$ of (3.3) for $\xi \neq 0$. The eigenvalue μ satisfies

$$0 = \det(\mu I - A(\xi)) = \mu^3 - \kappa\xi^2\mu^2 + (\alpha + \gamma_1\gamma_2)\xi^2\mu - \alpha\kappa\xi^4.$$

From here we conclude

$$\begin{aligned} -a^3 + (\alpha + \gamma_1\gamma_2)\xi^2 a &= 0, \\ \kappa\xi^2 a^2 - \alpha\kappa\xi^4 &= 0, \end{aligned}$$

which leads to a contradiction immediately.
The compactness of $Z_{mid}(\sigma, N)$ and the continuity of $\mathrm{Re}\,\mu(\xi)$ together with $\mathrm{Re}\,\mu(\xi) > 0$ for $|\xi| = N$ and all $\mu(\xi) \in \mathrm{spec}\,(A(\xi))$ yield the assertion. □

Hence, we have:

Proposition 3.4. *The solution* V *to the Cauchy problem of (3.3) satisfies in* $Z_{mid}(\sigma, N)$

$$|V(t,\xi)| \lesssim e^{-ct}|\hat{U}_0(\xi)|,$$

where c *is a positive constant.*

3.1.1.2. Results

We will now, based on the information derived in the last section, state with the help of our observations in the Sections 2.3, 2.4, 2.5 and 2.6 results on well-posedness, a L^p-L^q decay estimate, a diffusion phenomenon and on the propagation of singularities for our Cauchy problem.
Since the upcoming results are simple conclusions of the ones stated in the just mentioned sections, we can omit the proofs almost completely.

Theorem 3.5. *(Well-posedness result)*
We consider the Cauchy problem

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0, \\ \theta_t - \kappa \theta_{xx} + \gamma_2 u_{tx} = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \theta(0,x) = \theta_0(x) \end{cases}$$

in $\mathbb{R}_{\geq 0} \times \mathbb{R}$ *with* $\alpha, \kappa, \gamma_1\gamma_2 > 0$ *and assume*

$$u_0 \in H^{s+1}, u_1, \theta_0 \in H^s$$

for a fixed $s \in \mathbb{R}$. *Then there exists a unique solution satisfying*

$$\begin{aligned} &u \in C([0,\infty), H^{s+1}) \cap C^1([0,\infty), H^s), \\ &\theta \in C([0,\infty), H^s). \end{aligned}$$

Theorem 3.6. *(L^p-L^q decay estimate)*
We assume $u_0, u_1, \theta_0 \in \mathcal{S}$. *Then the following* L^p-L^q *decay estimate holds for solutions to the above Cauchy problem:*

$$\|(u_t, u_x, \theta)\|_{L^q} \lesssim (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|(\langle D \rangle u_0, u_1, \theta_0)\|_{L^{p,r_p}}.$$

Here $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $r_p = \frac{1}{p} - \frac{1}{q}$.

For the result on the diffusion phenomenon we define the parabolic reference system:

$$\begin{cases} W_t + M_1 W_x - M_2 W_{xx} = 0, \\ W(0,x) = W_0(x) \end{cases} \tag{3.8}$$

with

$$\begin{aligned} M_1 &= \operatorname{diag}\left(-\sqrt{\alpha + \gamma_1\gamma_2}, 0, \sqrt{\alpha + \gamma_1\gamma_2}\right), \\ M_2 &= \operatorname{diag}\left(\frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}, \frac{\alpha\kappa}{\alpha+\gamma_1\gamma_2}, \frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}\right), \\ W_0(x) &= \tilde{L}^{(1)} U_0(x) = \begin{pmatrix} -\gamma_2\alpha & -\gamma_2\sqrt{\alpha+\gamma_1\gamma_2} & \gamma_1\gamma_2 \\ 2\gamma_2\sqrt{\alpha} & 0 & 2\sqrt{\alpha} \\ -\gamma_2\alpha & \gamma_2\sqrt{\alpha+\gamma_1\gamma_2} & \gamma_1\gamma_2 \end{pmatrix} \begin{pmatrix} u_0' \\ u_1 \\ \theta_0 \end{pmatrix} \end{aligned}$$

(compare to Proposition 3.1). We can now state:

Theorem 3.7. *We assume* $u_0, u_1, \theta_0 \in \mathcal{S}$. *Then we obtain for the solution* U *to the Cauchy problem (3.2) the estimate*

$$\left\| U - \tilde{R}^{(1)} W \right\|_{L^q} \lesssim (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|(\langle D \rangle u_0, u_1, \theta_0)\|_{L^{p,r_p}}$$

for dual values $q, 1 < p \leq 2$, $r_p = \frac{1}{p} - \frac{1}{q}$ *and* $\tilde{R}^{(1)}$ *from Section 3.1.1.1.*

Proof. We apply the result of Theorem 2.24 to $\phi_{int}(D)\left(U - \tilde{R}^{(1)}W\right)$ and use this together with estimates for $(1-\phi_{int}(D))U$ and $(1-\phi_{int}(D))\tilde{R}^{(1)}W$, where we obtain exponential decay. □

The asymptotic profiles of solutions to the Cauchy problem (3.1) (from the viewpoint of decay estimates) are thus parabolic. More precise, they are given by solutions to three Cauchy problems for heat equations, which are 'shifted' in two cases.
Taking into account that the matrix T_{-1} from (2.51) is given by $T_{-1} = -(\hat{K}_{(1)} + \hat{K}_{(2)})$, we conclude on the propagation of singularities:

Theorem 3.8. *We consider the Cauchy problem (3.1) and assume for a fixed $s \in \mathbb{R}$:*

(i) $u_1 \pm \sqrt{\alpha}\, u_0', \theta_0 \in [H^s(\mathbb{R}) \cap H^{s+1}(\mathbb{R}\backslash\{x_0\})] \backslash H^{s+1}(\mathbb{R})$. *Then we obtain for any $t > 0$*

$$\begin{aligned}
&u_t(t,\cdot), u_x(t,\cdot) \in H^s(\mathbb{R}) \cap H^{s+1}(\mathbb{R}\backslash\{x_0 \pm \sqrt{\alpha}t\}),\\
&u_t(t,\cdot), u_x(t,\cdot) \notin H^{s+1}_{loc}(\{x_0 \pm \sqrt{\alpha}t\}),\\
&\theta(t,\cdot) \in H^{s+1}(\mathbb{R}) \cap H^{s+2}(\mathbb{R}\backslash\{x_0 \pm \sqrt{\alpha}t\}),\\
&\theta(t,\cdot) \notin H^{s+2}_{loc}(\{x_0 \pm \sqrt{\alpha}t\}).
\end{aligned}$$

(ii) $u_1, \theta_0 \in H^s(\mathbb{R}) \cap C^\infty(\mathbb{R}\backslash\{x_0\})$, $u_0 \in H^{s+1}(\mathbb{R}) \cap C^\infty(\mathbb{R}\backslash\{x_0\})$. *Then we obtain*

$$(u,\theta) \in C^\infty(((0,\infty)\times\mathbb{R})\backslash\{(t, x_0 \pm \sqrt{\alpha}t) : t > 0\}).$$

The propagation of weak singularities is thus dominated by the hyperbolic part with exactly the same characteristic lines. The well-known smoothing effect of the heat equation is not observed. However, there is a nonlocal combination of the components of U from (3.2), which shows that smoothing effect, as can be seen in the proof of Theorem 2.26 in (2.49). Similar phenomena were already observed in [RW98, RW99a, Wan02].

To convince the reader of the fact that our procedure does not only work for the well-known classical thermoelasticity model, we will now apply it (with detailed calculations) to the second sound model in 1D. Lots of the following calculations can be found in [YW06a].

3.1.2. Thermoelasticity with second sound

We consider the Cauchy problem for the linear thermoelastic system with second sound in one space variable, that is,

$$\begin{cases}
u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0,\\
\theta_t + q_x + \gamma_2 u_{tx} = 0,\\
\tau q_t + q + \kappa \theta_x = 0,\\
u(0,x) = u_0(x), u_t(0,x) = u_1(x), \theta(0,x) = \theta_0(x), q(0,x) = q_0(x)
\end{cases} \tag{3.9}$$

with the positiveness conditions $\alpha, \tau, \kappa, \gamma_1\gamma_2 > 0$.
We introduce, as in the last section, $u_\pm = u_t \pm \sqrt{\alpha}\, u_x$ and $U = (u_+, u_-, \theta, q)^T$ and transform (3.9) equivalently into an initial value problem for a hyperbolic system:

$$\begin{cases} U_t + A_0 U + A_1 U_x = 0, \\ U(0,x) = U_0(x) = (u_1 + \sqrt{\alpha}\, u_0', u_1 - \sqrt{\alpha}\, u_0', \theta_0, q_0)^T, \end{cases} \tag{3.10}$$

where

$$A_0 = \operatorname{diag}(0,0,0,1/\tau) \qquad \text{and} \qquad A_1 = \begin{pmatrix} -\sqrt{\alpha} & 0 & \gamma_1 & 0 \\ 0 & \sqrt{\alpha} & \gamma_1 & 0 \\ \frac{\gamma_2}{2} & \frac{\gamma_2}{2} & 0 & 1 \\ 0 & 0 & \frac{\kappa}{\tau} & 0 \end{pmatrix}.$$

With $d = 4$ system (3.10) is one of type (2.2) with matrices A_0 and A_1 from $\mathbb{R}^{d\times d}$.
After partial Fourier transformation we obtain the system

$$V_t + (A_0 + i\xi A_1)V = 0 \tag{3.11}$$

for $V(t,\xi) = \mathcal{F}_{x\to\xi}(U(t,\cdot))(\xi)$.

3.1.2.1. The diagonalization procedure

Diagonalization for small frequencies

Step 0: Diagonalization modulo $\mathcal{O}(\xi)$-terms

The eigenvalues of the diagonal matrix A_0 are

$$\lambda_{0,1} = \lambda_{0,2} = \lambda_{0,3} = 0 \, < \, \lambda_{0,4} = \frac{1}{\tau}.$$

The matrices of left and right eigenvectors are given by $\tilde{L}^{(0)} = \tilde{R}^{(0)} = I$, and $\tilde{V}^{(0)} = V$ satisfies system (2.9) with

$$\Lambda_0 = A_0,\ \tilde{A}_1^{(0)} = A_1 \quad \text{and} \quad \tilde{A}_2^{(0)} \equiv 0.$$

Step 1: Diagonalization modulo $\mathcal{O}(\xi^2)$-terms

Substep 1: We introduce the vector $V^{(1)} = (I + i\xi K_{(1)})\tilde{V}^{(0)}$ with

$$K_{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tau \\ 0 & 0 & \kappa & 0 \end{pmatrix}.$$

The matrices in (2.10) are therefore given by

$$A_1^{(1)} = \begin{pmatrix} -\sqrt{\alpha} & 0 & \gamma_1 & 0 \\ 0 & \sqrt{\alpha} & \gamma_1 & 0 \\ \frac{\gamma_2}{2} & \frac{\gamma_2}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_2^{(1)} = \begin{pmatrix} 0 & 0 & 0 & -\tau\gamma_1 \\ 0 & 0 & 0 & -\tau\gamma_1 \\ 0 & 0 & \kappa & 0 \\ -\frac{\kappa\gamma_2}{2} & -\frac{\kappa\gamma_2}{2} & 0 & -\kappa \end{pmatrix} \quad \text{and} \quad A_3^{(1)} = \mathcal{O}(\xi^3).$$

Substep 2: The matrix $A_1^{(1)}$ is symmetrizable with the eigenvalues

$$\lambda_{1,1} = -\sqrt{\alpha+\gamma_1\gamma_2} < \lambda_{1,2} = 0 < \lambda_{1,3} = \sqrt{\alpha+\gamma_1\gamma_2}, \qquad \lambda_{1,4} = 0,$$

and matrices

$$\tilde{L}^{(1)} = \begin{pmatrix} -\frac{\gamma_2}{2}\,a_+ & -\frac{\gamma_2}{2}\,a_- & \gamma_1\gamma_2 & 0 \\ \gamma_2 & -\gamma_2 & 2\sqrt{\alpha} & 0 \\ \frac{\gamma_2}{2}\,a_- & \frac{\gamma_2}{2}\,a_+ & \gamma_1\gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tilde{R}^{(1)} = \frac{1}{2(\alpha+\gamma_1\gamma_2)} \begin{pmatrix} -\frac{\gamma_1}{a_-} & \gamma_1 & \frac{\gamma_1}{a_+} & 0 \\ -\frac{\gamma_1}{a_+} & -\gamma_1 & \frac{\gamma_1}{a_-} & 0 \\ 1 & \sqrt{\alpha} & 1 & 0 \\ 0 & 0 & 0 & 2(\alpha+\gamma_1\gamma_2) \end{pmatrix},$$

$a_\pm = \sqrt{\alpha+\gamma_1\gamma_2} \pm \sqrt{\alpha}$, of corresponding left and right eigenvectors (one possible choice). The matrices in (2.14) for $\tilde{V}^{(1)} = \tilde{L}^{(1)} V^{(1)}$ are given by $\Lambda_1 = \operatorname{diag}(\lambda_{1,1}, \lambda_{1,2}, \lambda_{1,3}, 0)$,

$$\tilde{A}_2^{(1)} = \frac{\kappa}{2(\alpha+\gamma_1\gamma_2)} \begin{pmatrix} \gamma_1\gamma_2 & \gamma_1\gamma_2\sqrt{\alpha} & \gamma_1\gamma_2 & \frac{2\tau\gamma_1\gamma_2(\alpha+\gamma_1\gamma_2)^{\frac{3}{2}}}{\kappa} \\ 2\sqrt{\alpha} & 2\alpha & 2\sqrt{\alpha} & 0 \\ \gamma_1\gamma_2 & \gamma_1\gamma_2\sqrt{\alpha} & \gamma_1\gamma_2 & -\frac{2\tau\gamma_1\gamma_2(\alpha+\gamma_1\gamma_2)^{\frac{3}{2}}}{\kappa} \\ \sqrt{\alpha+\gamma_1\gamma_2} & 0 & -\sqrt{\alpha+\gamma_1\gamma_2} & -2(\alpha+\gamma_1\gamma_2) \end{pmatrix}$$

and $\tilde{A}_3^{(1)} = \mathcal{O}(\xi^3)$ for $\xi \to 0$.
The assumptions (A_1') and (B_1) are satisfied, and we therefore have full diagonalizability of (3.11) for small frequencies. We will now perform one more step of the procedure to obtain sufficient information on the asymptotic behavior of the eigenvalues of the coefficient matrix $A_0 + i\xi A_1$.

Step 2: Diagonalization modulo $\mathcal{O}(\xi^3)$-terms

From the fact that (A_1') and (B_1) hold and considering the structure of the diagonal matrices Λ_0 and Λ_1 we know that there are matrices $K_{(1\frac{1}{2})}$ and $K_{(2)}$ so that $\tilde{V}^{(2)} = (I + i\xi K_{(2)})(I + \xi^2 K_{(1\frac{1}{2})})\tilde{V}^{(1)}$ satisfies

$$\tilde{V}_t^{(2)} + \left(\Lambda_0 + i\xi\Lambda_1 + \xi^2\Lambda_2 + \tilde{A}_3^{(2)}\right)\tilde{V}^{(2)} = 0 \tag{3.12}$$

with $\Lambda_2 = \operatorname{diag}\left(\frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}, \frac{\alpha\kappa}{\alpha+\gamma_1\gamma_2}, \frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}, -\kappa\right)$ and $\tilde{A}_3^{(2)} = \mathcal{O}(\xi^3)$ for $\xi \to 0$.
In detail the calculations look like this:

Substep 1: The vector $V^{(1\frac{1}{2})} = (I + \xi^2 K_{(1\frac{1}{2})})\tilde{V}^{(1)}$ with

$$K_{(1\frac{1}{2})} = \frac{\tau}{2\sqrt{\alpha+\gamma_1\gamma_2}} \begin{pmatrix} 0 & 0 & 0 & -2\tau\gamma_1\gamma_2(\alpha+\gamma_1\gamma_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\tau\gamma_1\gamma_2(\alpha+\gamma_1\gamma_2) \\ \kappa & 0 & -\kappa & 0 \end{pmatrix}$$

satisfies (2.15) with $A_3^{(1\frac{1}{2})} = \mathcal{O}(\xi^3)$ and

$$A_2^{(1\frac{1}{2})} = \frac{\kappa}{2(\alpha+\gamma_1\gamma_2)} \begin{pmatrix} \gamma_1\gamma_2 & \gamma_1\gamma_2\sqrt{\alpha} & \gamma_1\gamma_2 & 0 \\ 2\sqrt{\alpha} & 2\alpha & 2\sqrt{\alpha} & 0 \\ \gamma_1\gamma_2 & \gamma_1\gamma_2\sqrt{\alpha} & \gamma_1\gamma_2 & 0 \\ 0 & 0 & 0 & -2(\alpha+\gamma_1\gamma_2) \end{pmatrix}.$$

Substep 2: We introduce the vector $V^{(2)} = (I + i\xi K_{(2)})V^{(1\frac{1}{2})}$ with

$$K_{(2)} = \frac{\kappa}{4(\alpha+\gamma_1\gamma_2)^{\frac{3}{2}}} \begin{pmatrix} 0 & 2\gamma_1\gamma_2\sqrt{\alpha} & \gamma_1\gamma_2 & 0 \\ -4\sqrt{\alpha} & 0 & 4\sqrt{\alpha} & 0 \\ -\gamma_1\gamma_2 & -2\gamma_1\gamma_2\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices in (2.18) are given by $A_3^{(2)} = \mathcal{O}(\xi^3)$ and

$$A_2^{(2)} = \operatorname{diag}\left(\frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}, \frac{\alpha\kappa}{\alpha+\gamma_1\gamma_2}, \frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}, -\kappa\right) = \Lambda_2.$$

Substep 3: We have

$$\lambda_{2,1} = \frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}, \lambda_{2,2} = \frac{\alpha\kappa}{\alpha+\gamma_1\gamma_2}, \ \lambda_{2,3} = \frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}, \ \lambda_{2,4} = -\kappa,$$

$\tilde{L}^{(2)} = \tilde{R}^{(2)} = I$, and $\tilde{V}^{(2)} = V^{(2)}$ satisfies (3.12).
Hence, we conclude:

Proposition 3.9. *(i) The characteristic roots $\mu_j = \mu_j(\xi)$ of the coefficient matrix $A(\xi) = A_0 + i\xi A_1$ from (3.11) behave for $|\xi| \le \sigma \ll 1$ as*

$$\begin{aligned} \mu_{1,3}(\xi) &= \mp i\sqrt{\alpha+\gamma_1\gamma_2}\,\xi + \frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}\,\xi^2 + \mathcal{O}(\xi^3), \\ \mu_2(\xi) &= \frac{\alpha\kappa}{\alpha+\gamma_1\gamma_2}\,\xi^2 + \mathcal{O}(\xi^3), \\ \mu_4(\xi) &= \tfrac{1}{\tau} - \kappa\,\xi^2 + \mathcal{O}(\xi^3). \end{aligned}$$

(ii) The solution to the Cauchy problem of (3.11) has in $Z_{int}(\sigma) = \{|\xi| \le \sigma \ll 1\}$ the representation

$$V(t,\xi) = T_{int}^{-1}(\xi)\operatorname{diag}(e^{-\mu_1(\xi)t}, e^{-\mu_2(\xi)t}, e^{-\mu_3(\xi)t}, e^{-\mu_4(\xi)t})T_{int}(\xi)\hat{U}_0(\xi),$$

where $T_{int}(\xi) = \tilde{L}(\xi)(I+i\xi K_{(2)})(I+\xi^2 K_{(1\frac{1}{2})})\tilde{L}^{(1)}(I+i\xi K_{(1)})$ with a matrix $\tilde{L}(\xi) = I + \mathcal{O}(\xi^2)$ for $\xi \to 0$.

Remark 3.1. Note that as the relaxation parameter τ goes to zero, and thus, as the second sound model of thermoelasticity formally converges to the classical model, the first three components of V in Proposition 3.9, *(ii)* converge (pointwise in $Z_{int}(\sigma)$) to the solution of (3.3) from Proposition 3.1, *(ii)*.

Diagonalization for large frequencies

Step 0: Diagonalization modulo $\mathcal{O}(\xi)$-terms

The matrix A_2 is identically zero, so are therefore all its eigenvalues $\hat{\lambda}_{2,j}$, and we simply rewrite system (3.11) with the use of $\hat{\tilde{L}}^{(0)} = \hat{\tilde{R}}^{(0)} = I$ and $\tilde{V}^{(0)} = V$ into

$$\tilde{V}_t^{(0)} + \left(i\xi\hat{\tilde{A}}_1^{(0)} + \hat{\tilde{A}}_0^{(0)}\right)\tilde{V}^{(0)} = 0$$

with $\hat{\hat{A}}_i^{(0)} = A_i$.

Step 1: Diagonalization modulo $\mathcal{O}(1)$-terms

Substep 1: We are practically skipping this substep as well and just rewrite the above system with $\hat{K}_{(1)} \equiv 0$, $V^{(1)} = \tilde{V}^{(0)} = V$ and $\hat{A}_i^{(1)} = \hat{\hat{A}}_i^{(0)} = A_i$ into

$$V_t^{(1)} + \left(i\xi \hat{A}_1^{(1)} + \hat{A}_0^{(1)}\right) V^{(1)} = 0.$$

Substep 2: We diagonalize the symmetrizable matrix $\hat{A}_1^{(1)} = A_1$. The eigenvalues, given by

$$\begin{aligned} \hat{\lambda}_{1,1/4} &= \mp\sqrt{\frac{1}{2}\left(\alpha + \frac{\kappa}{\tau} + \gamma_1\gamma_2 + \sqrt{\left(\alpha + \frac{\kappa}{\tau} + \gamma_1\gamma_2\right)^2 - 4\frac{\alpha\kappa}{\tau}}\right)}, \\ \hat{\lambda}_{1,2/3} &= \mp\sqrt{\frac{1}{2}\left(\alpha + \frac{\kappa}{\tau} + \gamma_1\gamma_2 - \sqrt{\left(\alpha + \frac{\kappa}{\tau} + \gamma_1\gamma_2\right)^2 - 4\frac{\alpha\kappa}{\tau}}\right)}, \end{aligned} \tag{3.13}$$

are real, distinct and nonzero.
Matrices of left and right eigenvectors are given by

$$\begin{aligned} \hat{\hat{L}}^{(1)^T} &= \left(\hat{\hat{l}}_1^{(1)^T}, \ldots, \hat{\hat{l}}_4^{(1)^T}\right) \qquad \text{with} \quad \hat{\hat{l}}_j^{(1)} = c_j\left(\frac{\gamma_2}{\hat{\lambda}_{1,j} + \sqrt{\alpha}}, \frac{\gamma_2}{\hat{\lambda}_{1,j} - \sqrt{\alpha}}, 2, \frac{2}{\hat{\lambda}_{1,j}}\right), \\ \hat{\hat{R}}^{(1)} &= \left(\hat{\hat{r}}_1^{(1)}, \ldots, \hat{\hat{r}}_4^{(1)}\right) \qquad \text{with} \quad \hat{\hat{r}}_j^{(1)} = \left(\frac{\gamma_1}{\hat{\lambda}_{1,j} + \sqrt{\alpha}}, \frac{\gamma_1}{\hat{\lambda}_{1,j} - \sqrt{\alpha}}, 1, \frac{\kappa}{\tau\hat{\lambda}_{1,j}}\right)^T \end{aligned}$$

and

$$c_j = \left(\frac{\gamma_1\gamma_2}{(\hat{\lambda}_{1,j} + \sqrt{\alpha})^2} + \frac{\gamma_1\gamma_2}{(\hat{\lambda}_{1,j} - \sqrt{\alpha})^2} + 2 + \frac{2\kappa}{\tau\hat{\lambda}_{1,j}^2}\right)^{-1} > 0$$

(cf. [YW06a]). The vector $\tilde{V}^{(1)} = \hat{\hat{L}}^{(1)} V^{(1)}$ thus satisfies

$$\tilde{V}_t^{(1)} + \left(i\xi\hat{\Lambda}_1 + \hat{\hat{A}}_0^{(1)}\right)\tilde{V}^{(1)} = 0,$$

$$\hat{\Lambda}_1 = \operatorname{diag}(\hat{\lambda}_{1,1}, \ldots, \hat{\lambda}_{1,4}), \quad \hat{\hat{A}}_0^{(1)} = (a_{ij})_{i,j=1}^4 \quad \text{with } a_{ij} = \frac{2\kappa c_i}{\tau^2\hat{\lambda}_{1,i}\,\hat{\lambda}_{1,j}}. \tag{3.14}$$

Once again, the assumptions ($\hat{\mathrm{A}}_1'$) and ($\hat{\mathrm{B}}_1$) are satisfied. Therefore, we have full diagonalizability of (3.11) for large frequencies.

Step 2: Diagonalization modulo $\mathcal{O}(\xi^{-1})$-terms

Substep 1: The vector $V^{\left(1\frac{1}{2}\right)} = \tilde{V}^{(1)}$, $\hat{K}_{\left(1\frac{1}{2}\right)} \equiv 0$, satisfies with $\hat{A}_0^{\left(1\frac{1}{2}\right)} = \hat{\hat{A}}_0^{(1)}$

$$V_t^{\left(1\frac{1}{2}\right)} + \left(i\xi\hat{\Lambda}_1 + \hat{A}_0^{\left(1\frac{1}{2}\right)}\right) V^{\left(1\frac{1}{2}\right)} = 0.$$

Substep 2: We introduce $V^{(2)} = (I + i\xi^{-1}\hat{K}_{(2)})V^{\left(1\frac{1}{2}\right)}$ with

$$\hat{K}_{(2)} = (k_{ij})_{i,j=1}^4, \quad k_{ij} = \begin{cases} \dfrac{a_{ij}}{\hat{\lambda}_{1,i} - \hat{\lambda}_{1,j}}, & i \neq j, \\ 0, & i = j. \end{cases}$$

The vector $V^{(2)}$ then satisfies

$$V_t^{(2)} + \left(i\xi\hat{\Lambda}_1 + \hat{A}_0^{(2)} + \hat{A}_{-1}^{(2)}\right) V^{(2)} = 0$$

with

$$\hat{A}_0^{(2)} = \operatorname{diag}(a_{11} \dots, a_{44}) =: \hat{\Lambda}_0 \quad \text{and} \quad \hat{A}_{-1}^{(2)} = \mathcal{O}(\xi^{-1}) \text{ for } |\xi| \to \infty.$$

The terms a_{jj} from (3.14) are all positive.

Substep 3: The matrix $\hat{A}_0^{(2)}$ is already diagonal. Hence, we have $\hat{\tilde{L}}^{(2)} = \hat{\tilde{R}}^{(2)} = I$, and $\tilde{V}^{(2)} = V^{(2)}$ satisfies

$$\tilde{V}_t^{(2)} + \left(i\xi\hat{\Lambda}_1 + \hat{\Lambda}_0 + \hat{\tilde{A}}_{-1}^{(2)}\right) \tilde{V}^{(2)} = 0$$

with $\hat{\tilde{A}}_{-1}^{(2)} = \hat{A}_{-1}^{(2)} = \mathcal{O}(\xi^{-1})$ for $|\xi| \to \infty$.

Proposition 3.10. *(i) The characteristic roots $\mu_j = \mu_j(\xi)$ of the coefficient matrix $A(\xi) = A_0 + i\xi A_1$ from (3.11) behave for $|\xi| \geq N \gg 1$ as*

$$\mu_j(\xi) = i\hat{\lambda}_{1,j}\,\xi + a_{jj} + \mathcal{O}(\xi^{-1}).$$

The numbers $\hat{\lambda}_{1,j}$ are taken from (3.13) and the positive a_{jj} from (3.14).

(ii) The solution to the Cauchy problem of (3.11) has in $Z_{ext}(N) = \{|\xi| \geq N \gg 1\}$ the representation

$$V(t,\xi) = T_{ext}^{-1}(\xi)\operatorname{diag}(e^{-\mu_1(\xi)t}, \dots, e^{-\mu_4(\xi)t})T_{ext}(\xi)\hat{U}_0(\xi)$$

with $T_{ext}(\xi) = \hat{\tilde{L}}(\xi)(I + i\xi^{-1}\hat{K}_{(2)})\hat{\tilde{L}}^{(1)}$ and a matrix $\hat{\tilde{L}}(\xi) = I + \mathcal{O}(\xi^{-2})$ for $|\xi| \to \infty$.

Diagonalization for bounded frequencies away from zero

Lemma 3.11. *The assumption (C) holds.*

Proof. A purely imaginary eigenvalue $\mu = i\,a$, $a \in \mathbb{R}$, of the coefficient matrix $A(\xi) = A_0 + i\xi A_1$ of (3.11) for a $\xi \neq 0$ satisfies

$$0 = \det(\mu I - A(\xi)) = \mu^4 - \frac{1}{\tau}\mu^3 + \left(\alpha + \frac{\kappa}{\tau} + \gamma_1\gamma_2\right)\xi^2\mu^2 - \frac{\alpha + \gamma_1\gamma_2}{\tau}\,\xi^2\mu + \frac{\alpha\kappa}{\tau}\,\xi^4$$

and thus

$$\begin{aligned} a^4 - \left(\alpha + \frac{\kappa}{\tau} + \gamma_1\gamma_2\right)\xi^2 a^2 + \frac{\alpha\kappa}{\tau}\,\xi^4 &= 0, \\ \frac{1}{\tau}\,a^3 - \frac{\alpha + \gamma_1\gamma_2}{\tau}\,\xi^2 a &= 0. \end{aligned}$$

We obtain a contradiction. The continuity of $\operatorname{Re}\mu(\xi)$ together with $\operatorname{Re}\mu(\xi) > 0$ for $|\xi| = N$ and all $\mu(\xi) \in \operatorname{spec}(A(\xi))$ and the compactness of $Z_{mid}(\sigma, N)$ prove the assertion. ☐

Hence, we have:

Proposition 3.12. *The solution V to the Cauchy problem of (3.11) satisfies in $Z_{mid}(\sigma, N)$*

$$|V(t,\xi)| \lesssim e^{-ct}|\hat{U}_0(\xi)|,$$

where c is a positive constant.

3.1.2.2. Results

From our foregoing observations we immediately conclude:

Theorem 3.13. *(Well-posedness result)*
We consider the Cauchy problem

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0, \\ \theta_t + q_x + \gamma_2 u_{tx} = 0, \\ \tau q_t + q + \kappa \theta_x = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \theta(0,x) = \theta_0(x), q(0,x) = q_0(x) \end{cases}$$

in $\mathbb{R}_{\geq 0} \times \mathbb{R}$ *with* $\alpha, \tau, \kappa, \gamma_1\gamma_2 > 0$ *and assume*

$$u_0 \in H^{s+1}, u_1, \theta_0, q_0 \in H^s$$

for a fixed $s \in \mathbb{R}$. *Then there exists a unique solution satisfying*

$$\begin{aligned} &u \in C([0,\infty), H^{s+1}) \cap C^1([0,\infty), H^s), \\ &\theta, q \in C([0,\infty), H^s). \end{aligned}$$

Theorem 3.14. *(L^p-L^q decay estimate)*
We assume $u_0, u_1, \theta_0, q_0 \in \mathcal{S}$. *Then we have the following* L^p-L^q *decay estimate for solutions to the above Cauchy problem:*

$$\|(u_t, u_x, \theta, q)\|_{L^q} \lesssim (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|(\langle D\rangle u_0, u_1, \theta_0, q_0)\|_{L^{p,r_p}} \tag{3.15}$$

for dual q, $1 < p \leq 2$ *and* $r_p = \frac{1}{p} - \frac{1}{q}$.

Next, we state a result on a diffusion phenomenon. The small frequencies are the decay-determining ones, and we therefore define the reference system:

$$\begin{cases} W_t + M_0 W + M_1 W_x - M_2 W_{xx} = 0, \\ W(0,x) = W_0(x) \end{cases} \tag{3.16}$$

with

$$\begin{aligned} M_0 &= \operatorname{diag}\left(0,0,0,\frac{1}{\tau}\right), \\ M_1 &= \operatorname{diag}\left(-\sqrt{\alpha+\gamma_1\gamma_2}, 0, \sqrt{\alpha+\gamma_1\gamma_2}, 0\right), \\ M_2 &= \operatorname{diag}\left(\frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}, \frac{\alpha\kappa}{\alpha+\gamma_1\gamma_2}, \frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}, 0\right), \\ W_0(x) &= \tilde{L}^{(1)} U_0(x) = \begin{pmatrix} -\gamma_2\alpha & -\gamma_2\sqrt{\alpha+\gamma_1\gamma_2} & \gamma_1\gamma_2 & 0 \\ 2\gamma_2\sqrt{\alpha} & 0 & 2\sqrt{\alpha} & 0 \\ -\gamma_2\alpha & \gamma_2\sqrt{\alpha+\gamma_1\gamma_2} & \gamma_1\gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0' \\ u_1 \\ \theta_0 \\ q_0 \end{pmatrix}. \end{aligned}$$

With this reference system we can now state:

Theorem 3.15. *We assume* $u_0, u_1, \theta_0, q_0 \in \mathcal{S}$. *Then we have for the solution* U *to the Cauchy problem (3.10) the estimate*

$$\left\| U - \tilde{R}^{(1)} W \right\|_{L^q} \lesssim (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \| (\langle D \rangle u_0, u_1, \theta_0, q_0) \|_{L^{p,r_p}}$$

for dual indices $q, 1 < p \leq 2$, $r_p = \frac{1}{p} - \frac{1}{q}$ *and* $\tilde{R}^{(1)}$ *from the diagonalization procedure in Section 3.1.2.1.*

Remark 3.2. Note that the first three equations and their initial data in (3.16) are the same as in the reference system (3.8) for the classical thermoelasticity.

Remark 3.3. The system in (3.9) is a strictly hyperbolic one. The propagation of weak singularities is well studied for such systems (cf. [Bea89]), and we will therefore omit it to state results in that direction here. However, the hyperbolicity allows it to directly study the propagation of strong singularities and the asymptotic behavior of jumper amplitudes of discontinuous solutions. For the 1D case such considerations were done in [RW05] for the Cauchy problem of the semilinear system even as the relaxation parameter τ goes to zero, and thus, some results for the classical hyperbolic-parabolic coupled thermoelasticity model are indicated as well.

At last, we will apply our procedure to the thermoelasticity models of type 2 and 3 in 1D.

3.1.3. Thermoelasticity of type 2 and 3

We consider the Cauchy problem

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0, \\ \theta_{tt} - \kappa \theta_{xx} - \delta \theta_{txx} + \gamma_2 u_{ttx} = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \theta(0,x) = \theta_0(x), \theta_t(0,x) = \theta_1(x) \end{cases} \tag{3.17}$$

with $\alpha, \kappa, \gamma_1\gamma_2 > 0$ and $\delta \geq 0$.
If we set $\delta = 0$, then (3.17) denotes the Cauchy problem for the thermoelasticity model of type 2. When assuming $\delta > 0$, then (3.17) denotes the Cauchy problem for the – in contrast to the thermoelasticity model of type 2 of dissipative nature – model of type 3.

Remark 3.4. In a first step we should try to transform the above Cauchy problem into one for a first order system with respect to the time t for an unknown $U = (u_+, u_-, \theta_+, \theta_-)^T$ with $u_\pm = u_t \pm \sqrt{\alpha}\, u_x$ and $\theta_\pm = (\theta + \gamma_2 u_x)_t \pm \sqrt{\kappa}\, \theta_x$. Doing this, we arrive at the system (3.30) with $m = 0$. The matrix A_0 is thus not diagonalizable, and our procedure won't work. We can moreover not even apply the considerations from Section 2.2.2.4 and have to come up with another idea.

The authors Reissig and Wang studied in [RW05] the Cauchy problem for the semilinear system of type 3 and introduced, following the considerations in [ZZ03], the transformation

$$\psi(t,x) = \int_0^t \theta(s,x)ds + \chi(x)$$

with a function $\chi = \chi(x)$ satisfying

$$\kappa\chi'' = \theta_1 - \delta\theta_0'' + \gamma_2 u_1'.$$

With the new unknown ψ the Cauchy problem (3.17) was transformed into

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1\psi_{tx} = 0, \\ \psi_{tt} - \kappa\psi_{xx} - \delta\psi_{txx} + \gamma_2 u_{tx} = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \psi(0,x) = \chi(x), \psi_t(0,x) = \theta_0(x). \end{cases}$$

Introducing the vector $U = (u_+, u_-, \psi_+, \psi_-)^T$ with $u_\pm = u_t \pm \sqrt{\alpha}\, u_x$, as usual, and $\psi_\pm = \psi_t \pm \sqrt{\kappa}\,\psi_x$, the above problem is equivalently transformed into an initial value problem

$$\begin{cases} U_t + A_1 U_x - A_2 U_{xx} = 0, \\ U(0,x) = U_0(x) \end{cases} \tag{3.18}$$

with

$$A_1 = \begin{pmatrix} -\sqrt{\alpha} & 0 & \frac{\gamma_1}{2} & \frac{\gamma_1}{2} \\ 0 & \sqrt{\alpha} & \frac{\gamma_1}{2} & \frac{\gamma_1}{2} \\ \frac{\gamma_2}{2} & \frac{\gamma_2}{2} & -\sqrt{\kappa} & 0 \\ \frac{\gamma_2}{2} & \frac{\gamma_2}{2} & 0 & \sqrt{\kappa} \end{pmatrix} \quad \text{and} \quad A_2 = \frac{\delta}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The system in (3.18) is one of type (2.2), and we can thus apply our procedure.
In fact, for $\delta = 0$, i.e., when considering the Cauchy problem for thermoelasticity of type 2, we have $A_2 \equiv 0$, and (3.18) is an initial value problem for a strictly hyperbolic system without any lower order terms. The eigenvalues are with the real, positive and distinct numbers

$$a_\pm = \frac{1}{\sqrt{2}}\sqrt{\alpha + \kappa + \gamma_1\gamma_2 \pm \sqrt{(\alpha+\kappa+\gamma_1\gamma_2)^2 - 4\alpha\kappa}} \tag{3.19}$$

given by $-a_+ < -a_- < a_- < a_+$ (cf. [RW05]). With $\tilde{L}^{(1)}$ and $\tilde{R}^{(1)}$ denoting matrices of corresponding left and right eigenvectors with $\tilde{L}^{(1)}\tilde{R}^{(1)} = I$ we obtain the explicit solution representation

$$U(t,x) = \mathcal{F}^{-1}_{\xi\to x}\left(\tilde{R}^{(1)}\operatorname{diag}\left(e^{ia_+\xi t}, e^{ia_-\xi t}, e^{-ia_-\xi t}, e^{-ia_+\xi t}\right)\tilde{L}^{(1)}\hat{U}_0(\xi)\right).$$

For $\delta > 0$, i.e., when considering the Cauchy problem for thermoelasticity of type 3, the calculations and results of the diagonalization procedure can be found in [RW05].
To obtain well-posedness for the Cauchy problem (3.17), H^s-regularity of U_0 should be guaranteed, or equivalently that

$$\mathcal{F}(U_0)(\xi) = \Big(\hat{u}_1 + i\sqrt{\alpha}\,\xi\hat{u}_0, \hat{u}_1 - i\sqrt{\alpha}\,\xi\hat{u}_0, \hat{\theta}_0 - i\frac{1}{\sqrt{\kappa}}(\xi^{-1}\hat{\theta}_1 + \delta\xi\hat{\theta}_0 + i\gamma_2\hat{u}_1), \\ \hat{\theta}_0 + i\frac{1}{\sqrt{\kappa}}(\xi^{-1}\hat{\theta}_1 + \delta\xi\hat{\theta}_0 + i\gamma_2\hat{u}_1)\Big)$$

belongs to the Fourier image of H^s. We thus need an additional assumption for θ_1, i.e., something like $|\xi|^{-1}\hat{\theta}_1 \in L^2(-1,1)$. Hence, we follow Reissig and Wang and work with homogeneous Sobolov spaces.
Let $\phi = \phi(\xi)$ be a nonnegative cut-off function around 0 with $\phi(\xi) = 1$ on $[-1,1]$.

Definition 3.1. *(Reissig and Wang, [RW05])*
The space $\dot{H}^{s_1,s_2}$, $s_1, s_2 \in \mathbb{R}$, denotes the collection of all distributions u from $\mathcal{Z}'$, the dual space to the subspace of the Schwartz space $\mathcal{S}$ consisting of functions with $d_\xi^k \hat{u}(0) = 0$ for all $k \in \mathbb{N}_0$, satisfying the property

$$\int_{\mathbb{R}} \phi(\xi)|\xi|^{2s_1}|\hat{u}|^2 d\xi + \int_{\mathbb{R}} (1-\phi(\xi))|\xi|^{2s_2}|\hat{u}|^2 d\xi < \infty.$$

The space $\mathcal{Z}'$ can be identified with the factor space $\mathcal{S}'/\mathcal{P}$, where $\mathcal{P}$ is the collection of all polynomials.
The following results are more or less simple corollaries of the ones given in [RW05]:

Theorem 3.16. *(Well-posedness result)*
We consider the Cauchy problem (3.17) in $\mathbb{R}_{\geq 0} \times \mathbb{R}$ with $\alpha, \kappa, \gamma_1\gamma_2 > 0$ and $\delta \geq 0$.
We assume for a fixed $s \in \mathbb{R}$ that $u_0 \in \dot{H}^{0,s+1}, u_1, \theta_0 \in \dot{H}^{0,s}$ as well as $\theta_1 \in \dot{H}^{-1,s-1}$ for $\delta = 0$ and $\theta_1 \in \dot{H}^{-1,s-2}$, $\theta_1 - \delta\theta_0'' \in \dot{H}^{0,s-1}$ for $\delta > 0$.
Then there exists a unique solution satisfying

$$u \in C([0,\infty), H^{s+1}) \cap C^1([0,\infty), H^s),$$
$$\theta \in C([0,\infty), H^s) \cap C^1([0,\infty), H^{s-2}).$$

For $\delta = 0$ we have moreover $\theta \in C^1([0,\infty), H^{s-1})$.

For solutions to the Cauchy problem for the thermoelasticity model of type 2 we do not have any decay, nor can we observe a diffusion phenomenon. We will also omit it to state anything on the propagation of singularities here and restrict our view from now on to the Cauchy problem for the thermoelasticity model of type 3, i.e., we assume $\delta > 0$.

Theorem 3.17. *(L^p-L^q decay estimates)*
We consider the Cauchy problem (3.17) with $\delta > 0$ and assume for a fixed $s \in \mathbb{R}$:

$$u_0 \in \dot{H}^{0,s+1}, u_1, \theta_0 \in \dot{H}^{0,s}, \theta_1 \in \dot{H}^{-1,s-2} \quad \textit{and} \quad \theta_1 - \delta\theta_0'' \in \dot{H}^{0,s-1}.$$

Then we have for the unique solution (u,θ):

(i) If we assume

$$|D|\phi(D)u_0, \phi(D)u_1, \phi(D)\theta_0, |D|^{-1}\phi(D)(\theta_1 - \delta\theta_0'') \in L^1,$$

then we have for dual indices q, $1 \leq p \leq 2$ the estimate

$$\|\phi(D)(u_t,u_x,\theta)\|_{L^q} \lesssim (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left\|\left(|D|\,\phi(D)u_0, \phi(D)(u_1,\theta_0), |D|^{-1}\,\phi(D)(\theta_1 - \delta\theta_0'')\right)\right\|_{L^p} \tag{3.20a}$$

(ii) Assuming the finiteness of the right-hand side of the upcoming estimate, we have

$$\|(1-\phi(D))(u_t,u_x,\theta)\|_{L^q} \lesssim e^{-ct} \\ \| \langle D\rangle (1-\phi(D))u_0, (1-\phi(D))(u_1,\theta_0), \langle D\rangle^{-1}(1-\phi(D))(\theta_1-\delta\theta_0'')\|_{L^{p,r_p}} \tag{3.20b}$$

for dual q, $1<p\le 2$, $r_p=\frac{1}{p}-\frac{1}{q}$ and c being a positive constant.

Remark 3.5. Suppose that the assumptions of Theorem 3.17 hold and that we have the finiteness of the right-hand side of the upcoming estimate, then we can by a few further calculations prove

$$\|(\theta_t,\theta_x)\|_{L^q} \lesssim (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left\|\left(\langle D\rangle u_0, u_1, \theta_0, \langle D\rangle^{-1}(\theta_1-\delta\theta_0'')\right)\right\|_{L^{p,r_p+2}} \tag{3.21}$$

for dual q, $1<p\le 2$ and $r_p=\frac{1}{p}-\frac{1}{q}$.

The reference system for which we can prove a diffusion phenomenon to (3.18) as in Theorem 2.24 with $n_s=m_s=2$ is a parabolic one, more specific, it takes the form

$$\begin{cases} W_t + M_1W_x - M_2W_{xx} = 0, \\ W(0,x) = \tilde{L}^{(1)}U_0(x) \end{cases}$$

with $M_1=\operatorname{diag}(-a_+,-a_-,a_-,a_+)$, $a_\pm$ (and $\tilde{L}^{(1)}$) coming from (3.19), as well as $M_2=\operatorname{diag}(b_+,b_-,b_-,b_+)$, where $b_\pm$ are positive numbers given by

$$b_\pm = \frac{\delta}{4}\left(1 \pm \frac{\gamma_1\gamma_2+\kappa-\alpha}{\sqrt{(\gamma_1\gamma_2+\alpha+\kappa)^2-4\alpha\kappa}}\right).$$

Hence, the asymptotic profiles of solutions to the Cauchy problem (3.17) with $\delta>0$ are (from the viewpoint of decay estimates) parabolic.
Concerning the propagation of weak singularities we can state (cf. [RW05]):

Theorem 3.18. *We consider the Cauchy problem (3.17) with $\delta>0$ and assume for a fixed $s\in\mathbb{R}$ that $\theta_1\in\dot{H}^{-1,s-2}$ and*

(i) that we have

$$u_1\pm\sqrt{\alpha}u_0', \theta_0 \in [H^s(\mathbb{R})\cap H^{s+1}(\mathbb{R}\backslash\{x_0\})]\backslash H^{s+1}(\mathbb{R}), \\ \theta_1-\delta\theta_0''\in H^{s-1}(\mathbb{R})\cap H^s(\mathbb{R}\backslash\{x_0\}), \theta_1-\delta\theta_0''+\gamma_2u_1'\notin H^s_{loc}(\{x_0\}).$$

Then we obtain for any $t>0$

$$u_t(t,\cdot),u_x(t,\cdot)\in H^s(\mathbb{R})\cap H^{s+1}(\mathbb{R}\backslash\{x_0\pm\sqrt{\alpha}t\}), \\ u_t(t,\cdot),u_x(t,\cdot)\notin H^{s+1}_{loc}(\{x_0\pm\sqrt{\alpha}t\}), \\ \theta(t,\cdot)\in H^{s+1}(\mathbb{R})\cap H^{s+2}(\mathbb{R}\backslash\{x_0\pm\sqrt{\alpha}t,x_0\}), \\ \theta(t,\cdot)\notin H^{s+2}_{loc}(\{x_0\pm\sqrt{\alpha}t\}), \theta(t,\cdot)\notin H^{s+2}_{loc}(\{x_0\}).$$

(ii) or that we have

$$u_0 \in H^{s+1}(\mathbb{R}) \cap C^\infty(\mathbb{R}\backslash\{x_0\}),$$
$$u_1, \theta_0 \in H^s(\mathbb{R}) \cap C^\infty(\mathbb{R}\backslash\{x_0\}),$$
$$\theta_1 - \delta\theta_0'' \in H^{s-1}(\mathbb{R}) \cap C^\infty(\mathbb{R}\backslash\{x_0\}).$$

Then we obtain

$$(u,\theta) \in C^\infty\left(((0,\infty)\times\mathbb{R})\backslash\left[\{(t, x_0 \pm \sqrt{\alpha}t) : t > 0\} \cup \{(t, x_0) : t > 0\}\right]\right).$$

Proof. We discuss only *(i)*. The assumptions

$$\theta_1 - \delta\theta_0'' \in H^{s-1}(\mathbb{R}) \cap H^s(\mathbb{R}\backslash\{x_0\}), \quad \theta_1 - \delta\theta_0'' + \gamma_2 u_1' \notin H^s_{\text{loc}}(\{x_0\}) \tag{3.22}$$

guarantee that

$$\phi_{ext}(D)\chi' \in [H^s(\mathbb{R}) \cap H^{s+1}(\mathbb{R}\backslash\{x_0\})]\,\backslash H^{s+1}(\mathbb{R}).$$

Everything else follows from Theorem 2.26 and from the fact that the matrix T_{-1} from (2.51) (given by $T_{-1} = -\left(\hat{K}_{(1)} + \hat{K}_{(2)}\right)$, with the $\hat{K}_{(i)}$ coming from the procedure) has only nonzero entries apart from the diagonal. □

The picture obtained here is thus very similar to the one for the classical model, i.e., as in Theorem 3.8.

3.2. Selected results for thermoelasticity models with dissipation or mass

As mentioned in Section 1.1, we are not only interested in studying thermoelasticity models of type 1, 2 and 3 and second sound models of thermoelasticity itself, but also with additional lower order terms, i.e., with a dissipation or a mass term. We will in this section cover all such cases and state selected results, where the problem has a parabolic structure from the point of view of decay estimates.

3.2.1. Classical thermoelasticity

Classical thermoelasticity with dissipation

The results for the classical thermoelasticity model without any additional lower order terms were given in Section 3.1.1. If we add an additional dissipation term into the first equation, i.e., we are now devoting ourselves to the study of

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1\theta_x + m u_t = 0, \\ \theta_t - \kappa\theta_{xx} + \gamma_2 u_{tx} = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \theta(0,x) = \theta_0(x), \end{cases}$$

$\alpha, \kappa, \gamma_1\gamma_2, m > 0$, then (with regard to the results discussed in Section 3.1.1.2) not too many significant changes should be expected.
In fact, we have to study the same system for the vector of unknowns $U = (u_+, u_-, \theta)^T$, $u_\pm = u_t \pm \sqrt{\alpha}\, u_x$, as in (3.2) with an additional summand A_0U,

$$A_0 = \frac{m}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and we obtain exactly the same results on well-posedness and decay estimates as for the case $m = 0$, i.e., as in the Theorems 3.5 and 3.6. The dissipation term thus has no essential helping character so far.
The reference system for the diffusion phenomenon takes the form

$$\begin{cases} W_t + M_0W - M_2W_{xx} = 0, \\ W(0,x) = W_0(x) \end{cases} \tag{3.23}$$

with

$$\begin{aligned} M_0 &= \operatorname{diag}(0, 0, m), \\ M_2 &= \operatorname{diag}(\lambda_-, \lambda_+, 0), \\ W_0(x) &= \underbrace{\frac{1}{a}\begin{pmatrix} \alpha\gamma_2 & 0 & \alpha - m\lambda_- \\ -\alpha\gamma_2 & 0 & -\alpha + m\lambda_+ \\ 0 & \sqrt{2}\,a & 0 \end{pmatrix}}_{=:L} \begin{pmatrix} u_0' \\ u_1 \\ \theta_0 \end{pmatrix} \end{aligned}$$

and positive numbers

$$\lambda_\mp = \frac{\alpha + \gamma_1\gamma_2 + m\kappa \mp a}{2m}, \qquad a = \sqrt{(\alpha + \gamma_1\gamma_2 + m\kappa)^2 - 4m\alpha\kappa}. \tag{3.24}$$

Replacing $\tilde{R}^{(1)}$ by the inverse of L we obtain the same result as in Theorem 3.7.
Since the additional dissipation term has no essential influence on the large frequencies, we moreover obtain the same results concerning the propagation of weak singularities as for the problem without any additional lower order terms, i.e., as in Theorem 3.8.

Classical thermoelasticity with mass

We will now devote our attention to the Cauchy problem for the classical thermoelasticity model with an additional mass term, that is,

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1\theta_x + m^2u = 0, \\ \theta_t - \kappa\theta_{xx} + \gamma_2u_{tx} = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \theta(0,x) = \theta_0(x), \end{cases}$$

$\alpha, \kappa, \gamma_1\gamma_2, m > 0$. In phase space the above system is given by

$$\begin{cases} \hat{u}_{tt} + \alpha\left(\xi^2 + \frac{m^2}{\alpha}\right)\hat{u} + i\gamma_1\xi\hat{\theta} = 0, \\ \hat{\theta}_t + \kappa\xi^2\hat{\theta} + i\gamma_2\xi\hat{u}_t = 0. \end{cases}$$

With the help of $V = (\hat{u}_+, \hat{u}_-, \hat{\theta})^T$, $\hat{u}_\pm = \hat{u}_t \pm i\sqrt{\alpha}\,\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}}\,\hat{u}$, $\langle\xi\rangle^2_{\frac{m}{\sqrt{\alpha}}} = \xi^2 + \frac{m^2}{\alpha}$, this can equivalently be rewritten into the first-order system

$$V_t + \left(i\xi B_1^{(1)} + i\,\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}}\,B_1^{(2)} + \xi^2 B_2\right)V = 0 \tag{3.25}$$

with the matrices

$$B_1^{(1)} = \begin{pmatrix} 0 & 0 & \gamma_1 \\ 0 & 0 & \gamma_1 \\ \frac{\gamma_2}{2} & \frac{\gamma_2}{2} & 0 \end{pmatrix},$$

$B_1^{(2)} = \operatorname{diag}(-\sqrt{\alpha}, \sqrt{\alpha}, 0)$ and $B_2 = \operatorname{diag}(0, 0, \kappa)$.
Using the asymptotic expansions $\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}} = \sqrt{\frac{m^2}{\alpha} + \xi^2} = \frac{m}{\sqrt{\alpha}}\left(1 + \frac{\alpha}{2m^2}\,\xi^2 - \frac{\alpha^2}{8m^4}\,\xi^4\right) + \mathcal{O}(\xi)^6$ for small frequencies and $\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}} = |\xi|\left(1 + \frac{m^2}{2\alpha}\xi^{-2}\right) + \mathcal{O}(\xi^{-3})$ for $|\xi| \to \infty$, we can rewrite (3.25) in $Z_{int}(\sigma)$ as

$$V_t + \left(A_0 + i\xi A_1 + \xi^2 A_2 + \xi^4 A_4 + R_s(\xi)\right)V = 0 \tag{3.26}$$

with $R_s(\xi) = \mathcal{O}(\xi^6)$ and

$$\begin{aligned} A_0 &= i\operatorname{diag}(-m, m, 0), \qquad & A_2 &= \operatorname{diag}(-i\tfrac{\alpha}{2m}, i\tfrac{\alpha}{2m}, \kappa), \\ A_1 &= B_1^{(1)}, & A_4 &= i\operatorname{diag}(\tfrac{\alpha^2}{8m^3}, -\tfrac{\alpha^2}{8m^3}, 0). \end{aligned}$$

In $Z_{ext}(N)$ we can rewrite (3.25) as

$$V_t + \left(\xi^2 A_2 + i\xi A_1 + R_l(\xi)\right)V = 0 \tag{3.27}$$

with matrices $R_l(\xi) = \mathcal{O}(\xi^{-1})$, $A_2 = B_2$ and

$$A_1 = \begin{pmatrix} -\operatorname{sgn}(\xi)\sqrt{\alpha} & 0 & \gamma_1 \\ 0 & \operatorname{sgn}(\xi)\sqrt{\alpha} & \gamma_1 \\ \frac{\gamma_2}{2} & \frac{\gamma_2}{2} & 0 \end{pmatrix},$$

where $\operatorname{sgn}(\xi)$ denotes the sign of ξ, i.e., $\operatorname{sgn}(\xi) = 1$ for $\xi > 0$ and $\operatorname{sgn}(\xi) = -1$ for $\xi < 0$. The systems (3.26) and (3.27) are of type (2.53) and (almost) of type (2.54), respectively. We apply our procedure and obtain concerning the behavior of the eigenvalues to the coefficient matrix in (3.25):

Lemma 3.19. *For the characteristic roots $\mu_j = \mu_j(\xi)$ of the coefficient matrix $B(\xi) = i\xi B_1^{(1)} + i\,\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}}\,B_1^{(2)} + \xi^2 B_2$ from (3.25) we have:*

(i) For $|\xi| \leq \sigma \ll 1$ they behave as

$$\begin{aligned} \mu_{1,2}(\xi) &= \mp im \mp i\frac{\alpha+\gamma_1\gamma_2}{2m}\xi^2 + \left(\frac{\kappa\gamma_1\gamma_2}{2m^2} \pm i\frac{(\alpha+\gamma_1\gamma_2)^2}{8m^3}\right)\xi^4 + \mathcal{O}(\xi^5), \\ \mu_3(\xi) &= \kappa\xi^2 - \frac{\kappa\gamma_1\gamma_2}{m^2}\xi^4 + \mathcal{O}(\xi^5) \end{aligned}$$

(ii) and for $|\xi| \geq N \gg 1$ as

$$\begin{aligned} \mu_{1,2}(\xi) &= \mp i\sqrt{\alpha}|\xi| + \frac{\gamma_1\gamma_2}{2\kappa} + \mathcal{O}(\xi^{-1}), \\ \mu_3(\xi) &= \kappa\xi^2 - \frac{\gamma_1\gamma_2}{\kappa} + \mathcal{O}(\xi^{-1}). \end{aligned}$$

(iii) The assumption (C) is satisfied.

We obtain the same well-posedness result as in Theorem 3.5 and, taking into consideration that the decay-determining numbers $n_{s,j}$ from Section 2.4.1 are given by $n_{s,j} = q_j = 2$ for $j = 1,2$ and $n_{s,3} = p_3 = 2$, we obtain the same result on decay estimates as for the Cauchy problem without any lower order terms, i.e., as in Theorem 3.6. More specific, we observe, similar to (1.11), that the solution u itself satisfies the estimate in Theorem 3.6. We have analogous results on the propagation of weak singularities as in Theorem 3.8 (i.e., the same results when replacing in *(i)* u_0' by $i\langle D\rangle_{\frac{m}{\sqrt{\alpha}}} u_0$ and $u_x(t,\cdot)$ by $\langle D\rangle_{\frac{m}{\sqrt{\alpha}}} u(t,\cdot)$) and obtain for the diffusion phenomenon with $U(t,x) = \mathcal{F}^{-1}_{\xi\to x}(V(t,\xi))$, $m_s = \max_{j=1,2,3} p_j = 4$ and the reference system

$$\begin{cases} W_t + M_0 W - M_2 W_{xx} + M_4 W_{xxxx} = 0, \\ W(0,x) = (u_1 + i\sqrt{\alpha}\langle D\rangle_{\frac{m}{\sqrt{\alpha}}} u_0, u_1 - i\sqrt{\alpha}\langle D\rangle_{\frac{m}{\sqrt{\alpha}}} u_0, \theta_0)^T \end{cases} \tag{3.28}$$

with

$$\begin{aligned} M_0 &= i\,\mathrm{diag}\,(-m, m, 0), \\ M_2 &= \mathrm{diag}\left(-i\frac{\alpha+\gamma_1\gamma_2}{2m}, i\frac{\alpha+\gamma_1\gamma_2}{2m}, \kappa\right), \\ M_4 &= \mathrm{diag}\left(\frac{\kappa\gamma_1\gamma_2}{2m^2} + i\frac{(\alpha+\gamma_1\gamma_2)^2}{8m^3}, \frac{\kappa\gamma_1\gamma_2}{2m^2} - i\frac{(\alpha+\gamma_1\gamma_2)^2}{8m^3}, 0\right): \end{aligned}$$

Theorem 3.20. *We assume $u_0, u_1, \theta_0 \in \mathcal{S}$. Then we obtain the estimate*

$$\|U - W\|_{L^q} \lesssim (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{4}} \|(\langle D\rangle u_0, u_1, \theta_0)\|_{L^{p,r_p}}$$

for dual values $q, 1 < p \leq 2$ and $r_p = \frac{1}{p} - \frac{1}{q}$.

Remark 3.6. When considering the Cauchy problem for classical thermoelasticity with an additional mass and an additional dissipation term, i.e.,

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1\theta_x + m_1^2 u + m_2 u_t = 0, \\ \theta_t - \kappa\theta_{xx} + \gamma_2 u_{tx} = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \theta(0,x) = \theta_0(x) \end{cases}$$

with $\alpha, \kappa, \gamma_1\gamma_2, m_1, m_2 > 0$, then one might - having the situation for the wave equation with mass and dissipation in mind - hope for an exponential decay. However, it is the coupling with the parabolic heat equation that annihilates this hope. Calculations show that we obtain only a parabolic decay.
Moreover, special considerations have to be done (cf. the wave equation case) when $m_2 = 2m_1$. In that case our conditions fail when diagonalizing in $Z_{int}(\sigma)$, and we have to apply ideas from Section 2.2.2.4. For a more detailed discussion we would like the reader to refer to Section 3.3.

3.2.2. Thermoelasticity with second sound

Thermoelasticity with second sound and dissipation

The results for the model without any additional lower order terms were given in Section 3.1.2. We will now turn to the study of

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x + m u_t = 0, \\ \theta_t + q_x + \gamma_2 u_{tx} = 0, \\ \tau q_t + q + \kappa \theta_x = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \theta(0,x) = \theta_0(x), q(0,x) = q_0(x). \end{cases}$$

Regarding the results on well-posedness, decay estimates and on the propagation of singularities from Section 3.1.2.2, we should not expect significant changes. In particular, we have already obtained a parabolic decay rate for the above problem with $m = 0$ in Theorem 3.14.
We equivalently transform the above Cauchy problem into the one given by (3.10) for $U = (u_+, u_-, \theta, q)^T$ when replacing A_0 by

$$A_0 = \begin{pmatrix} \frac{m}{2} & \frac{m}{2} & 0 & 0 \\ \frac{m}{2} & \frac{m}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\tau} \end{pmatrix}.$$

The results on well-posedness and decay estimates are exactly the same as in the Theorems 3.13 and 3.14. The reference system for the diffusion phenomenon (cf. with the reference system for classical thermoelasticity with dissipation from the previous section and Remark 3.1) now takes the following form:

$$\begin{cases} W_t + M_0 W - M_2 W_{xx} = 0, \\ W(0,x) = W_0(x) \end{cases}$$

with

$$\begin{aligned} M_0 &= \operatorname{diag}\left(0, 0, m, \frac{1}{\tau}\right), \\ M_2 &= \operatorname{diag}\left(\lambda_-, \lambda_+, 0, 0\right), \\ W_0(x) &= L \cdot (u_0', u_1, \theta_0, q_0)^T \end{aligned}$$

and the positive numbers $\lambda_\mp$ from (3.24). The matrix L is in the cases $m \neq \frac{1}{\tau}$ given by

$$L = \frac{1}{a}\begin{pmatrix} \alpha\gamma_2 & 0 & \alpha - m\lambda_- & 0 \\ -\alpha\gamma_2 & 0 & -\alpha + m\lambda_+ & 0 \\ 0 & \sqrt{2}\,a & 0 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

and if $m = \frac{1}{\tau}$, i.e., we have a connection between the reciprocal of the relaxation time and the dissipation, it is given by

$$L = \frac{1}{a}\begin{pmatrix} \alpha\gamma_2 & 0 & \alpha - m\lambda_- & 0 \\ -\alpha\gamma_2 & 0 & -\alpha + m\lambda_+ & 0 \\ 0 & \sqrt{2}\,a & 0 & -\frac{\sqrt{2}\,\gamma_1\tau\, a}{\kappa-\lambda_-} \\ 0 & \sqrt{2}\,a & 0 & \frac{\sqrt{2}\,(\kappa-\lambda_-)a}{\gamma_2\kappa} \end{pmatrix}.$$

Replacing $\tilde{R}^{(1)}$ by the inverse of L, we obtain the same results as in Theorem 3.15.

Thermoelasticity with second sound and mass

We consider the Cauchy problem

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1\theta_x + m^2 u = 0, \\ \theta_t + q_x + \gamma_2 u_{tx} = 0, \\ \tau q_t + q + \kappa\theta_x = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \theta(0,x) = \theta_0(x), q(0,x) = q_0(x) \end{cases}$$

and follow the lines of our considerations for classical thermoelasticity with an additional mass term from Section 3.2.1. The system that we have to study in phase space is

$$V_t + \left(B_0 + i\xi B_1^{(1)} + i\,\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}}\, B_1^{(2)}\right) V = 0$$

for $V = (\hat{u}_+, \hat{u}_-, \hat{\theta}, \hat{q})^T$, $\hat{u}_\pm = \hat{u}_t \pm i\sqrt{\alpha}\,\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}}\,\hat{u}$, and matrices $B_0 = \operatorname{diag}(0,0,0,\frac{1}{\tau})$, $B_1^{(2)} = \operatorname{diag}(-\sqrt{\alpha}, \sqrt{\alpha}, 0, 0)$ and

$$B_1^{(1)} = \begin{pmatrix} 0 & 0 & \gamma_1 & 0 \\ 0 & 0 & \gamma_1 & 0 \\ \frac{\gamma_2}{2} & \frac{\gamma_2}{2} & 0 & 1 \\ 0 & 0 & \frac{\kappa}{\tau} & 0 \end{pmatrix}.$$

We obtain the same well-posedness result as in Theorem 3.13 and the same result on decay estimates as in Theorem 3.14 with the difference that the solution u itself, measured in the L^q-norm, satisfies the estimate (3.15) as well. The reference system for the diffusion phenomenon (cf. with (3.28) and Remark 3.1) takes the form:

$$\begin{cases} W_t + M_0 W - M_2 W_{xx} + M_4 W_{xxxx} = 0, \\ W(0,x) = (u_1 + i\sqrt{\alpha}\,\langle D\rangle_{\frac{m}{\sqrt{\alpha}}}\, u_0, u_1 - i\sqrt{\alpha}\,\langle D\rangle_{\frac{m}{\sqrt{\alpha}}}\, u_0, \theta_0, q_0)^T \end{cases}$$

with

$$\begin{aligned} M_0 &= \operatorname{diag}\left(-i\,m, i\,m, 0, \frac{1}{\tau}\right), \\ M_2 &= \operatorname{diag}\left(-i\frac{\alpha+\gamma_1\gamma_2}{2m}, i\frac{\alpha+\gamma_1\gamma_2}{2m}, \kappa, 0\right), \\ M_4 &= \operatorname{diag}\left(m_+, m_-, 0, 0\right) \end{aligned}$$

and

$$m_\pm = \frac{\kappa\gamma_1\gamma_2}{2m^2(1+m^2\tau^2)} \pm i\left(\frac{(\alpha+\gamma_1\gamma_2)^2}{8m^3} - \frac{\kappa\tau\gamma_1\gamma_2}{2m(1+m^2\tau^2)}\right).$$

For $U(t,x) = \mathcal{F}^{-1}_{\xi\to x}\left(V(t,\xi)\right)$ and $\mathcal{S}$-data u_0, u_1, θ_0, q_0 we obtain an estimate as in Theorem 3.20.

3.2.3. Thermoelasticity of type 2 and 3

Thermoelasticity of type 2 and 3 with dissipation

We devote our attention to the Cauchy problems

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1\theta_x + m u_t = 0, \\ \theta_{tt} - \kappa\theta_{xx} - \delta\theta_{txx} + \gamma_2 u_{ttx} = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \theta(0,x) = \theta_0(x), \theta_t(0,x) = \theta_1(x) \end{cases} \tag{3.29}$$

for the thermoelasticity models of type 2 ($\delta = 0$) and 3 ($\delta > 0$) with additional dissipation terms.

Introducing the vector of unknowns $U = (u_+, u_-, \theta_+, \theta_-)^T$ with $u_\pm = u_t \pm \sqrt{\alpha}\,u_x$ and $\theta_\pm = (\theta+\gamma_2 u_x)_t \pm \sqrt{\kappa}\,\theta_x$, the initial value problem (3.29) can equivalently be transformed into one for

$$U_t + A_0 U + A_1 U_x - A_2 U_{xx} - A_3 U_{xxx} = 0 \tag{3.30}$$

with

$$A_0 = \begin{pmatrix} \frac{m}{2} & \frac{m}{2} & \frac{\gamma_1}{2\sqrt{\kappa}} & -\frac{\gamma_1}{2\sqrt{\kappa}} \\ \frac{m}{2} & \frac{m}{2} & \frac{\gamma_1}{2\sqrt{\kappa}} & -\frac{\gamma_1}{2\sqrt{\kappa}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\sqrt{\kappa}\gamma_2}{2} & -\frac{\sqrt{\kappa}\gamma_2}{2} & \frac{\delta}{2} & \frac{\delta}{2} \\ \frac{\sqrt{\kappa}\gamma_2}{2} & \frac{\sqrt{\kappa}\gamma_2}{2} & \frac{\delta}{2} & \frac{\delta}{2} \end{pmatrix}, A_3 = -\frac{\delta\gamma_2}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

and $A_1 = \operatorname{diag}(-\sqrt{\alpha}, \sqrt{\alpha}, -\sqrt{\kappa}, \sqrt{\kappa})$. The system (3.30) is of type (2.2) for $\delta = 0$ and of type (2.52) with $m = 3$ for $\delta > 0$ and can be treated with our diagonalization procedure for all but large frequencies (i.e., assumption (C) holds as well).

For large frequencies the first matrices to diagonalize would be A_3 for $\delta > 0$ and A_2 for $\delta = 0$, but the eigenspace to the 4-fold eigenvalue 0 is in both cases only three-dimensional. Diagonalizability is thus not given, and our procedure won't work.

To overcome the non-symmetry between the terms $\gamma_1\theta_x$ and $\gamma_2 u_{ttx}$ however, we can differentiate in the first equation of (3.29) with respect to t and obtain after introducing $v := u_t$ the initial value problem

$$\begin{cases} v_{tt} - \alpha v_{xx} + \gamma_1 \theta_{tx} + m v_t = 0, \\ \theta_{tt} - \kappa \theta_{xx} - \delta \theta_{txx} + \gamma_2 v_{tx} = 0, \\ v(0,x) = u_1(x), v_t(0,x) = \alpha u_0''(x) - \gamma_1 \theta_0'(x) - m u_1(x), \\ \theta(0,x) = \theta_0(x), \theta_t(0,x) = \theta_1(x), \end{cases}$$

which can equivalently be transformed into

$$\begin{cases} \tilde{U}_t + A_0 \tilde{U} + A_1 \tilde{U}_x - A_2 \tilde{U}_{xx} = 0, \\ \tilde{U}(0,x) = \tilde{U}_0(x) \end{cases} \tag{3.31}$$

for $\tilde{U} = (v_+, v_-, \tilde{\theta}_+, \tilde{\theta}_-)^T$, $v_\pm = v_t \pm \sqrt{\alpha}\, v_x$, $\tilde{\theta}_\pm = \theta_t \pm \sqrt{\kappa}\, \theta_x$, and matrices

$$A_0 = \frac{m}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} -\sqrt{\alpha} & 0 & \frac{\gamma_1}{2} & \frac{\gamma_1}{2} \\ 0 & \sqrt{\alpha} & \frac{\gamma_1}{2} & \frac{\gamma_1}{2} \\ \frac{\gamma_2}{2} & \frac{\gamma_2}{2} & -\sqrt{\kappa} & 0 \\ \frac{\gamma_2}{2} & \frac{\gamma_2}{2} & 0 & \sqrt{\kappa} \end{pmatrix} \text{ and } A_2 = \frac{\delta}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The problem (3.31) is for $m = 0$ the one given by (3.18) and can be studied for large frequencies for both cases $\delta = 0$ and $\delta > 0$.
Taking into consideration that the partial Fourier images of the components of U can be written as functions of the partial Fourier image of $\tilde{U}$,

$$\hat{u}_\pm = g_\pm^{(1)}(\hat{v}_+, \hat{v}_-, \hat{\tilde{\theta}}_+, \hat{\tilde{\theta}}_-, \xi) \quad \text{and} \quad \hat{\theta}_\pm = g_\pm^{(2)}(\hat{v}_+, \hat{v}_-, \hat{\tilde{\theta}}_+, \hat{\tilde{\theta}}_-) \tag{3.32}$$

with $g_\pm^{(1)}$ ($\mathcal{F}(\tilde{U})$ treated as constant) having the behavior $\mathcal{O}(\xi^{-1})$ for $|\xi| \to \infty$, we immediately obtain results on well-posedness and decay estimates:

Theorem 3.21. *(Well-posedness result)*
We consider the Cauchy problem (3.29) in $\mathbb{R}_{\geq 0} \times \mathbb{R}$ with $\alpha, \kappa, \gamma_1\gamma_2, m > 0$, $\delta \geq 0$ and assume for a fixed $s \in \mathbb{R}$

$$u_0 \in H^{s+1}, u_1, \theta_0 \in H^s, \theta_1 \in H^{s-1}.$$

Then there exists a unique solution satisfying

$$u \in C([0,\infty), H^{s+1}) \cap C^1([0,\infty), H^s),$$
$$\theta \in C([0,\infty), H^s) \cap C^1([0,\infty), H^{s-1}).$$

Remark 3.7. Besides changes that are due to our approach - in particular we observe a regularity improvement for the solution θ in the case $\delta > 0$ - we see that the dissipation term has a helping character with respect to the regularity of the data for small frequencies (no need of homogeneous Sobolov spaces) when comparing the above result with Theorem 3.16.

Theorem 3.22. *Assume $u_0, u_1, \theta_0, \theta_1 \in \mathcal{S}$. Then the following L^p-L^q decay estimates hold for solutions (u, θ) to the Cauchy problem (3.29):*

$$\|(u_t, u_x)\|_{L^q} \lesssim (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left\|(\langle D\rangle^2 u_0, \langle D\rangle u_1, \langle D\rangle \theta_0, \theta_1)\right\|_{L^{p,r_p-1}} \tag{3.33}$$

and

$$\|(\theta_t, \theta_x)\|_{L^q} \lesssim (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left\|(\langle D\rangle^2 u_0, \langle D\rangle u_1, \langle D\rangle \theta_0, \theta_1)\right\|_{L^{p,r_p}}$$

for dual q, $1 < p \leq 2$ and $r_p = \frac{1}{p} - \frac{1}{q}$.

Remark 3.8. 1. The difference in the regularity is due to the difference in the behavior of $g_\pm^{(1)}$ and $g_\pm^{(2)}$ from (3.32).
2. In the case $\delta = 0$, i.e., when considering the thermoelasticity model of type 2, the decay is generated from the dissipation term only. It has an essential helping influence.

The reference system for which we can prove a diffusion phenomenon as in Theorem 2.24 with $n_s = m_s = 2$ is given by:

$$\begin{cases} W_t + M_0 W + M_1 W_x - M_2 W_{xx} = 0, \\ W(0,x) = LU_0(x) \end{cases}$$

with some constant matrix L coming from the procedure, U_0 being the initial data to (3.30) and

$$\begin{aligned} M_0 &= \operatorname{diag}(0,0,0,m), \\ M_1 &= \operatorname{diag}\left(-\sqrt{\kappa}, \sqrt{\kappa}, 0, 0\right), \\ M_2 &= \operatorname{diag}\left(\frac{\gamma_1\gamma_2}{2m} + \frac{\delta}{2}, \frac{\gamma_1\gamma_2}{2m} + \frac{\delta}{2}, \frac{\alpha}{m}, 0\right). \end{aligned}$$

We will postpone our considerations for the Cauchy problem to the thermoelasticity model of type 2 with an additional mass term to Chapter 4 and restrict our view in the following to the thermoelasticity model of type 3.

Thermoelasticity of type 3 with mass

We consider the Cauchy problem

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x + m^2 u = 0, \\ \theta_{tt} - \kappa \theta_{xx} - \delta \theta_{txx} + \gamma_2 u_{ttx} = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \theta(0,x) = \theta_0(x), \theta_t(0,x) = \theta_1(x) \end{cases} \tag{3.34}$$

with the usual assumptions $\alpha, \kappa, \delta, \gamma_1\gamma_2, m > 0$ and are now interested whether the mass term has a similar positive influence as the dissipation term did.
We proceed as in the previous section and divide our considerations into some for the region of all but large and some for the region of large frequencies, starting off with $Z_{int}(\sigma) \cup Z_{mid}(\sigma, N) = \{|\xi| \leq N\}$, $N \gg 1$.

Therefore we transform the system in (3.34) after applying partial Fourier transformation equivalently into

$$V_t + \left(B_0 + i\xi B_1^{(1)} + i\,\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}}\, B_1^{(2)} + \xi^2 B_2 + i\xi^3 B_3\right) V = 0 \tag{3.35}$$

for $V = (\hat{u}_+, \hat{u}_-, \hat{\theta}_+, \hat{\theta}_-)^T$, $\hat{u}_\pm = \hat{u}_t \pm i\sqrt{\alpha}\,\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}}\,\hat{u}$, $\hat{\theta}_\pm = (\hat{\theta} + i\gamma_2\xi\hat{u})_t \pm i\sqrt{\kappa}\,\xi\,\hat{\theta}$ and matrices

$$B_0 = \frac{\gamma_1}{2\sqrt{\kappa}}\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\sqrt{\kappa}\gamma_2}{2} & -\frac{\sqrt{\kappa}\gamma_2}{2} & \frac{\delta}{2} & \frac{\delta}{2} \\ \frac{\sqrt{\kappa}\gamma_2}{2} & \frac{\sqrt{\kappa}\gamma_2}{2} & \frac{\delta}{2} & \frac{\delta}{2} \end{pmatrix}, B_3 = -\frac{\delta\gamma_2}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

$B_1^{(1)} = \operatorname{diag}(0, 0, -\sqrt{\kappa}, \sqrt{\kappa})$ and $B_1^{(2)} = \operatorname{diag}(-\sqrt{\alpha}, \sqrt{\alpha}, 0, 0)$.
Using the asymptotic expansion $\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}} = \frac{m}{\sqrt{\alpha}}\left(1 + \frac{\alpha}{2m^2}\,\xi^2 - \frac{\alpha^2}{8m^4}\,\xi^4\right) + \mathcal{O}(\xi)^6$ for small frequencies, we rewrite (3.35) in $Z_{int}(\sigma)$ as

$$V_t + \left(A_0 + i\xi A_1 + \xi^2 A_2 + i\xi^3 A_3 + \xi^4 A_4 + R_s(\xi)\right) V = 0, \tag{3.36}$$

where $R_s(\xi) = \mathcal{O}(\xi^6)$, $A_1 = B_1^{(1)}$, $A_3 = B_3$, $A_4 = i\frac{\alpha^2}{8m^3}\operatorname{diag}(1, -1, 0, 0)$,

$$A_0 = \begin{pmatrix} -im & 0 & \frac{\gamma_1}{2\sqrt{\kappa}} & -\frac{\gamma_1}{2\sqrt{\kappa}} \\ 0 & im & \frac{\gamma_1}{2\sqrt{\kappa}} & -\frac{\gamma_1}{2\sqrt{\kappa}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -i\frac{\alpha}{2m} & 0 & 0 & 0 \\ 0 & i\frac{\alpha}{2m} & 0 & 0 \\ -\frac{\sqrt{\kappa}\gamma_2}{2} & -\frac{\sqrt{\kappa}\gamma_2}{2} & \frac{\delta}{2} & \frac{\delta}{2} \\ \frac{\sqrt{\kappa}\gamma_2}{2} & \frac{\sqrt{\kappa}\gamma_2}{2} & \frac{\delta}{2} & \frac{\delta}{2} \end{pmatrix}.$$

We can now apply our procedure and obtain:

Lemma 3.23. *(i) The characteristic roots $\mu_j = \mu_j(\xi)$ of the coefficient matrix from (3.35) behave for $|\xi| \le \sigma \ll 1$ as*

$$\begin{aligned} \mu_{1,2}(\xi) &= \pm i\sqrt{\kappa}\,\xi + \frac{\delta}{2}\,\xi^2 \mp i\left(\frac{\gamma_1\gamma_2\sqrt{\kappa}}{2m^2} + \frac{\delta^2}{8\sqrt{\kappa}}\right)\xi^3 - \frac{\delta\gamma_1\gamma_2}{2m^2}\,\xi^4 + \mathcal{O}(\xi^5), \\ \mu_{3,4}(\xi) &= \pm im \pm i\frac{\alpha + \gamma_1\gamma_2}{2m}\,\xi^2 + \left(\frac{\delta\gamma_1\gamma_2}{2m^2} \mp i\left(\frac{(\alpha + \gamma_1\gamma_2)^2}{8m^3} - \frac{\kappa\gamma_1\gamma_2}{2m^3}\right)\right)\xi^4 + \mathcal{O}(\xi^5). \end{aligned}$$

(ii) The assumption (C) holds.

The first matrix to diagonalize, when studying (3.35) in $Z_{ext}(N) = \{|\xi| \ge N \gg 1\}$, would be B_3. This matrix is not diagonalizable, and we are thus proceeding as in our considerations in the previous section. We differentiate in the first equation of (3.34) with respect to t and obtain with $v = u_t$ the problem

$$\begin{cases} v_{tt} - \alpha v_{xx} + \gamma_1\theta_{tx} + m^2 v = 0, \\ \theta_{tt} - \kappa\theta_{xx} - \delta\theta_{txx} + \gamma_2 v_{tx} = 0, \\ v(0,x) = u_1(x), v_t(0,x) = \alpha u_0''(x) - \gamma_1\theta_0'(x) - m^2 u_0(x), \\ \theta(0,x) = \theta_0(x), \theta_t(0,x) = \theta_1(x). \end{cases} \tag{3.37}$$

Applying partial Fourier transformation and using the asymptotic expansion $\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}} = |\xi|\left(1+\frac{m^2}{2\alpha}\,\xi^{-2}\right)+\mathcal{O}(\xi^{-3})$ for $|\xi|\to\infty$, we equivalently transform the system in (3.37) into

$$\tilde{V}_t + \left(\xi^2 A_2 + i\xi A_1 + R_l(\xi)\right)\tilde{V} = 0 \tag{3.38}$$

for a vector function $\tilde{V} = (\hat{v}_+, \hat{v}_-, \hat{\hat{\theta}}_+, \hat{\hat{\theta}}_-)^T$ with $\hat{v}_\pm = \hat{v}_t \pm i\sqrt{\alpha}\,\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}}\,\hat{v}$, $\hat{\hat{\theta}}_\pm = \hat{\theta}_t \pm i\sqrt{\kappa}\,\xi\hat{\theta}$ and matrices $R_l(\xi) = \mathcal{O}(\xi^{-1})$ for $|\xi|\to\infty$,

$$A_1 = \begin{pmatrix} -\operatorname{sgn}(\xi)\sqrt{\alpha} & 0 & \frac{\gamma_1}{2} & \frac{\gamma_1}{2} \\ 0 & \operatorname{sgn}(\xi)\sqrt{\alpha} & \frac{\gamma_1}{2} & \frac{\gamma_1}{2} \\ \frac{\gamma_2}{2} & \frac{\gamma_2}{2} & -\sqrt{\kappa} & 0 \\ \frac{\gamma_2}{2} & \frac{\gamma_2}{2} & 0 & \sqrt{\kappa} \end{pmatrix} \quad \text{and} \quad A_2 = \frac{\delta}{2}\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&1\\0&0&1&1 \end{pmatrix},$$

$\operatorname{sgn}(\xi)$ denotes, as before, the sign of ξ. To this system we can apply our procedure.
We observe, as in the last section, that we can write the components of V as functions of the components of $\tilde{V}$, i.e.,

$$\hat{u}_\pm = g_\pm^{(1)}(\hat{v}_+, \hat{v}_-, \hat{\hat{\theta}}_+, \hat{\hat{\theta}}_-, \xi) \quad \text{and} \quad \hat{\theta}_\pm = g_\pm^{(2)}(\hat{v}_+, \hat{v}_-, \hat{\hat{\theta}}_+, \hat{\hat{\theta}}_-), \tag{3.39}$$

where $g_\pm^{(1)}$ ($\tilde{V}$ treated as constant) behave as $\mathcal{O}(\xi^{-1})$ for $|\xi|\to\infty$, and therewith obtain the same well-posedness result as in Theorem 3.21 and the same result on decay estimates as in Theorem 3.22 with the difference that the solution u itself, measured in the L^q-norm, satisfies the estimate (3.33) as well. Hence, the additional mass term has (just like the additional dissipation term did) a positive influence with respect to the regularity of the data for small frequencies (no need of homogeneous Sobolev spaces).
Concerning the diffusion phenomenon, we obtain with $U(t,x) = \mathcal{F}^{-1}_{\xi\to x}(V(t,\xi))$, $n_s = 2$ and $m_s = 4$ a result as in Theorem 2.24 for the reference system

$$\begin{cases} W_t + M_0 W + M_1 W_x - M_2 W_{xx} + M_4 W_{xxxx} = 0, \\ W(0,x) = LU_0(x) \end{cases}$$

with some constant matrix L coming from the procedure, $U_0 = \mathcal{F}^{-1}(V_0)$, V_0 denoting the initial data to (3.35),

$$\begin{aligned} M_0 &= i\operatorname{diag}(0,0,m,-m), \\ M_1 &= \operatorname{diag}\left(\sqrt{\kappa}, -\sqrt{\kappa}, 0, 0\right), \\ M_2 &= \operatorname{diag}\left(\frac{\delta}{2}, \frac{\delta}{2}, i\frac{\alpha+\gamma_1\gamma_2}{2m}, -i\frac{\alpha+\gamma_1\gamma_2}{2m}\right), \\ M_4 &= \operatorname{diag}(0,0,m_-,m_+) \end{aligned}$$

and

$$m_\mp = \frac{\delta\gamma_1\gamma_2}{2m^2} \mp i\left(\frac{(\alpha+\gamma_1\gamma_2)^2}{8m^3} - \frac{\kappa\gamma_1\gamma_2}{2m^3}\right).$$

3.3. An exceptional model

Having Remark 3.6 in mind, we will consider in this section the Cauchy problem

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x + m^2 u + 2m u_t = 0, \\ \theta_t - \kappa \theta_{xx} + \gamma_2 u_{tx} = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \theta(0,x) = \theta_0(x) \end{cases}$$

for classical thermoelasticity with an additional mass and an additional dissipation term that are related to each other as above.

After partial Fourier transformation the above problem is equivalently transformed into (cf. (3.25))

$$\begin{cases} V_t + \left(B_0 + i\xi B_1^{(1)} + i \langle \xi \rangle_{\frac{m}{\sqrt{\alpha}}} B_1^{(2)} + \xi^2 B_2 \right) V = 0, \\ V(0,\xi) = V_0(\xi) \end{cases} \tag{3.40}$$

for $V = (\hat{u}_+, \hat{u}_-, \hat{\theta})^T$, $\hat{u}_\pm = \hat{u}_t \pm i\sqrt{\alpha} \langle \xi \rangle_{\frac{m}{\sqrt{\alpha}}} \hat{u}$, and matrices

$$B_0 = \begin{pmatrix} m & m & 0 \\ m & m & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_1^{(1)} = \begin{pmatrix} 0 & 0 & \gamma_1 \\ 0 & 0 & \gamma_1 \\ \frac{\gamma_2}{2} & \frac{\gamma_2}{2} & 0 \end{pmatrix},$$

$B_1^{(2)} = \operatorname{diag}(-\sqrt{\alpha}, \sqrt{\alpha}, 0)$ and $B_2 = \operatorname{diag}(0, 0, \kappa)$.

We can proceed as in our considerations in Section 3.2.1 for classical thermoelasticity with an additional mass term. For small frequencies $\xi \in Z_{int}(\sigma)$ we have to study a system

$$V_t + \left(A_0 + i\xi A_1 + \xi^2 A_2 + R_s(\xi)\right) V = 0 \tag{3.41}$$

with $R_s(\xi) = \mathcal{O}(\xi^4)$,

$$A_0 = B_0 + i\frac{m}{\sqrt{\alpha}} B_1^{(2)} = \begin{pmatrix} (1-i)m & m & 0 \\ m & (1+i)m & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$A_1 = B_1^{(1)}$ and $A_2 = i\frac{\sqrt{\alpha}}{2m} B_1^{(2)} + B_2 = \operatorname{diag}(-i\frac{\alpha}{2m}, i\frac{\alpha}{2m}, \kappa)$. For large frequencies $\xi \in Z_{ext}(N)$ we obtain a system as in (3.27) with the additional summand $B_0 V$.

There occur no difficulties in the diagonalization procedure for all but small frequencies. In particular, assumption (C) is satisfied, and we have the usual solution representation (as in Proposition 2.13, *(ii)*) in $Z_{ext}(N)$ with eigenvalues having the same qualitative behavior as the ones from Lemma 3.19, *(ii)*.

3.3.1. Diagonalization for small frequencies

Step 0: Diagonalization modulo $\mathcal{O}(\xi)$-terms

The matrix A_0 is not diagonalizable and thus already (A_0) is not satisfied. We will therefore apply ideas from Section 2.2.2.4. The eigenvalues of A_0 are given by $\lambda_{0,1} = \lambda_{0,2} = m$, $\lambda_{0,3} = 0$ and $\tilde{V}^{(0)} = \left(\tilde{S}^{(0)}\right)^{-1} V$,

$$\tilde{S}^{(0)} = \begin{pmatrix} \frac{1}{m} & -i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

satisfies

$$\tilde{V}_t^{(0)} + \left(J_0 + i\xi \tilde{A}_1^{(0)} + \xi^2 \tilde{A}_2^{(0)} + \left(\tilde{S}^{(0)}\right)^{-1} R_s(\xi)\tilde{S}^{(0)}\right)\tilde{V}^{(0)} = 0$$

with

$$J_0 = \begin{pmatrix} m & 0 & 0 \\ 1 & m & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tilde{A}_1^{(0)} = \begin{pmatrix} 0 & 0 & (1+i)m\gamma_1 \\ 0 & 0 & \gamma_1 \\ \frac{\gamma_2}{2m} & \frac{(1-i)\gamma_2}{2} & 0 \end{pmatrix}, \tilde{A}_2^{(0)} = \begin{pmatrix} -i\frac{\alpha}{2m} & -\alpha & 0 \\ 0 & i\frac{\alpha}{2m} & 0 \\ 0 & 0 & \kappa \end{pmatrix}.$$

Hence, we have not diagonalized but (let's say) $(2,1)$-block-diagonalized modulo $\mathcal{O}(\xi)$-terms as good as possible.

Step 1: Diagonalization modulo $\mathcal{O}(\xi^2)$-terms

Taking account of the structures of J_0 and $\tilde{A}_1^{(0)}$, we observe that there is a matrix $K_{(1)}$ such that $\tilde{V}^{(1)} = (I + i\xi K_{(1)})\tilde{V}^{(0)}$ satisfies

$$\tilde{V}_t^{(1)} + \left(J_0 + \xi^2 \tilde{A}_2^{(1)} + \tilde{A}_3^{(1)}\right)\tilde{V}^{(1)} = 0 \tag{3.42}$$

with $\tilde{A}_3^{(1)}(\xi) = \mathcal{O}(\xi^3)$. The matrices are explicitly given by

$$K_{(1)} = \begin{pmatrix} 0 & 0 & (1+i)\gamma_1 \\ 0 & 0 & -i\frac{\gamma_1}{m} \\ -i\frac{\gamma_2}{2m^2} & -\frac{(1-i)\gamma_2}{2m} & 0 \end{pmatrix}, \quad \tilde{A}_2^{(1)} = \begin{pmatrix} -\frac{\gamma_1\gamma_2 + i(\alpha+\gamma_1\gamma_2)}{2m} & -(\alpha+\gamma_1\gamma_2) & 0 \\ i\frac{\gamma_1\gamma_2}{2m^2} & \frac{\gamma_1\gamma_2 + i(\alpha+\gamma_1\gamma_2)}{2m} & 0 \\ 0 & 0 & \kappa \end{pmatrix}.$$

When looking at the matrix $\tilde{A}_2^{(1)}$, we further see that it is not possible to diagonalize it (or to transform it to the Jordan canonical form of J_0) without losing the almost diagonal structure of J_0.
However, having the previous step and the analyticity of the coefficient matrix from (3.42) near $\xi = 0$ in mind, we know that there is a $\sigma > 0$ such that we can perform a perfect $(2,1)$-block-diagonalization (of $\tilde{A}_3^{(1)}$) in $Z_{int}(\sigma) = \{|\xi| \leq \sigma \ll 1\}$ with the help on an analytic block-diagonalizer $I + K(\xi)$, $K(\xi) = \mathcal{O}(\xi^3)$ (cf. [Kat80]).
The components of $\tilde{V} = (\tilde{V}_1, \tilde{V}_2)^T = (I + K(\xi))\tilde{V}^{(1)}$ thus satisfy

$$\begin{cases} \partial_t \tilde{V}_1 + (M + E)\tilde{V}_1 = 0, \\ \tilde{V}_1(0,\xi) = \tilde{V}_{0,1}(\xi) \end{cases}$$

(cf. (2.24)) with $(\tilde{V}_{0,1}, \tilde{V}_{0,2})^T = TV_0$, $T = T(\xi) = (I + K(\xi))(I + i\xi K_{(1)})\left(\tilde{S}^{(0)}\right)^{-1}$, $E(\xi) = \mathcal{O}(\xi^2)$ and

$$M = \begin{pmatrix} m & 0 \\ 1 & m \end{pmatrix}$$

and

$$\begin{cases} \partial_t \tilde{V}_2 + \mu(\xi)\tilde{V}_2 = 0, \\ \tilde{V}_2(0,\xi) = \tilde{V}_{0,2}(\xi) \end{cases}$$

with $\mu(\xi) = \kappa\xi^2 + \mathcal{O}(\xi^3)$ for $\xi \to 0$.
Solution representations are given by $\tilde{V}_2(t,\xi) = e^{-\mu(\xi)t}\tilde{V}_{0,2}(\xi)$ and

$$\tilde{V}_1(t,\xi) = F(t)\tilde{V}_{0,1}(\xi) - \int_0^t F(t-s)E(\xi)\tilde{V}_1(s,\xi)ds$$

(cf. (2.25)) with

$$F(t) = e^{-mt}\begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}.$$

3.3.2. Results

An application of Gronwall's lemma yields

$$|\tilde{V}_1(t,\xi)| \lesssim |F(t)\tilde{V}_{0,1}(\xi)| + \int_0^t |F(s)\tilde{V}_{0,1}(\xi)||F(t-s)E(\xi)|e^{c\int_s^t |F(t-\tau)E(\xi)|d\tau} ds,$$

which gives for $\xi \in Z_{int}(\sigma)$ the estimate

$$|\tilde{V}_1(t,\xi)| \lesssim e^{-ct}|\tilde{V}_{0,1}(\xi)|$$

for some positive constant c.
The solution to (3.41) is given by $V(t,\xi) = T^{-1}(\xi)\tilde{V}(t,\xi)$. This together with the above considerations (and the application of the usual arguments in $Z_{mid}(\sigma,N)$ and $Z_{ext}(N)$) yields a decay estimate as in Theorem 3.6 with the difference that the solution u itself, measured in the L^q-norm, satisfies the estimate as well (and the parabolic decay rate being determined by $\tilde{V}_2$ in $Z_{int}(\sigma)$). Moreover, we obtain the same well-posedness result as in Theorem 3.5 and analogous results on the propagation of singularities as in Theorem 3.8 (the same results when replacing in *(i)* u_0' by $i\langle D\rangle_{\frac{m}{\sqrt{\alpha}}} u_0$ and $u_x(t,\cdot)$ by $\langle D\rangle_{\frac{m}{\sqrt{\alpha}}} u(t,\cdot)$).
For a result on a diffusion phenomenon we define the reference system

$$\begin{cases} W_t + J_0 W - M_2 W_{xx} = 0, \\ W(0,x) = \left(\tilde{S}^{(0)}\right)^{-1}(u_1 + i\sqrt{\alpha}\langle D\rangle_{\frac{m}{\sqrt{\alpha}}} u_0, u_1 - i\sqrt{\alpha}\langle D\rangle_{\frac{m}{\sqrt{\alpha}}} u_0, \theta_0)^T \end{cases} \tag{3.43}$$

with $M_2 = \operatorname{diag}(0,0,\kappa)$. With the notation $U(t,x) = \mathcal{F}^{-1}_{\xi\to x}(V(t,\xi))$ and an analogous proof to the one for Theorem 2.24 we obtain:

Theorem 3.24. *We assume $u_0, u_1, \theta_0 \in \mathcal{S}$. Then the estimate*

$$\left\|U - \tilde{S}^{(0)}W\right\|_{L^q} \lesssim (1+t)^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\|(\langle D\rangle u_0, u_1, \theta_0)\|_{L^{p,r_p}}$$

holds for dual values $q, 1 < p \leq 2$ and $r_p = \frac{1}{p} - \frac{1}{q}$.

4. Applications to hyperbolic structured models with mass

We consider in this chapter the Cauchy problem for the thermoelasticity model of type 2 with an additional mass term, that is,

$$\begin{cases} u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x + m^2 u = 0, \\ \theta_{tt} - \kappa \theta_{xx} + \gamma_2 u_{ttx} = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \theta(0,x) = \theta_0(x), \theta_t(0,x) = \theta_1(x) \end{cases} \tag{4.1}$$

with the usual assumptions $\alpha, \kappa, \gamma_1\gamma_2, m > 0$.
Noting that the solutions u and θ of (4.1) both satisfy

$$v_{tttt} - (\kappa + \alpha + \gamma_1\gamma_2) v_{ttxx} + m^2 v_{tt} - \kappa m^2 v_{xx} + \alpha\kappa v_{xxxx} = 0, \tag{4.2}$$

i.e., an equation of 4th order with partial derivatives in t of only even order, we can calculate the characteristic roots of the system in (4.1) explicitly. They are given by

$$\mu_{1,2}(\xi) = \pm \frac{i}{\sqrt{2}} \sqrt{(\kappa + \alpha + \gamma_1\gamma_2)\xi^2 + m^2 - \sqrt{\left((\alpha - \kappa + \gamma_1\gamma_2)\,\xi^2 + m^2\right)^2 + 4\kappa\gamma_1\gamma_2\xi^4}},$$

$$\mu_{3,4}(\xi) = \pm \frac{i}{\sqrt{2}} \sqrt{(\kappa + \alpha + \gamma_1\gamma_2)\xi^2 + m^2 + \sqrt{\left((\alpha - \kappa + \gamma_1\gamma_2)\,\xi^2 + m^2\right)^2 + 4\kappa\gamma_1\gamma_2\xi^4}}. \tag{4.3}$$

The eigenvalues are always purely imaginary. We therefore have to divide our considerations into some for the regions of small and large frequencies, where we are able to apply the procedure from Chapter 2, and some for the remaining region of bounded frequencies with a positive distance to zero, where we have to study the behavior of the eigenvalues from the formulas in (4.3) directly.

4.1. The regions of small and large frequencies

The Cauchy problem (4.1) is given by (3.34) when setting $\delta = 0$ there. Hence, we can follow our considerations for the thermoelasticity model of type 3 with an additional mass term in Section 3.2.3 closely.
It is the initial value problem for (3.36) with $\delta = 0$ that we need to study in $Z_{int}(\sigma) = \{|\xi| \leq \sigma \ll 1\}$, and we obtain for the characteristic roots from (4.3) the asymptotic

behavior that is given by the results in Lemma 3.23, *(i)*, when setting $\delta = 0$ and replacing ξ by $|\xi|$ there.
When studying the system (3.35) with $\delta = 0$ for large frequencies, we observe that the first matrix to diagonalize, namely B_2, has a 4-fold eigenvalue 0 with an only three-dimensional eigenspace. We therefore differentiate, as before, in the first equation of (4.1) with respect to t, introduce $v = u_t$, and should now study the initial value problem for (3.38) with $\delta = 0$ in $Z_{ext}(N)$.
The matrix $R_l(\xi)$ can be rewritten as $R_l(\xi) = i\xi^{-1}A_{-1} + \tilde{R}_l(\xi)$ with $\tilde{R}_l(\xi) = \mathcal{O}(\xi^{-3})$ for $|\xi| \to \infty$ and $A_{-1} = \frac{m^2}{2\sqrt{\alpha}} \operatorname{sgn}\xi \cdot \operatorname{diag}(-1, 1, 0, 0)$, and applying our procedure, we find the following asymptotic behavior of the characteristic roots from (4.3):

Lemma 4.1. *The characteristic roots $\mu_j = \mu_j(\xi)$ from (4.3) behave for $|\xi| \geq N \gg 1$ as*

$$\begin{aligned} \mu_{1,2}(\xi) &= \pm i\lambda_-|\xi| \pm ic_+ |\xi|^{-1} + \mathcal{O}(\xi^{-2}), \\ \mu_{3,4}(\xi) &= \pm i\lambda_+|\xi| \pm ic_- |\xi|^{-1} + \mathcal{O}(\xi^{-2}) \end{aligned}$$

with distinct positive numbers

$$\lambda_\pm = \frac{1}{\sqrt{2}}\sqrt{\kappa + \alpha + \gamma_1\gamma_2 \pm \sqrt{(\alpha - \kappa + \gamma_1\gamma_2)^2 + 4\kappa\gamma_1\gamma_2}} \tag{4.4}$$

and positive numbers

$$c_\pm = \pm\frac{m^2(\lambda_\pm^2 - \alpha)\lambda_\mp}{2\alpha(\lambda_+^2 - \lambda_-^2)}.$$

4.2. The region of bounded frequencies with a positive distance to zero

We are now interested in the behavior of the characteristic roots from the system in (4.1) in $Z_{mid}(\sigma, N) = \{\sigma \leq |\xi| \leq N\}$. By studying the formulas in (4.3) directly, we obtain:

Lemma 4.2. *The purely imaginary eigenvalues $\mu_j = \mu_j(\xi)$, given by (4.3), have in $Z_{mid}(\sigma, N) = \{\sigma \leq |\xi| \leq N\}$ the following properties:*

(i) They are distinct.

(ii) We have $0 < C_1 \leq |d_\xi \mu_j| \leq C_2$.

(iii) The second derivatives $d_\xi^2 \mu_j$ have for $j = 1, 2$ in the case

$$4\alpha(\alpha - \kappa) + (5\kappa + 7\alpha)\gamma_1\gamma_2 + 3(\gamma_1\gamma_2)^2 \geq 0 \tag{4.5}$$

exactly one pair of roots $\pm\xi_0$ and otherwise at the most 4 such pairs.
If (4.5) holds, then $\pm\xi_0$ are points of inflection of μ_j, more accurate, $d_\xi^3 \mu_j(\pm\xi_0) \neq 0$.

(iv) The second derivatives $d_\xi^2\mu_j$ have for $j = 3, 4$ in the case

$$\alpha(\alpha-\kappa)+(\kappa+2\alpha)\gamma_1\gamma_2+(\gamma_1\gamma_2)^2 \geq 0 \tag{4.6}$$

no real roots.

Proof. The distinctiveness of the eigenvalues follows directly from the formulas in (4.3) and the upper bound in *(ii)* for $|d_\xi\mu_j|$ from the Lemmas 3.23, 4.1 and continuity arguments. To prove the statement on the lower bound for $j = 1, 2$, we need to guarantee that for any $\xi \neq 0$

$$\frac{d\mu_1}{d\xi} = \left(\frac{d}{d\xi}\,\mu_1^2\right)\frac{1}{2\mu_1} \neq 0.$$

Let us assume that $\frac{d}{d\xi}\,\mu_1^2(\xi) = 0$ for a $\xi \neq 0$. Then we obtain the equality

$$\begin{aligned}\left((\alpha-\kappa+\gamma_1\gamma_2)^2+4\kappa(\alpha+\gamma_1\gamma_2)\right)&\left(\left((\alpha-\kappa+\gamma_1\gamma_2)\,\xi^2+m^2\right)^2+4\kappa\gamma_1\gamma_2\xi^4\right)\\ =\ &\left(\left((\alpha-\kappa+\gamma_1\gamma_2)\,\xi^2+m^2\right)(\alpha-\kappa+\gamma_1\gamma_2)+4\kappa\gamma_1\gamma_2\xi^2\right)^2.\end{aligned}$$

By basic calculations we conclude the equivalence to

$$0 \ = \ \alpha\left(\left((\alpha-\kappa+\gamma_1\gamma_2)\xi^2+m^2\right)^2+4\kappa\gamma_1\gamma_2\xi^4\right)+m^4\gamma_1\gamma_2 > 0.$$

Thus, we have constructed a contradiction, and continuity arguments yield the statements in *(ii)* for $j = 1, 2$. Similar considerations for $d_\xi\mu_3$ yield the remaining statements in *(ii)*.
By some calculations one can moreover conclude that $d_\xi^2\mu_1(\xi) = 0$ is equivalent to

$$a\,\xi^8 - b\,\xi^4 - c\,\xi^2 - d = 0$$

with

$$\begin{aligned} a &= \alpha((\kappa-\alpha+\gamma_1\gamma_2)^2+4\alpha\gamma_1\gamma_2)^2,\\ b &= 3m^4(2\alpha+\gamma_1\gamma_2)((\kappa-\alpha+\gamma_1\gamma_2)^2+4\alpha\gamma_1\gamma_2),\\ c &= 2m^6(4\alpha(\alpha-\kappa)+(5\kappa+7\alpha)\gamma_1\gamma_2+3(\gamma_1\gamma_2)^2),\\ d &= 3m^8(\alpha+\gamma_1\gamma_2).\end{aligned}$$

The numbers a, b and d are positive. If we assume that $c \geq 0$, then the above equation has exactly two solutions $\pm\xi_0$ and $d_\xi^2\mu_1$ therefore exactly these two roots .
We assume now that (4.5) holds. Taking the information from the Lemmas 3.23 (with $\delta = 0$ and ξ replaced by $|\xi|$) and 4.1 into consideration, we know that $\mu_1''(\xi)$ changes its sign, when passing through one of its roots $\pm\xi_0$. Hence, $\pm\xi_0$ are points of inflection of $\mu_1 = \mu_1(\xi)$.
Further, we know that μ_1 satisfies the polynomial equation

$$\mu_1^4+((\kappa+\alpha+\gamma_1\gamma_2)\xi^2+m^2)\mu_1^2+\kappa m^2\xi^2+\alpha\kappa\xi^4 = 0. \tag{4.7}$$

Let us assume that for an $\eta \neq 0$ we have

$$\mu_1''(\eta) = \mu_1'''(\eta) = \mu_1^{(4)}(\eta) = 0. \tag{4.8}$$

We differentiate four times with respect to ξ in (4.7) and evaluate the obtained in η. Then we get

$$(\mu_1'(\eta))^4 + (\kappa + \alpha + \gamma_1\gamma_2)(\mu_1'(\eta))^2 + \kappa\alpha = 0.$$

Hence, $\mu_1'(\eta)$ takes one of the values

$$\mu_1'(\eta)_{(1,2)} = \pm i\lambda_- \qquad \text{or} \qquad \mu_1'(\eta)_{(3,4)} = \pm i\lambda_+$$

with the positive and distinct numbers $\lambda_\pm$ from (4.4). Let us assume that $\eta > 0$. Then we know that $-i\mu_1'(\eta) > 0$. Thus, $\mu_1'(\eta)$ can not take the values $\mu_1'(\eta)_{(2)}$ and $\mu_1'(\eta)_{(4)}$. By some calculations we can furthermore show that $\sqrt{\kappa} < \lambda_+$. With this, $\lambda_+ > \lambda_-$ and using the information from the Lemmas 3.23 and 4.1 as well as the properties of μ_1'', we have proved that $\mu_1'(\eta)$ can not take any of the values $\mu_1'(\eta)_{(i)}$. Analogous calculations of course hold for $\eta < 0$. Consequently, (4.8) can not occur, and $\mu_1''' = \mu_1'''(\xi)$ is not vanishing in the roots of μ_1''.
The statement in *(iv)* is proved by similar calculations. □

We have a clear understanding of the behavior of the roots from (4.3) only under the conditions (4.5) and (4.6). Let us therefore introduce the assumption

(D) $\qquad 4\alpha(\alpha - \kappa) + (\kappa + 2\alpha)\gamma_1\gamma_2 + (\gamma_1\gamma_2)^2 \geq 0$

and note that it implies (4.5) and (4.6) and is satisfied when $\alpha \geq \kappa$. The latter should be, at least to the authors knowledge, satisfied in all practically useful cases.

Remark 4.1. Assuming (D) and reconsidering the results from the Lemmas 3.23 (part *(i)* with $\delta = 0$ and ξ replaced by $|\xi|$), 4.1 and 4.2, one comes to the conclusion that while the last two roots behave very similar to roots of the classical Klein-Gordon equation, the first two roots are in their behavior in between of that of roots of the classical Klein-Gordon and the classical wave equation.

Using the partial differential equation (4.2), we can write down explicit solution representations for u and θ consisting of Fourier multipliers of the type

$$\mathcal{F}^{-1}\left(a(\xi)e^{i\mu_j(\xi)t}\mathcal{F}(w)(\xi)\right)$$

with in $Z_{mid}(\sigma, N)$ smooth amplitudes $a = a(\xi)$, $j \in \{1, \dots, 4\}$ and w being from the set $\{u_0, u_1, \theta_0, \theta_1\}$.

4.3. Results

With the observation in (3.39) we immediately obtain a well-posedness result as in Theorem 3.21.
Now we are concerned with L^p-L^q decay estimates. For the regions of small and large frequencies we can apply results from Chapter 2. For the region of bounded frequencies with a positive distance to zero we need to make some extra considerations and therefore first study Fourier multipliers of the type

$$\mathcal{F}^{-1}\left(a(\xi)e^{i\mu(\xi)t}\chi(\xi)\mathcal{F}(w)(\xi)\right),$$

where $\chi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subset [c_1, c_2]$, $0 < c_1 < c_2 < \infty$ or $-\infty < c_1 < c_2 < 0$, $a = a(\xi)$ is smooth on $\operatorname{supp}\chi$ and $\mu = \mu(\xi)$ is a real-valued, smooth function with $|d_\xi^k \mu| \geq c > 0$ on $\operatorname{supp}\chi$ for a $k \geq 2$.

Proposition 4.3. *We obtain the decay estimate*

$$\left\|\mathcal{F}^{-1}\left(a(\xi)e^{i\mu(\xi)t}\chi(\xi)\mathcal{F}(w)(\xi)\right)\right\|_{L^q} \lesssim (1+t)^{-\frac{1}{k}\left(\frac{1}{p}-\frac{1}{q}\right)}\|w\|_{L^p}$$

for $1 \leq p \leq 2$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The L^2-L^2 estimate is given by

$$\left\|\mathcal{F}^{-1}\left(a(\xi)e^{i\mu(\xi)t}\chi(\xi)\mathcal{F}(w)(\xi)\right)\right\|_{L^2} \lesssim \|w\|_{L^2}. \tag{4.9}$$

For the L^1-L^∞ estimates we have for small times $t \leq 1$

$$\left\|\mathcal{F}^{-1}\left(a(\xi)e^{i\mu(\xi)t}\chi(\xi)\mathcal{F}(w)(\xi)\right)\right\|_{L^\infty} \lesssim \|w\|_{L^1} \tag{4.10}$$

and for large times $t \geq 1$ with

$$\left\|\mathcal{F}^{-1}\left(e^{i\mu(\xi)t}a(\xi)\chi(\xi)\right)\right\|_{L^\infty} \lesssim t^{-\frac{1}{k}},$$

where we have used Corollary B.4, the estimate

$$\left\|\mathcal{F}^{-1}\left(e^{i\mu(\xi)t}a(\xi)\chi(\xi)\mathcal{F}(w)(\xi)\right)\right\|_{L^\infty} \lesssim t^{-\frac{1}{k}}\|w\|_{L^1}. \tag{4.11}$$

Combining (4.10) and (4.11) with (4.9) and applying the Riesz-Thorin interpolation theorem proves our assertions. □

Now we can state with the help of the Lemmas 3.23 (part *(i)* with $\delta = 0$), 4.1 and the results of Chapter 2 as well as the considerations from the last section and the above Proposition 4.3:

Theorem 4.4. *Assume* $u_0, u_1, \theta_0, \theta_1 \in \mathcal{S}$ *and that the assumption (D) is satisfied. Then the following* L^p-L^q *decay estimates hold for solutions* (u, θ) *to the Cauchy problem (4.1):*

$$\|(u, u_t, u_x)\|_{L^q} \lesssim (1+t)^{-\frac{1}{3}\left(\frac{1}{p}-\frac{1}{q}\right)}\left\|\left(\langle D\rangle^2 u_0, \langle D\rangle u_1, \langle D\rangle \theta_0, \theta_1\right)\right\|_{L^{p,r_p-1}}$$

and

$$\|(\theta_t, \theta_x)\|_{L^q} \lesssim (1+t)^{-\frac{1}{3}\left(\frac{1}{p}-\frac{1}{q}\right)} \left\|(\langle D\rangle^2 u_0, \langle D\rangle u_1, \langle D\rangle \theta_0, \theta_1)\right\|_{L^{p,r_p}}$$

for dual q, $1 < p \leq 2$ *and* $r_p = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$.

Remark 4.2. The decay is generated from the mass term. It therefore has an essential helping character for the thermoelasticity model of type 2. However, if we have an additional dissipation term instead, then that generates an even stronger decay (cf. with Theorem 3.22).

5. Thermoelasticity in 3D

5.1. Theoretical considerations

5.1.1. The problem

The reasonable replacement of (2.4) for studying linear thermoelasticity models in 3D for isotropic media as discussed in Section 1.1 (after applying a Helmholtz decomposition) is the system

$$\tilde{V}_t + (B_0(\eta) + i|\xi|B_1(\eta) + |\xi|^2 B_2(\eta))\tilde{V} = 0 \tag{5.1}$$

for a d-dimensional unknown $\tilde{V} = \tilde{V}(t,\xi)$, $(t,\xi) \in \mathbb{R}_{>0} \times \mathbb{R}^3$, and matrices $B_k = B_k(\eta)$, $\eta = \xi/|\xi| \in \mathbb{S}^2$, that are given by smooth functions $B_k : \mathbb{S}^2 \to \mathbb{C}^{d\times d}$ (compare e.g. to (5.18)).

However, the η-dependence of the matrices B_k in (5.1) comes for the considered problems - aside from the models involving second sound thermoelasticity - only from the coupling terms and has in any case such a special structure that there are matrices $L = L(\eta)$ and $R = R(\eta)$ with $LR = I$ that are - at least for overlapping open subsets of $\mathbb{S}^2$ - smooth in η such that $V = L\tilde{V}$ satisfies a system

$$V_t + (A_0 + i|\xi|A_1 + |\xi|^2 A_2)V = 0, \tag{5.2}$$

where the matrices $A_k \in \mathbb{C}^{d\times d}$ are independent of η. I.e., the η-dependence of the matrices B_k occurs only, because we are displaying the coefficient matrix $B(\xi) = B_0(\eta) + i|\xi|B_1(\eta) + |\xi|^2 B_2(\eta)$ in (5.1) in the wrong basis.

The initial value problem that we will therefore study in detail in Section 5.1 is

$$\begin{cases} \tilde{V}_t + (B_0(\eta) + i|\xi|B_1(\eta) + |\xi|^2 B_2(\eta))\tilde{V} = 0, \\ \tilde{V}(0,\xi) = \tilde{V}_0(\xi) \end{cases} \tag{5.3}$$

for a d-dimensional unknown $\tilde{V} = \tilde{V}(t,\xi)$, $(t,\xi) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^3$, and matrices $B_k = B_k(\eta)$ that are given in such a way that the coefficient matrix in (5.3) can be rewritten into one as in (5.2) via a similarity transformation involving a matrix that is - at least for overlapping open subsets of $\mathbb{S}^2$ - smooth in η.

The equivalent problem to (5.3) in physical space then is

$$\begin{cases} U_t + B(D)U = 0, \\ U(0,x) = U_0(x) \end{cases} \tag{5.4}$$

for a d-dimensional unknown $U = U(t,x)$, $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^3$, and B(D) being a pseudo-differential operator having a symbol $B(\xi) = B_0(\eta) + i|\xi|B_1(\eta) + |\xi|^2 B_2(\eta)$ with the same features as the coefficient matrix in (5.3).
We note that the only significant change when comparing (2.4) with (5.2) is that $\xi \in \mathbb{R}^1$ is replaced by $|\xi| \in \mathbb{R}^1_{\geq 0}$.
We can therefore apply the diagonalization procedure from Section 2.2 to (5.2). With the same assumptions $(\mathrm{A_n})$, $(\mathrm{A_n'})$, $(\mathrm{B_n})$ and $(\hat{\mathrm{A}}_\mathrm{n})$, $(\hat{\mathrm{A}}_\mathrm{n}')$, $(\hat{\mathrm{B}}_\mathrm{n})$ we obtain in particular the same results as in Lemma 2.6 and as in the Propositions 2.9, 2.12 and 2.13, when replacing ξ there by $|\xi|$ in the right spots. Moreover, replacing the previous condition (C) by

$$\textbf{(C)} \quad \exists C = C(\sigma, N) > 0 \, \forall \xi \in Z_{mid}(\sigma, N) \, \forall \mu(|\xi|) \in \operatorname{spec}(B(\xi)) : \operatorname{Re} \mu(|\xi|) \geq C > 0,$$

we obtain a result as in Proposition 2.14.
In the upcoming sections we will study the initial value problem (5.4) in detail and try to reproduce results that we have obtained for (2.2) in our 1D considerations in Chapter 2. Let us therefore first agree on the convention that when assuming the existence of a number n such that $(\mathrm{A_n})$, $(\mathrm{A_n'})$, $(\mathrm{B_n})$, $(\hat{\mathrm{A}}_\mathrm{n})$, $(\hat{\mathrm{A}}_\mathrm{n}')$ or $(\hat{\mathrm{B}}_\mathrm{n})$ are satisfied for (5.4) we mean that this is the case when applying the diagonalization procedure to the associated system of type (5.2).

5.1.2. Well-posedness

With the condition

$$\forall \xi \in Z_{ext}(N) \, \forall \mu(|\xi|) \in \operatorname{spec}(B(\xi)) : \operatorname{Re} \mu(|\xi|) \geq 0 \tag{5.5}$$

for the eigenvalues $\mu = \mu(|\xi|)$ of the symbol of $B(D)$ in (5.4) and an analogous proof to the one for Theorem 2.15 we obtain:

Theorem 5.1. *(H^s well-posedness)*
We consider the Cauchy problem (5.4). Suppose that for some n the assumptions $(\hat{A}_n)$ and $(\hat{B}_n)$ hold together with (5.5) and that we have $U_0 \in H^s$ for a fixed $s \in \mathbb{R}$.
Then there exists a (in $C([0,\infty), \mathcal{S}')$) unique solution satisfying

$$U \in C([0,\infty), H^s).$$

Further, we get for an arbitrary $T > 0$ the a priori estimate

$$\|U\|_{C([0,T],H^s)} \leq C_T \|U_0\|_{H^s}$$

with a T-dependent constant C_T.

5.1.3. L^p-L^q decay estimates

We will, as in the 1D considerations, state and prove decay estimates for solutions to the Cauchy problem (5.4) in the regions of small, bounded and large frequencies and therefore introduce functions $\phi_{int}, \phi_{mid}, \phi_{ext} \in C^\infty(\mathbb{R}^3)$ having their support in $Z_{int}(\sigma)$, $Z_{mid}(\sigma/2, 2N)$ and $Z_{ext}(N)$, respectively, so that $\phi_{int} + \phi_{mid} + \phi_{ext} \equiv 1$.
In contrary to our 1D considerations in Section 2.4 we will not try to draw complete pictures for all possible situations in the regions of small and large frequencies, but provide only the estimates that we need for the applications that we have in mind.

5.1.3.1. L^p-L^q decay estimates for small frequencies

We assume $U_0 \in \mathcal{S}$, the existence of a number n so that the assumptions ($\mathrm{A_n}$) and ($\mathrm{B_n}$) hold and with $B(\xi)$ denoting the symbol of $B(D)$ in (5.4)

$$\forall \xi \in Z_{int}(\sigma)\, \forall \mu(|\xi|) \in \operatorname{spec}(B(\xi)) : \operatorname{Re}\mu(|\xi|) \geq 0.$$

Then we have the solution representation

$$\phi_{int}(D)U = \mathcal{F}^{-1}\left(R(\eta)T_{int}^{-1}(|\xi|)\operatorname{diag}(e^{-\mu_1(|\xi|)t}, \ldots, e^{-\mu_d(|\xi|)t})T_{int}(|\xi|)L(\eta)\phi_{int}(\xi)\hat{U}_0(\xi)\right). \tag{5.6}$$

Due to our assumption, the matrices $L = L(\eta)$ and $R = R(\eta)$ may be smooth only in overlapping open subsets of $\mathbb{S}^2$. If that is the case, then we proceed - without actually carrying it out - locally in open subsets $O_j \subset \mathbb{S}^2$, $\mathbb{S}^2 = \bigcup_{j=1}^k O_j$, in which we can choose $L = L(\eta)$ and $R = R(\eta)$ smooth in η and, using a partition of unity, later on glue the results together. We go on by writing

$$\phi_{int}(D)U = \left(\sum_{j,r=1}^{d} \mathcal{F}^{-1}\left(c_{jrk}(\xi)e^{-\mu_j(|\xi|)t}\phi_{int}(\xi)\hat{U}_{0,r}(\xi)\right)\right)_{k=1}^{d}$$

with $\hat{U}_0(\xi) = (\hat{U}_{0,1}(\xi), \ldots, \hat{U}_{0,d}(\xi))^T$ and functions $c_{jrk} = c_{jrk}(\xi)$ being made up of entries of $T_{int}(|\xi|)L(\eta)$ and its inverse.
The multipliers for which we have to derive estimates are therefore

$$\mathcal{F}^{-1}\left(a(\xi)e^{-\mu(|\xi|)t}\phi_{int}(\xi)\hat{v}(\xi)\right), \tag{5.7}$$

where $v \in \mathcal{S}$ and $a = a(\xi)$ is given in such a way that $a(\xi)\phi_{int}(\xi) \in C^2(\mathbb{R}^3\backslash\{0\})$ satisfies for any $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 2$ the estimate

$$\left|\partial_\xi^\alpha(a\phi_{int})(\xi)\right| \leq M|\xi|^{-|\alpha|} \qquad \text{for all } \xi \neq 0$$

and is thus a multiplier in L^q for $1 < q < \infty$ (cf. Lemma B.7). We further assume $\mu = \mu(\rho)$ to be analytic and to satisfy $\operatorname{Re}\mu(\rho) \geq 0$ near $\rho = 0$. In particular, μ allows asymptotic expansions

$$\mu(|\xi|) = \lambda_0 + i\lambda_1|\xi| + \lambda_2|\xi|^2 + \ldots + a_n\lambda_n|\xi|^n + \mathcal{O}(|\xi|^{n+1}) \tag{5.8}$$

for any n on $\operatorname{supp}\phi_{int}$, the numbers λ_k are in general complex, $a_n = i$ for odd n and $a_n = 1$ for even n (cf. with Lemma 2.6).
As in the 1D considerations, we should distinguish the two cases in which μ is purely imaginary and in which μ has a non-vanishing real part on $\operatorname{supp}\phi_{int}\backslash\{0\}$. We will restrict here solely to the latter one, i.e., we consider only situations in which

$$a_k\lambda_k \in i\mathbb{R} \text{ for } 0 \leq k \leq m-1, \quad \operatorname{Re}(a_m\lambda_m) > 0, \tag{5.9}$$

for a number $m \in \mathbb{N}_0$.

Proposition 5.2. *For the multiplier in (5.7) we obtain in the situations (5.9) for dual values q, $1 \leq p \leq 2$ the estimates*

$$\left\|\mathcal{F}^{-1}\left(a(\xi)e^{-\mu(|\xi|)t}\phi_{int}(\xi)\hat{v}(\xi)\right)\right\|_{L^q} \lesssim e^{-ct}\|v\|_{L^p}$$

with a positive constant c if $m = 0$ and

$$\left\|\mathcal{F}^{-1}\left(a(\xi)e^{-\mu(|\xi|)t}\phi_{int}(\xi)\hat{v}(\xi)\right)\right\|_{L^q} \lesssim (1+t)^{-\frac{3}{m}\left(\frac{1}{p}-\frac{1}{q}\right)}\|v\|_{L^p}$$

otherwise.

Proof. We derive first, as in previous considerations, L^2-L^2 and L^1-L^∞ estimates. The derivation of the L^2-L^2 estimates is straightforward using that $a(\xi)\phi_{int}(\xi)$ is bounded. For the L^1-L^∞ estimates we use analogous arguments as in the proof of Theorem 2.16 in the cases $n_{s,j} = p_j < \infty$. The assertions then follow from applying the Riesz-Thorin interpolation theorem. □

From our considerations in 1D we know that it might also be the imaginary part of $\mu = \mu(|\xi|)$ that is determining the decay of the multiplier (5.7) in the situations (5.9), i.e., when μ has a non-vanishing real part. Hence, the estimates in Proposition 5.2 might not be optimal. Such questions we want to discuss for each application separately. Let us nevertheless consider one situation in which we can improve the results from Proposition 5.2, that is, we assume additionally to (5.9)

$$\lambda_k = 0 \text{ for } 0 < k < l, \quad 0 \neq a_l\lambda_l \in i\mathbb{R}, \quad \lambda_k = 0 \text{ for } l < k < m, \tag{5.10}$$

$l \geq 2$, $m \geq 4$ and $l < m$.

Proposition 5.3. *For the multiplier in (5.7) we obtain in the situations (5.10) for arbitrary but fixed positive numbers ε and dual values q, $1+\varepsilon \leq p \leq 2$ the estimate*

$$\left\|\mathcal{F}^{-1}\left(a(\xi)e^{-\mu(|\xi|)t}\phi_{int}(\xi)\hat{v}(\xi)\right)\right\|_{L^q} \leq C_\varepsilon(1+t)^{-\frac{3}{l}\left(\frac{1}{p}-\frac{1}{q}\right)}\|v\|_{L^{p_\varepsilon}},$$

where C_ε is an ε-dependent constant and $p_\varepsilon = p\left(1-\varepsilon\frac{p}{1+(1+p)\varepsilon}\right)$.

Remark 5.1. The constant ε may be arbitrarily small. Hence, we find the desired estimate for any fixed dual q, $1 < p \le 2$, but with v measured in a L^{p_ε}- instead of the L^p-norm, $\varepsilon \le p-1$ small. Here, we pay for including quite complex situations in the setting (5.10).

Proof. Since it is the imaginary part of μ that is determining the decay we will have to make use of the theory of oscillatory integrals.
Using our assumption on $a = a(\xi)$ we obtain

$$\left\|\mathcal{F}^{-1}\left(a(\xi)e^{-\mu(|\xi|)t}\phi_{int}(\xi)\hat{v}(\xi)\right)\right\|_{L^q} \lesssim \left\|\mathcal{F}^{-1}\left(e^{ic|\xi|^l t}\mathcal{F}_{x\to\xi}(f(t,\cdot))(\xi)\right)\right\|_{L^q}$$

with $c = -\mathrm{Im}\,(a_l\lambda_l) \neq 0$, $f(t,x) = \mathcal{F}^{-1}_{\xi\to x}\left(e^{-\nu(|\xi|)t}\psi(\xi)\hat{v}(\xi)\right)$, $\psi = \psi(\xi) \in C_0^\infty(\mathbb{R}^3)$ with $\psi \equiv 1$ on $\operatorname{supp}\phi_{int}$, $\operatorname{supp}\psi$ sufficiently small, and $\nu = \nu(|\xi|)$ in particular satisfying

$$\nu(|\xi|) = d|\xi|^m + \mathcal{O}(|\xi|^{m+1}), \qquad \mathrm{Re}\,(d) = \mathrm{Re}\,(a_m\lambda_m) > 0$$

on $\operatorname{supp}\psi$. An application of Corollary B.6 implies

$$\left\|\mathcal{F}^{-1}\left(e^{ic|\xi|^l t}\hat{f}(t,\xi)\right)\right\|_{L^q} \lesssim t^{-\frac{3}{l}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f(t,\cdot)\|_{L^p}$$

for dual q, $1 < p \le 2$ and $t \ge 1$. Deriving L^2-L^2, L^1-L^∞ estimates and interpolating for small times $0 < t \le 1$ we arrive at

$$\left\|\mathcal{F}^{-1}\left(e^{ic|\xi|^l t}\hat{f}(t,\xi)\right)\right\|_{L^q} \lesssim (1+t)^{-\frac{3}{l}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f(t,\cdot)\|_{L^p}$$

for $t > 0$.
It is now our goal to estimate $\|f(t,\cdot)\|_{L^p}$ by a term independent of t. Young's inequality implies for any $\varepsilon > 0$

$$\begin{aligned}\|f(t,\cdot)\|_{L^p} &\le \left\|\mathcal{F}^{-1}\left(e^{-\nu(|\xi|)t}\psi(\xi)\right) * v\right\|_{L^p}\\ &\le \left\|\mathcal{F}^{-1}\left(e^{-\nu(|\xi|)t}\psi(\xi)\right)\right\|_{L^{1+\varepsilon}}\|v\|_{L^{p_\varepsilon}},\end{aligned}$$

where $p_\varepsilon = p\,\frac{1+\varepsilon}{1+\varepsilon+p\varepsilon} = p\left(1-\varepsilon\frac{p}{1+(1+p)\varepsilon}\right)$ and $p \ge 1+\varepsilon$ must be guaranteed.
For the left factor in the above right-hand side we estimate

$$\left\|\mathcal{F}^{-1}\left(e^{-\nu(|\xi|)t}\psi(\xi)\right)\right\|^{1+\varepsilon}_{L^{1+\varepsilon}} \lesssim 1 + \int_{|x|\ge 1}\left|\int_{\mathbb{R}^3} e^{ix\cdot\xi}e^{-\nu(|\xi|)t}\psi(\xi)d\xi\right|^{1+\varepsilon}dx.$$

For the latter summand we use $Le^{ix\cdot\xi} = e^{ix\cdot\xi}$ with $L = \frac{1}{i|x|^2}\,x\cdot\nabla_\xi$, partial integration and calculations as in

$$\int_{|\xi|\le c}|\xi|^{m-k}t\,e^{-|\xi|^m t}d\xi \lesssim (1+t)^{\frac{k-3}{m}}$$

for $k < m$. More specific, we partially integrate exactly three times, note that the appearing decay-determining terms are the above ones with $k=3$ and that $1/|x|^3 \in L^{1+\varepsilon}(\mathbb{R}^3\backslash\{|x|\le 1\})$ for any positive $\varepsilon > 0$. Thus, we have proved

$$\left\|\mathcal{F}^{-1}\left(e^{-\nu(|\xi|)t}\psi(\xi)\right)\right\|_{L^{1+\varepsilon}} \le C_\varepsilon$$

for some ε-dependent constant C_ε. □

5.1.3.2. L^p-L^q decay estimates for large frequencies

We assume, in analogy to our considerations for small frequencies, $U_0 \in \mathcal{S}$, the existence of a number n so that the assumptions ($\hat{\mathrm{A}}_\mathrm{n}$) and ($\hat{\mathrm{B}}_\mathrm{n}$) hold and

$$\forall \xi \in Z_{ext}(N)\, \forall \mu(|\xi|) \in \operatorname{spec}(B(\xi)) : \operatorname{Re}\mu(|\xi|) \geq 0.$$

We thus obtain solution representations consisting of Fourier multipliers

$$\mathcal{F}^{-1}\left(a(\xi)e^{-\mu(|\xi|)t}\phi_{ext}(\xi)\hat{v}(\xi)\right), \tag{5.11}$$

where $v \in \mathcal{S}$, $a = a(\xi)$ is bounded on $\operatorname{supp}\phi_{ext}$ and $\mu = \mu(|\xi|)$ allows an asymptotic expansion

$$\mu(|\xi|) = \hat{\lambda}_2|\xi|^2 + i\hat{\lambda}_1|\xi| + \hat{\lambda}_0 + \mathcal{O}(|\xi|^{-1})$$

on $\operatorname{supp}\phi_{ext}$ together with $\operatorname{Re}\mu(|\xi|) \geq 0$.
For the applications that we have in mind it is sufficient to restrict not only to the situations in which μ has a positive real part in the whole zone $Z_{ext}(N)$, but to the ones, where it generates an exponential decay, i.e., we assume

$$a_k\hat{\lambda}_{2-k} \in i\mathbb{R} \text{ for } 0 \leq k < m, \quad \operatorname{Re}(a_m\hat{\lambda}_{2-m}) > 0 \tag{5.12}$$

and $m \in \{0, 1, 2\}$. With analogous calculations as in the proofs of Theorem 2.18 (for the cases $p_j \leq 0$) and Theorem 2.17 we obtain:

Proposition 5.4. *For the Fourier multiplier in (5.11) we obtain in the situations (5.12) for dual values q, $1 < p \leq 2$ the estimate*

$$\left\|\mathcal{F}^{-1}\left(a(\xi)e^{-\mu(|\xi|)t}\phi_{ext}(\xi)\hat{v}(\xi)\right)\right\|_{L^q} \lesssim e^{-ct}\|v\|_{L^{p,r_p}}$$

with a positive constant c and $r_p = 3\left(\frac{1}{p} - \frac{1}{q}\right)$.

5.1.3.3. L^p-L^q decay estimates for bounded frequencies away from zero

With the help of the assumption (C) and a result as in Proposition 2.14 we immediately obtain:

Proposition 5.5. *We assume $U_0 \in \mathcal{S}$ and (C). Then we have for solutions U to (5.4) the estimate*

$$\|\phi_{mid}(D)U\|_{L^q} \lesssim e^{-ct}\|U_0\|_{L^p}$$

with a positive constant c, $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

5.1.4. Diffusion phenomena

The Cauchy problem (5.4) includes in particular hyperbolic-parabolic coupled systems. As we have done it in the 1D considerations for the Cauchy problem to system (2.2), we could therefore ask whether we can find an underlying asymptotic parabolic structure to solutions of (5.4) in such cases, that is, whether we can prove diffusion phenomena.
We consider only the decay-determining region of the phase space and, since we want to provide here, as in the previous section, only results that we need for the applications that we have in mind instead of trying to cover all possible situations, restrict our view solely to the region of small frequencies.
Suppose now $U_0 \in \mathcal{S}$, that a number n exists such that ($\mathrm{A_n}$) and ($\mathrm{B_n}$) hold and hence that we have a solution representation as in (5.6) with asymptotic expansions for all eigenvalues $\mu_j = \mu_j(|\xi|)$ as in (5.8), that is,

$$\mu_j(|\xi|) = \lambda_{0,j} + i\lambda_{1,j}|\xi| + \lambda_{2,j}|\xi|^2 + \ldots + a_n\lambda_{n,j}|\xi|^n + \mathcal{O}(|\xi|^{n+1}).$$

We are looking for an underlying parabolic structure for large times t and therefore restrict further onto the cases, where all eigenvalues have a positive real part in $Z_{int}(\sigma)\backslash\{0\}$. We define p_j to be the smallest number k for which $\mathrm{Re}\,(a_k\lambda_{k,j}) > 0$, $m_s := \max_{j=1,\ldots,d} p_j$, and exclude the cases in which $m_s = 0$, i.e., the ones in which we obtain an exponential decay for solutions to (5.4) in the region of small frequencies.
Now we define the reference system:

$$\begin{cases} W_t + \displaystyle\sum_{k=0}^{m_s} a_k M_k |D|^k W = 0, \\ W(0,x) = \tilde{L}^{(c)} \cdot \ldots \cdot \tilde{L}^{(0)} \mathcal{F}^{-1}\Big(L(\eta)\hat{U}_0(\xi)\Big), \end{cases} \tag{5.13}$$

where $a_k = i$ for odd k, $a_k = 1$ for even k, $M_k = \mathrm{diag}(m_1^{(k)}, \ldots, m_d^{(k)})$ with $m_j^{(k)} = 0$ if $k > p_j$ and $m_j^{(k)} = \lambda_{k,j}$ otherwise and $|D|^k$ denotes an operator with symbol $|\xi|^k$. The matrix $L = L(\eta)$ is the one from the similarity transformation that transfers the system in (5.3) into one as in (5.2), the matrices $\tilde{L}^{(i)}$ are from the diagonalization procedure and c is the minimal number for which ($\mathrm{B_n}$) holds.

Lemma 5.6. *Assume that $U_0 \in H^s$, $s \in \mathbb{R}$. Then there exists a (in $C([0,\infty),\mathcal{S}')$) unique solution to (5.13) with regularity*

$$W \in C([0,\infty), H^s).$$

With an analogous proof to the one for Theorem 2.24 we obtain:

Theorem 5.7. *Assume $U_0 \in \mathcal{S}$, the existence of a number n so that (A_n) and (B_n) hold and that all eigenvalues $\mu_j = \mu_j(|\xi|)$ have a positive real part in $Z_{int}(\sigma)\backslash\{0\}$. Define the number m_s as above and assume it to be positive.*
Then we obtain for the solution U to the Cauchy problem (5.4) the estimate

$$\left\|\mathcal{F}^{-1}\Big(\phi_{int}(\xi)\Big(\hat{U} - R(\eta)\tilde{R}^{(0)} \cdot \ldots \cdot \tilde{R}^{(c)}\hat{W}\Big)\Big)\right\|_{L^q} \lesssim (1+t)^{-\frac{3}{m_s}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{m_s}} \|U_0\|_{L^p},$$

where $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$ *and the matrices* $R(\eta)$ *and* $\tilde{R}^{(i)}$ *are the inverse ones to* $L(\eta)$ *and* $\tilde{L}^{(i)}$.

The results of Proposition 5.2 further imply in the situation of Theorem 5.7 the estimates

$$\|\phi_{int}(D)U\|_{L^q} \lesssim (1+t)^{-\frac{3}{m_s}\left(\frac{1}{p}-\frac{1}{q}\right)}\|U_0\|_{L^p}$$

and

$$\|\phi_{int}(D)W\|_{L^q} \lesssim (1+t)^{-\frac{3}{m_s}\left(\frac{1}{p}-\frac{1}{q}\right)}\|U_0\|_{L^p}$$

for $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assuming that the above two estimates are optimal - and such discussions we postpone again to the applications - the difference, as stated in Theorem 5.7, is decaying faster and the asymptotic profiles (at least from the viewpoint of decay estimates) of solutions to (5.4) are thus given by solutions to (5.13).
In our considerations in 1D in Section 2.5.1 we included more precisely the case in which the decay of $\phi_{int}(D)U$ was partly determined by the imaginary parts of the eigenvalues. Suppose now that additionally to all the above assumptions there is at least one eigenvalue that does not only fit into the situation (5.9) but also (5.10). We denote for each such eigenvalue by $n_{s,j}$ the number l from (5.10). For all other eigenvalues we set $n_{s,j} = p_j$ and assume further that $n_s := \max_{j=1,\dots,d} n_{s,j} < m_s$.
With the same reference system and by applying the arguments used in the proof of Proposition 5.3 together with analogous arguments to the ones in the proof of Theorem 2.24 we obtain:

Corollary 5.8. *Assume* $U_0 \in \mathcal{S}$, *the existence of a number* n *so that* (A_n) *and* (B_n) *hold, all eigenvalues* $\mu_j = \mu_j(|\xi|)$ *to have a positive real part in* $Z_{int}(\sigma)\backslash\{0\}$ *and that at least one fits into the situations (5.9) and (5.10). Define the numbers* n_s *and* m_s *as above and assume* $n_s < m_s$.
Then we obtain for the solution U *to the Cauchy problem (5.4) the estimate*

$$\begin{aligned}\left\|\mathcal{F}^{-1}\left(\phi_{int}(\xi)\left(\hat{U} - R(\eta)\tilde{R}^{(0)}\cdot\ldots\cdot\tilde{R}^{(c)}\,\hat{W}\right)\right)\right\|_{L^q} \\ \leq\ C_\varepsilon(1+t)^{-\frac{3}{n_s}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{m_s}}\left(\|U_0\|_{L^p} + \|U_0\|_{L^{p_\varepsilon}}\right).\end{aligned}$$

Here ε *is an arbitrary but fixed positive number,* $1+\varepsilon \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, C_ε *is an* ε*-dependent constant and* $p_\varepsilon = p\left(1 - \varepsilon\frac{p}{1+(1+p)\varepsilon}\right)$.

Hence, solutions to (5.13) describe asymptotic profiles (from the viewpoint of decay estimates) of solutions U to (5.4) also in cases, where the decay of $\phi_{int}(D)U$ is partly determined by the imaginary parts of the eigenvalues.

5.1.5. Propagation of singularities

Before starting to state results on the propagation of singularities, let us remind the reader of the notion of (H^s-)wave front sets. Let $\Omega \subset \mathbb{R}^d$ be an open set, $T \in \mathcal{D}'(\Omega)$ a distribution and $(x_0, \xi_0) \in \Omega \times \mathbb{R}^d \backslash \{0\}$. With the help of cut-off functions $\chi \in C_0^\infty(\Omega)$, $\chi \equiv 1$ in a neighborhood of x_0, and $\psi \in C^\infty(\mathbb{R}^d)$, $\psi \equiv 1$ in a conical neighborhood of ξ_0 and $\psi \equiv 0$ outside a larger conical neighborhood of ξ_0, we define the symbol

$$p(x, \xi) = \chi(x)\psi(\xi) \in S^0_{1,0}.$$

If there are cut-off functions χ and ψ as above such that

$$p(x, D)T \in C^\infty(\Omega) \qquad (p(x, D)T \in H^s_{\text{loc}}(\Omega)),$$

then we call the distribution T in (x_0, ξ_0) microlocally regular (microlocally in H^s).

Definition 5.1. *Let $T \in \mathcal{D}'(\Omega)$, $\Omega \subset \mathbb{R}^d$ open. The set of all pairs $(x_0, \xi_0) \in \Omega \times \mathbb{R}^d \backslash \{0\}$, in which T is not microlocally regular (microlocally in H^s), is called (H^s-)wave front set of T and is denoted by $WF(T)$ ($WF_s(T)$).*

We proceed as in the 1D case. Suppose that the assumptions of Theorem 5.1 hold and thus that we have well-posedness for (5.4). We define with $\tilde{L} = \tilde{L}(\eta)$, given by

$$\tilde{L} := \hat{\tilde{L}}^{(\hat{c})} \cdot \ldots \cdot \hat{\tilde{L}}^{(1)} \hat{\tilde{L}}^{(0)} L(\eta),$$

where the above matrices are taken from the diagonalization procedure for large frequencies, and $\hat{c}$ is the minimal number for which $(\hat{\mathrm{B}}_\mathrm{n})$ holds, the vectors

$$\hat{W}_0 = (\hat{w}_{0,1}, \ldots, \hat{w}_{0,d})^T = \tilde{L}\,\mathcal{F}(U_0) \qquad \text{and} \quad \hat{W} = (\hat{w}_1, \ldots, \hat{w}_d)^T = \tilde{L}\,\mathcal{F}(U).$$

We further assume that the numbers $\hat{\lambda}_{k,j}$ from the asymptotic expansions of the eigenvalues

$$\mu_j(|\xi|) = \hat{\lambda}_{2,j}|\xi|^2 + i\hat{\lambda}_{1,j}|\xi| + \hat{\lambda}_{0,j} + \ldots + a_m \hat{\lambda}_{-m,j}|\xi|^{-m} + \mathcal{O}(|\xi|^{-m-1}) \tag{5.14}$$

in $Z_{ext}(N)$ are all real. That is for instance true when $(\hat{\mathrm{A}}_\mathrm{n}')$ is satisfied instead of $(\hat{\mathrm{A}}_\mathrm{n})$ and all matrices A_i from the associated system of type (5.2) are from $\mathbb{R}^{d\times d}$.

Theorem 5.9. *We consider the Cauchy problem (5.4) and assume the matrices A_i from the associated system of type (5.2) to be from $\mathbb{R}^{d\times d}$. Suppose that for some n the assumptions $(\hat{A}_n')$ and $(\hat{B}_n)$ hold together with (5.5) and that for a fixed $s \in \mathbb{R}$ we have for all $j = 1, \ldots, d$*

$$w_{0,j} \in \left[H^s(\mathbb{R}^3) \cap H^{s+1}(\mathbb{R}^3 \backslash \{x_0\})\right] \backslash H^{s+1}(\mathbb{R}^3).$$

Then we have for $j \in \{1, \ldots, d\}$ with (cf. (5.14)):

$\underline{\hat{\lambda}_{2,j} = 0}$: $w_j(t, \cdot) \in \left[H^s(\mathbb{R}^3) \cap H^{s+1}(\mathbb{R}^3 \backslash \{x : |x - x_0| = |\hat{\lambda}_{1,j}|t\})\right] \backslash H^{s+1}(\mathbb{R}^3)$,

$\underline{\hat{\lambda}_{2,j} > 0}$: $w_j(t, \cdot) \in H^{s+1}(\mathbb{R}^3) \cap H^{s+2}(\mathbb{R}^3 \backslash \{x : |x - x_0| = |\hat{\lambda}_{1,k}|t, k \in K\})$,

$K := \{k \in \{1, \ldots, d\} : \hat{\lambda}_{2,k} = 0\}$,

for all $t > 0$.

Proof. Let $l, s \in \mathbb{R}$ be two fixed numbers with $l > s$ and $T \in H^s(\mathbb{R}^3)$. By using the definition of H^l-wave front sets,

$$\mathcal{F}^{-1}_{\xi \to x}(e^{-i\nu|\xi|t}\mathcal{F}(f)) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{i\left(x - \nu \frac{\xi}{|\xi|} t\right)\cdot \xi} \mathcal{F}(f)(\xi) d\xi$$

(for appropriate f) and the fact that $e^{i|D|}$ and $e^{-i|D|}$ are inverse operators on H^s we conclude that for any $t > 0$ we have

$$WF_l\left(\mathcal{F}^{-1}_{\xi \to x}(e^{-i\nu|\xi|t}\mathcal{F}(T))\right) = \left\{ \left(x + \nu \frac{\xi}{|\xi|} t, \xi \right) : (x,\xi) \in WF_l(T) \right\}.$$

The above fact together with analogous considerations to the ones in the proof of Theorem 2.26 prove the statements in Theorem 5.9. □

Remark 5.2. Analogous remarks to the ones in Remark 2.21 hold true.

We can state a slightly more precise result:

Corollary 5.10. *We consider the Cauchy problem (5.4), assume the matrices A_i from the associated system of type (5.2) to be from $\mathbb{R}^{d\times d}$, that a number n exists for which the assumptions ($\hat{A}'_n$) and ($\hat{B}_n$) hold and (5.5). We further assume that for a fixed $s \in \mathbb{R}$ we have $U_0 \in H^s(\mathbb{R}^3)$. Then we have for a fixed $x_0 \in \mathbb{R}^3$ and $j \in \{1, \dots, d\}$ with:*

$\underline{\hat{\lambda}_{2,j} = 0}$: $w_j(t,\cdot) \in H^s(\mathbb{R}^3)$,

$$WF_{s+1}(w_j(t,\cdot)) = \{(x_0 + \hat{\lambda}_{1,j}\, \xi_0|\xi_0|^{-1}t, \xi_0) : (x_0,\xi_0) \in WF_{s+1}(w_{0,j})\},$$

$\underline{\hat{\lambda}_{2,j} > 0}$: $w_j(t,\cdot) \in H^{s+1}(\mathbb{R}^3)$,

$$WF_{s+2}(w_j(t,\cdot)) \subset \{(x_0 + \hat{\lambda}_{1,k}\, \xi_0|\xi_0|^{-1}t, \xi_0) : (x_0,\xi_0) \in WF_{s+1}(w_{0,k}), k \in K\},$$

$$K := \{k \in \{1,\dots,d\} : \hat{\lambda}_{2,k} = 0\},$$

for all $t > 0$, assuming that t_{jk} does never vanish for all $k \in K$, $k \neq j$ and $T_{-1} = (t_{lm})^d_{l,m=1}$ from the corresponding expansion to the one in (2.51).
If we further assume that $|\hat{\lambda}_{1,k}| \neq |\hat{\lambda}_{1,l}|$ for all $l,k \in K$, $k \neq l$, then we can replace '$\subset$' by '$=$' in the relation for the wave front sets for j with $\hat{\lambda}_{2,j} > 0$.

The considerations in the proof of Theorem 5.9 also hold true for C^∞-wave front sets instead of H^l-wave front sets. Hence, we obtain:

Corollary 5.11. *We consider the Cauchy problem (5.4), assume the matrices A_i from the associated system of type (5.2) to be from $\mathbb{R}^{d\times d}$, the existence of a number n for which the assumptions ($\hat{A}'_n$) and ($\hat{B}_n$) hold and (5.5). We further assume that for a fixed $s \in \mathbb{R}$*

$$U_0 \in H^s(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3 \backslash \{x_0\}).$$

Then we obtain

$$U \in C^\infty(((0,\infty) \times \mathbb{R}^3) \backslash I),$$

where

$$I = \{(t,x) \in (0,\infty) \times \mathbb{R}^3 : |x - x_0| = |\hat{\lambda}_{1,k}| t \textit{ for such } k \textit{ with } \hat{\lambda}_{2,k} = 0\}.$$

5.2. Applications

We apply the results of the foregoing section onto the Cauchy problems for the classical thermoelasticity models for isotropic media in 3D with and without dissipation or mass terms.
The results of the discussions in the next section, i.e., for classical thermoelasticity in 3D without any lower order terms, are partially known. Concerning decay estimates, they may be found in [JR00], and for results on the propagation of singularities we refer the reader to [RW99b] and [Wan03b].

5.2.1. Classical thermoelasticity without dissipation or mass

We study the Cauchy problem for (1.1b), i.e.,

$$\begin{cases} U_{tt} - \mu\Delta U - (\mu+\lambda)\nabla\nabla^T U + \gamma_1 \nabla\theta = 0, \\ \theta_t - \kappa\Delta\theta + \gamma_2 \nabla^T U_t = 0, \\ U(0,x) = U_0(x), U_t(0,x) = U_1(x), \theta(0,x) = \theta_0(x) \end{cases} \tag{5.15}$$

with the assumptions $\kappa, \mu, \lambda + 2\mu, \gamma_1\gamma_2 > 0$.
Let us start our calculations by using the Helmholtz decomposition (cf. [Lei86])

$$L^2 = \overline{\nabla H^1} \oplus \mathcal{D}_0$$

with the spaces

$$\begin{aligned} \nabla H^1 &= \{\nabla f | f \in H^1(\mathbb{R}^3)\}, \\ \mathcal{D}_0 &= \{U \in L^2(\mathbb{R}^3) | \forall \phi \in C_0^\infty(\mathbb{R}^3) : (\nabla\phi, U)_{L^2} = 0\} \end{aligned}$$

and thus by decomposing our solution U into a potential and a solenoidal part

$$U = U^{po} \oplus U^{so}.$$

The vector U^{po} is rotation-free and U^{so} divergence-free in a weak sense.
Using that $\nabla\nabla^T U = \nabla \times (\nabla \times U) + \Delta U$, we can decouple (5.15) into

$$\begin{cases} U_{tt}^{so} - \mu\Delta U^{so} = 0, \\ U^{so}(0,x) = U_0^{so}(x), U_t^{so}(0,x) = U_1^{so}(x) \end{cases}$$

and

$$\begin{cases} U^{po}_{tt} - (\lambda + 2\mu)\Delta U^{po} + \gamma_1 \nabla\theta = 0, \\ \theta_t - \kappa\Delta\theta + \gamma_2 \nabla^T U^{po}_t = 0, \\ U^{po}(0,x) = U^{po}_0(x), U^{po}_t(0,x) = U^{po}_1(x), \theta(0,x) = \theta_0(x). \end{cases} \tag{5.16}$$

Remark 5.3. Using the above orthogonal decomposition of L^2, one can show that an element $u \in H^s$ with $s < 0$ decomposes into two distributions

$$u = u_1 + u_2,$$

where $u_1 \in \overline{\nabla H^{s+1}}$ (with $\overline{\nabla H^{s+1}}$ being the closure of $\nabla H^{s+1} = \{\nabla f | f \in H^{s+1}\}$ in H^s) and $u_2 \in H^s$ with $\operatorname{div} u_2 = 0$ in the sense of H^s. Thus, we do not need to restrict our solution concept for decoupling (5.15).

The vector U^{so} satisfies a Cauchy problem for a free wave equation. Hence, we do not need to be concerned about well-posedness for this part of the solution. Further, we can not expect any diffusion phenomena, singularities propagate in the usual hyperbolic way, and we obtain the Strichartz decay estimate

$$\|(U^{so}_t, \nabla_x U^{so})\|_{L^q} \lesssim (1+t)^{-\left(\frac{1}{p}-\frac{1}{q}\right)} \|(\langle D \rangle\, U^{so}_0, U^{so}_1)\|_{L^{p,r_p}} \tag{5.17}$$

for dual indices q, $1 < p \le 2$ and $r_p = 3\left(\frac{1}{p} - \frac{1}{q}\right)$, assuming $U^{so}_0, U^{so}_1 \in \mathcal{S}$.
We will now have to study the Cauchy problem (5.16) in detail, and we will do this by transforming it after partial Fourier transformation equivalently into the initial value problem

$$\begin{cases} \tilde{V}_t + (i|\xi| B_1(\eta) + |\xi|^2 B_2)\tilde{V} = 0, \\ \tilde{V}(0,\xi) = \tilde{V}_0(\xi) \end{cases} \tag{5.18}$$

for the vector $\tilde{V} = \left((\hat{U}^{po}_+)^T, (\hat{U}^{po}_-)^T, \hat{\theta}\right)^T$, $\hat{U}^{po}_\pm = \hat{U}^{po}_t \pm i\sqrt{\alpha}\,|\xi|\hat{U}^{po}$, $\alpha := \lambda + 2\mu > 0$, and matrices

$$B_1(\eta) = \begin{pmatrix} -\sqrt{\alpha}\, I_3 & 0_{(3\times 3)} & \gamma_1\eta \\ 0_{(3\times 3)} & \sqrt{\alpha}\, I_3 & \gamma_1 \eta \\ \frac{\gamma_2}{2}\,\eta^T & \frac{\gamma_2}{2}\,\eta^T & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \operatorname{diag}(0,0,0,0,0,0,\kappa).$$

Here and hereafter we denote by $0_{(k\times l)}$ a zero matrix of dimension $k \times l$ and by I_n the identity matrix of dimension $n \times n$. The problem (5.18) is of type (5.3) with real-valued matrices B_k .

5.2.1.1. The diagonalization procedure

We introduce the matrices $L = L(\eta)$ and $R = R(\eta)$ of left and right eigenvectors of B_1 corresponding to the eigenvalues

$$\lambda_{1,1} = \lambda_{1,2} = -\sqrt{\alpha}, \lambda_{1,3} = \lambda_{1,4} = \sqrt{\alpha}; \lambda_{1,5} = -\sqrt{\alpha + \gamma_1\gamma_2}, \lambda_{1,6} = 0, \lambda_{1,7} = \sqrt{\alpha + \gamma_1\gamma_2}$$

with $LR = I$ by

$$L = \begin{pmatrix} r_1(\eta)^T & 0_{(1\times 3)} & 0 \\ r_2(\eta)^T & 0_{(1\times 3)} & 0 \\ 0_{(1\times 3)} & r_1(\eta)^T & 0 \\ 0_{(1\times 3)} & r_2(\eta)^T & 0 \\ -\frac{\gamma_2}{2}\,a_+\eta^T & -\frac{\gamma_2}{2}\,a_-\eta^T & \gamma_1\gamma_2 \\ \gamma_2\eta^T & -\gamma_2\eta^T & 2\sqrt{\alpha} \\ \frac{\gamma_2}{2}\,a_-\eta^T & \frac{\gamma_2}{2}\,a_+\eta^T & \gamma_1\gamma_2 \end{pmatrix}, R^T = \begin{pmatrix} r_1(\eta)^T & 0_{(1\times 3)} & 0 \\ r_2(\eta)^T & 0_{(1\times 3)} & 0 \\ 0_{(1\times 3)} & r_1(\eta)^T & 0 \\ 0_{(1\times 3)} & r_2(\eta)^T & 0 \\ -\frac{\gamma_1}{2a_-(\alpha+\gamma_1\gamma_2)}\,\eta^T & -\frac{\gamma_1}{2a_+(\alpha+\gamma_1\gamma_2)}\,\eta^T & \frac{1}{2(\alpha+\gamma_1\gamma_2)} \\ \frac{\gamma_1}{2(\alpha+\gamma_1\gamma_2)}\,\eta^T & -\frac{\gamma_1}{2(\alpha+\gamma_1\gamma_2)}\,\eta^T & \frac{\sqrt{\alpha}}{2(\alpha+\gamma_1\gamma_2)} \\ \frac{\gamma_1}{2a_+(\alpha+\gamma_1\gamma_2)}\,\eta^T & \frac{\gamma_1}{2a_-(\alpha+\gamma_1\gamma_2)}\,\eta^T & \frac{1}{2(\alpha+\gamma_1\gamma_2)} \end{pmatrix},$$

where $a_\pm = \sqrt{\alpha+\gamma_1\gamma_2} \pm \sqrt{\alpha}$ and $r_1(\eta)$ and $r_2(\eta)$ satisfy

$$\operatorname{span}\{r_1(\eta), r_2(\eta)\} = \eta^\perp.$$

The vectors $r_1(\eta)$ and $r_2(\eta)$ cannot be parameterized globally on the sphere $\mathbb{S}^2$ in a continuous way (cf. the famous hairy ball theorem from topology). For the following we proceed locally in two open subsets $O_1, O_2 \subset \mathbb{S}^2$ with $\mathbb{S}^2 = O_1 \cup O_2$ such that in these O_j the vectors $r_1(\eta)$ and $r_2(\eta)$ can be chosen as continuous functions of η that are orthogonal and normed. Using a partition of unity, we later on glue the results together. A direct computation yields that the components V_1 and V_2 of the vector $V = (V_1^T, V_2^T)^T := L\tilde{V}$ satisfy

$$\begin{cases} \partial_t V_1 + i|\xi| D\, V_1 = 0, \\ V_1(0,\xi) = V_{0,1}(\xi) \end{cases}$$

with $D = \sqrt{\alpha}\,\mathrm{diag}\,(-1,-1,1,1)$, $V_0 = (V_{0,1}^T, V_{0,2}^T)^T = L\tilde{V}_0$ and

$$\begin{cases} \partial_t V_2 + (i|\xi|A_1 + |\xi|^2 A_2)V_2 = 0, \\ V_2(0,\xi) = V_{0,2}(\xi) \end{cases} \tag{5.19}$$

with $A_1 = \sqrt{\alpha+\gamma_1\gamma_2}\,\mathrm{diag}\,(-1,0,1)$ and

$$A_2 = \frac{\kappa}{2(\alpha+\gamma_1\gamma_2)} \begin{pmatrix} \gamma_1\gamma_2 & \gamma_1\gamma_2\sqrt{\alpha} & \gamma_1\gamma_2 \\ 2\sqrt{\alpha} & 2\alpha & 2\sqrt{\alpha} \\ \gamma_1\gamma_2 & \gamma_1\gamma_2\sqrt{\alpha} & \gamma_1\gamma_2 \end{pmatrix}. \tag{5.20}$$

However, noting that for a rotation-free vector W we have $\eta \times \hat{W} = 0$ for $\eta \in \mathbb{S}^2$, we observe that $V_{0,1}$ vanishes completely and so does V_1. Hence, the solution to (5.18) is given by

$$\tilde{V}(t,\xi) = R(\eta)V(t,\xi) = R(\eta)\begin{pmatrix} 0_{(4\times 1)} \\ V_2(t,\xi) \end{pmatrix}, \tag{5.21}$$

and to find solution representations for V_2, we can apply our diagonalization procedure to (5.19). For the latter we will use for convenience the same notations that would be used when applying the procedure to the whole system for V.

Diagonalization for small frequencies

We study (5.19) in $Z_{int}(\sigma) = \{|\xi| \leq \sigma \ll 1\}$ and note that we can formally rewrite the system by setting $\tilde{V}^{(1)} = V_2$, $\Lambda_1 = A_1$, $\tilde{A}_2^{(1)} = A_2$ and by replacing $|\xi|$ with ξ as the one from (3.5) from the diagonalization procedure for small frequencies in Section 3.1.1.1 for classical thermoelasticity in 1D.
Hence, we obtain the results from Proposition 3.1 when replacing ξ by $|\xi|$ there. In particular, we obtain in $Z_{int}(\sigma)$ the solution representation

$$V_2(t,\xi) = T_{int}^{-1}(|\xi|)\,\mathrm{diag}\left(e^{-\mu_1(|\xi|)t}, e^{-\mu_2(|\xi|)t}, e^{-\mu_3(|\xi|)t}\right) T_{int}(|\xi|)V_{0,2}(\xi)$$

with $T_{int}(|\xi|) = M(|\xi|)(I + i|\xi|K_{(2)})$, $K_{(2)}$ from (3.6) and $M(|\xi|) = I + \mathcal{O}(|\xi|^2)$ for $|\xi| \to 0$.

Diagonalization for large frequencies

We analyze (5.19) in $Z_{ext}(N) = \{|\xi| \geq N \gg 1\}$ and start with:

Step 0: Diagonalization modulo $\mathcal{O}(|\xi|)$-terms

The eigenvalues of the symmetrizable matrix A_2 are given by

$$\hat{\lambda}_{2,1} = \hat{\lambda}_{2,2} = 0 \;<\; \hat{\lambda}_{2,3} = \kappa.$$

A possible choice for the matrices $\hat{\tilde{L}}^{(0)}$ and $\hat{\tilde{R}}^{(0)}$ of corresponding left and right eigenvectors with $\hat{\tilde{L}}^{(0)}\hat{\tilde{R}}^{(0)} = I$ is

$$\hat{\tilde{L}}^{(0)} = \begin{pmatrix} -2\sqrt{\alpha} & \gamma_1\gamma_2 & 0 \\ -1 & 0 & 1 \\ 1 & \sqrt{\alpha} & 1 \end{pmatrix} \quad \text{and} \quad \hat{\tilde{R}}^{(0)} = \frac{1}{2(\alpha+\gamma_1\gamma_2)} \begin{pmatrix} -\sqrt{\alpha} & -\gamma_1\gamma_2 & \gamma_1\gamma_2 \\ 2 & -2\sqrt{\alpha} & 2\sqrt{\alpha} \\ -\sqrt{\alpha} & 2\alpha+\gamma_1\gamma_2 & \gamma_1\gamma_2 \end{pmatrix}.$$

The vector $\tilde{V}^{(0)} = \hat{\tilde{L}}^{(0)}V_2$ satisfies

$$\tilde{V}_t^{(0)} + \left(|\xi|^2\hat{\Lambda}_2 + i|\xi|\hat{\tilde{A}}_1^{(0)}\right)\tilde{V}^{(0)} = 0$$

with

$$\hat{\Lambda}_2 = \mathrm{diag}(0,0,\kappa) \quad \text{and} \quad \hat{\tilde{A}}_1^{(0)} = \frac{1}{\sqrt{\alpha+\gamma_1\gamma_2}} \begin{pmatrix} -\alpha & -\gamma_1\gamma_2\sqrt{\alpha} & \gamma_1\gamma_2\sqrt{\alpha} \\ -\sqrt{\alpha} & \alpha & \gamma_1\gamma_2 \\ 0 & \alpha+\gamma_1\gamma_2 & 0 \end{pmatrix}.$$

Step 1: Diagonalization modulo $\mathcal{O}(1)$-terms

Substep 1: The vector $V^{(1)} = (I + i|\xi|^{-1}\hat{K}_{(1)})\tilde{V}^{(0)}$ with

$$\hat{K}_{(1)} = \frac{1}{\kappa\sqrt{\alpha+\gamma_1\gamma_2}} \begin{pmatrix} 0 & 0 & -\gamma_1\gamma_2\sqrt{\alpha} \\ 0 & 0 & -\gamma_1\gamma_2 \\ 0 & \alpha+\gamma_1\gamma_2 & 0 \end{pmatrix}$$

satisfies

$$V_t^{(1)} + \left(|\xi|^2\hat{\Lambda}_2 + i|\xi|\hat{A}_1^{(1)} + \hat{A}_0^{(1)} + \hat{A}_{-1}^{(1)}\right) V^{(1)} = 0$$

with

$$\hat{A}_1^{(1)} = \frac{1}{\sqrt{\alpha+\gamma_1\gamma_2}} \begin{pmatrix} -\alpha & -\gamma_1\gamma_2\sqrt{\alpha} & 0 \\ -\sqrt{\alpha} & \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \hat{A}_0^{(1)} = \frac{1}{\kappa} \begin{pmatrix} 0 & \gamma_1\gamma_2\sqrt{\alpha} & \gamma_1\gamma_2\sqrt{\alpha} \\ 0 & \gamma_1\gamma_2 & 0 \\ \sqrt{\alpha} & -\alpha & -\gamma_1\gamma_2 \end{pmatrix}$$

and $\hat{A}_{-1}^{(1)} = \mathcal{O}(|\xi|^{-1})$ for $|\xi| \to \infty$.

Substep 2: The block-diagonal matrix $\hat{A}_1^{(1)}$ is symmetrizable with the eigenvalues

$$\hat{\lambda}_{1,1} = -\sqrt{\alpha},\ \hat{\lambda}_{1,2} = \sqrt{\alpha},\ \hat{\lambda}_{1,3} = 0$$

and matrices

$$\hat{\tilde{L}}^{(1)} = \begin{pmatrix} 1 & a_- & 0 \\ -1 & a_+ & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \hat{\tilde{R}}^{(1)} = \frac{1}{2\sqrt{\alpha+\gamma_1\gamma_2}} \begin{pmatrix} \frac{\gamma_1\gamma_2}{a_-} & -\frac{\gamma_1\gamma_2}{a_+} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2\sqrt{\alpha+\gamma_1\gamma_2} \end{pmatrix}$$

of corresponding left and right eigenvectors with $\hat{\tilde{L}}^{(1)}\hat{\tilde{R}}^{(1)} = I$.
The vector $\tilde{V}^{(1)} = \hat{\tilde{L}}^{(1)}V^{(1)}$ satisfies

$$\tilde{V}_t^{(1)} + \left(|\xi|^2\hat{\Lambda}_2 + i|\xi|\hat{\Lambda}_1 + \hat{\tilde{A}}_0^{(1)} + \hat{\tilde{A}}_{-1}^{(1)}\right) \tilde{V}^{(1)} = 0$$

with $\hat{\Lambda}_1 = \operatorname{diag}(-\sqrt{\alpha}, \sqrt{\alpha}, 0)$,

$$\hat{\tilde{A}}_0^{(1)} = \frac{1}{2\kappa} \begin{pmatrix} \gamma_1\gamma_2 & \gamma_1\gamma_2 & 2\gamma_1\gamma_2\sqrt{\alpha} \\ \gamma_1\gamma_2 & \gamma_1\gamma_2 & -2\gamma_1\gamma_2\sqrt{\alpha} \\ \sqrt{\alpha} & -\sqrt{\alpha} & -2\gamma_1\gamma_2 \end{pmatrix} \quad \text{and} \quad \hat{\tilde{A}}_{-1}^{(1)} = \mathcal{O}(|\xi|^{-1}).$$

The conditions $(\hat{\mathrm{A}}_1')$ and $(\hat{\mathrm{B}}_1)$ are satisfied, i.e., we have full diagonalizability of (5.19) for large frequencies.

Step 2: Diagonalization modulo $\mathcal{O}(|\xi|^{-1})$-terms

Taking into account that $(\hat{\mathrm{A}}_1')$ and $(\hat{\mathrm{B}}_1)$ hold and considering the diagonal entries of $\hat{\tilde{A}}_0^{(1)}$, we conclude that there exist matrices $\hat{K}_{(1\frac{1}{2})}$ and $\hat{K}_{(2)}$ such that the vector $\tilde{V}^{(2)} = (I + i|\xi|^{-1}\hat{K}_{(2)})(I + |\xi|^{-2}\hat{K}_{(1\frac{1}{2})})\tilde{V}^{(1)}$ satisfies

$$\tilde{V}_t^{(2)} + \left(|\xi|^2\hat{\Lambda}_2 + i|\xi|\hat{\Lambda}_1 + \hat{\Lambda}_0 + \hat{\tilde{A}}_{-1}^{(2)}\right) \tilde{V}^{(2)} = 0$$

with $\hat{\Lambda}_0 = \operatorname{diag}\left(\frac{\gamma_1\gamma_2}{2\kappa}, \frac{\gamma_1\gamma_2}{2\kappa}, -\frac{\gamma_1\gamma_2}{\kappa}\right)$ and $\hat{\tilde{A}}_{-1}^{(2)} = \mathcal{O}(|\xi|^{-1})$ for $|\xi| \to \infty$.

Proposition 5.12. *(i) The characteristic roots $\mu_j = \mu_j(|\xi|)$ of the coefficient matrix $A(|\xi|) = i|\xi|A_1 + |\xi|^2A_2$ from (5.19) behave for $|\xi| \geq N \gg 1$ as*

$$\begin{aligned} \mu_{1,2}(|\xi|) &= \mp i\sqrt{\alpha}\,|\xi| + \frac{\gamma_1\gamma_2}{2\kappa} + \mathcal{O}(|\xi|^{-1}), \\ \mu_3(|\xi|) &= \kappa|\xi|^2 - \frac{\gamma_1\gamma_2}{\kappa} + \mathcal{O}(|\xi|^{-1}). \end{aligned}$$

(ii) The solution to the Cauchy problem of (5.19) has in $Z_{ext}(N) = \{|\xi| \geq N \gg 1\}$ *the representation*

$$V_2(t,\xi) = T_{ext}^{-1}(|\xi|)\,\mathrm{diag}(e^{-\mu_1(|\xi|)t}, e^{-\mu_2(|\xi|)t}, e^{-\mu_3(|\xi|)t})T_{ext}(|\xi|)V_{0,2}(\xi),$$

where $T_{ext}(|\xi|) = \hat{M}(|\xi|)(I+i|\xi|^{-1}\hat{K}_{(2)})(I+|\xi|^{-2}\hat{K}_{(1\frac{1}{2})})\hat{\tilde{L}}^{(1)}(I+i|\xi|^{-1}\hat{K}_{(1)})\hat{\tilde{L}}^{(0)}$ *with a matrix* $\hat{M}(|\xi|) = I + \mathcal{O}(|\xi|^{-2})$ *for* $|\xi| \to \infty$.

Diagonalization for bounded frequencies away from zero

We are analyzing (5.19) in $Z_{mid}(\sigma, N) = \{\sigma \leq |\xi| \leq \mathrm{N}\}$ and see with an analogous proof that a result as in Lemma 3.3 holds true. Hence, we have:

Proposition 5.13. *The solution* V_2 *to the Cauchy problem (5.19) satisfies in* $Z_{mid}(\sigma, N)$

$$|V_2(t,\xi)| \lesssim e^{-ct}|V_{0,2}(\xi)|,$$

where c is a positive constant.

5.2.1.2. Results

We are now interested in stating results for the Cauchy problem (5.16) and start off with:

Theorem 5.14. *(Well-posedness result)*
We consider the Cauchy problem (5.16) in $\mathbb{R}_{\geq 0} \times \mathbb{R}^3$ *with* $\lambda+2\mu, \kappa, \gamma_1\gamma_2 > 0$ *and assume* $U_0^{po} \in H^{s+1}$, $U_1^{po}, \theta_0 \in H^s$ *for a fixed* $s \in \mathbb{R}$. *Then there exists a unique solution satisfying*

$$\begin{aligned} U^{po} &\in C([0,\infty), H^{s+1}) \cap C^1([0,\infty), H^s), \\ \theta &\in C([0,\infty), H^s). \end{aligned}$$

Theorem 5.15. *(L^p-L^q decay estimate)*
We assume $U_0^{po}, U_1^{po}, \theta_0 \in \mathcal{S}$. *Then the following* L^p-L^q *decay estimate holds for solutions to the Cauchy problem (5.16):*

$$\|(U_t^{po}, \nabla_x U^{po}, \theta)\|_{L^q} \lesssim (1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|(\langle D\rangle\, U_0^{po}, U_1^{po}, \theta_0)\|_{L^{p,r_p}}.$$

Here $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $r_p = 3\left(\frac{1}{p} - \frac{1}{q}\right)$.

Proof. From (5.21) we know

$$\tilde{V}(t,\xi) = \begin{pmatrix} R_1(\eta) & R_2(\eta) \end{pmatrix} \begin{pmatrix} 0_{(4\times 1)} \\ V_2(t,\xi) \end{pmatrix} = R_2(\eta)V_2(t,\xi). \tag{5.22}$$

Hence, we have to derive estimates for $\mathcal{F}^{-1}\left(R_2(\eta)V_2(t,\xi)\right)$ measured in L^q only. The results of the diagonalization procedure in the foregoing section together with the ones from the Propositions 5.2, 5.4 and the fact that the result from Proposition 5.13 is sufficient for obtaining one as in Proposition 5.5 yield the statement of Theorem 5.15. □

Remark 5.4. Combining the result from (5.17) with the one from Theorem 5.15, we observe that while we obtain a parabolic decay rate when considering the Cauchy problem for classical thermoelasticity without any lower order terms in 1D (cf. Theorem 3.6), we have only hyperbolic decay for the corresponding 3D problem. However, a parabolic decay rate is obtained if the initial data of the displacement, that is, U_0 and U_1 from (5.15), are rotation-free.

For a result on a diffusion phenomenon let us define the reference system (cf. (3.8)):

$$\begin{cases} \partial_t W_2 + (iM_1|D| - M_2\Delta)W_2 = 0, \\ W_2(0,x) = W_{0,2}(x) \end{cases} \tag{5.23}$$

with

$$\begin{aligned} M_1 &= \operatorname{diag}\left(-\sqrt{\alpha+\gamma_1\gamma_2}, 0, \sqrt{\alpha+\gamma_1\gamma_2}\right), \\ M_2 &= \operatorname{diag}\left(\frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}, \frac{\alpha\kappa}{\alpha+\gamma_1\gamma_2}, \frac{\kappa\gamma_1\gamma_2}{2(\alpha+\gamma_1\gamma_2)}\right), \\ W_{0,2}(x) &= \begin{pmatrix} -\gamma_2\alpha & -\gamma_2\sqrt{\alpha+\gamma_1\gamma_2} & \gamma_1\gamma_2 \\ 2\gamma_2\sqrt{\alpha} & 0 & 2\sqrt{\alpha} \\ -\gamma_2\alpha & \gamma_2\sqrt{\alpha+\gamma_1\gamma_2} & \gamma_1\gamma_2 \end{pmatrix} \begin{pmatrix} \nabla^T U_0^{po} \\ \mathcal{F}^{-1}(\eta^T \hat{U}_1^{po}) \\ \theta_0 \end{pmatrix}. \end{aligned}$$

We introduce $W(t,x) = \begin{pmatrix} 0_{(4\times 1)} \\ W_2(t,x) \end{pmatrix}$ and being aware of (5.22) conclude with the considerations in Section 5.1.4 (and some as in the proof of Theorem 5.15 for frequencies with a positive distance to zero):

Theorem 5.16. *Assume $U_0^{po}, U_1^{po}, \theta_0 \in \mathcal{S}$. Then we obtain for the solution $\tilde{V}$ to the Cauchy problem (5.18) the estimate*

$$\left\| \mathcal{F}^{-1}\left(\tilde{V} - R(\eta)\hat{W}\right)\right\|_{L^q} \lesssim (1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|(\langle D\rangle\, U_0^{po}, U_1^{po}, \theta_0)\|_{L^{p,r_p}}$$

for dual values $q, 1 < p \le 2$ and $r_p = 3\left(\frac{1}{p} - \frac{1}{q}\right)$.

Similar to the 1D case (cf. with Theorem 3.7) we thus conclude that the asymptotic profiles of solutions to the Cauchy problem (5.16) (from the viewpoint of decay estimates) are parabolic.
Concerning the propagation of singularities, we can state (cf. Theorem 3.8):

Theorem 5.17. *We consider the Cauchy problem (5.16) and assume for a fixed $s \in \mathbb{R}$:*

(i) $U_1^{po} \pm i\sqrt{\lambda+2\mu}\,|D|U_0^{po}, \theta_0 \in [H^s(\mathbb{R}^3) \cap H^{s+1}(\mathbb{R}^3\backslash\{x_0\})]\backslash H^{s+1}(\mathbb{R}^3)$.
Then we obtain for any $t > 0$

$$\begin{aligned} &U_t^{po}(t,\cdot), |D|U^{po}(t,\cdot) \in H^s(\mathbb{R}^3) \cap H^{s+1}(\mathbb{R}^3\backslash\{x : |x-x_0| = \sqrt{\lambda+2\mu}\,t\}), \\ &U_t^{po}(t,\cdot), |D|U^{po}(t,\cdot) \notin H^{s+1}_{loc}(\{x : |x-x_0| = \sqrt{\lambda+2\mu}\,t\}), \\ &\theta(t,\cdot) \in H^{s+1}(\mathbb{R}^3) \cap H^{s+2}(\mathbb{R}^3\backslash\{x : |x-x_0| = \sqrt{\lambda+2\mu}\,t\}), \\ &\theta(t,\cdot) \notin H^{s+2}_{loc}(\{x : |x-x_0| = \sqrt{\lambda+2\mu}\,t\}). \end{aligned}$$

(ii) $U_1^{po}, \theta_0 \in H^s(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3 \backslash \{x_0\})$, $U_0^{po} \in H^{s+1}(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3 \backslash \{x_0\})$. *Then we obtain*

$$(U^{po}, \theta) \in C^\infty(((0,\infty) \times \mathbb{R}^3) \backslash I),$$

where I denotes the forward light cone

$$I = \{(t,x) \in (0,\infty) \times \mathbb{R}^3 : |x - x_0| = \sqrt{\lambda + 2\mu}\, t\}.$$

Singularities in the initial data to the Cauchy problem (5.16) for the potential part U^{po} of the displacement and θ are thus propagated similar to the ones for the Cauchy problem of classical thermoelasticity in 1D.

5.2.2. Classical thermoelasticity with dissipation

We will now devote our attention to the Cauchy problem

$$\begin{cases} U_{tt} - \mu\Delta U - (\mu + \lambda)\nabla\nabla^T U + \gamma_1 \nabla\theta + mU_t = 0, \\ \theta_t - \kappa\Delta\theta + \gamma_2 \nabla^T U_t = 0, \\ U(0,x) = U_0(x), U_t(0,x) = U_1(x), \theta(0,x) = \theta_0(x) \end{cases}$$

for classical thermoelasticity with an additional dissipation term.
Applying the Helmholtz decomposition, the above initial value problem decouples into Cauchy problems for

$$U_{tt}^{so} - \mu\Delta U^{so} + mU_t^{so} = 0$$

and

$$\begin{cases} U_{tt}^{po} - (\lambda + 2\mu)\Delta U^{po} + \gamma_1 \nabla\theta + mU_t^{po} = 0, \\ \theta_t - \kappa\Delta\theta + \gamma_2 \nabla^T U_t^{po} = 0 \end{cases} \tag{5.24}$$

with corresponding initial data.
The vector U^{so} now satisfies a Cauchy problem for a damped wave equation. Hence, we obtain in particular the decay estimate (cf. (1.13))

$$\|(U_t^{so}, \nabla_x U^{so})\|_{L^q} \lesssim (1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|(\langle D \rangle\, U_0^{so}, U_1^{so})\|_{L^{p,r_p}} \tag{5.25}$$

for dual indices q, $1 < p \leq 2$, $r_p = 3\left(\frac{1}{p} - \frac{1}{q}\right)$, assuming $U_0^{so}, U_1^{so} \in \mathcal{S}$, and a diffusion phenomenon as in (1.15).
The Cauchy problem for (5.24) can equivalently be transformed into the initial value problem given by (5.18) with an additional summand $B_0\tilde{V}$,

$$B_0 = \begin{pmatrix} \frac{m}{2}\, I_3 & \frac{m}{2}\, I_3 & 0_{(3\times1)} \\ \frac{m}{2}\, I_3 & \frac{m}{2}\, I_3 & 0_{(3\times1)} \\ 0_{(1\times3)} & 0_{(1\times3)} & 0 \end{pmatrix}.$$

The component V_1 of the vector $V = (V_1^T, V_2^T)^T := L\tilde{V}$, where L denotes the matrix from Section 5.2.1.1, vanishes with the same arguments as before, and V_2 satisfies

$$\begin{cases} \partial_t V_2 + (A_0 + i|\xi|A_1 + |\xi|^2 A_2)V_2 = 0, \\ V_2(0,\xi) = V_{0,2}(\xi), \end{cases} \tag{5.26}$$

where $V_{0,2}$ denotes the vector given by the last three components of $L\tilde{V}_0$,

$$A_0 = \frac{m}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

and A_1 and A_2 denote the same matrices as in (5.19). Hence, we have again a representation for $\tilde{V}$ as in (5.21). For finding solution representations for V_2, we apply our diagonalization procedure and obtain in particular the following behavior of the eigenvalues to the coefficient matrix of (5.26):

Lemma 5.18. *(i) The characteristic roots $\mu_j = \mu_j(|\xi|)$ behave for $|\xi| \le \sigma \ll 1$ as*

$$\begin{aligned} \mu_{1,2}(|\xi|) &= \lambda_{\mp}\,|\xi|^2 + \mathcal{O}(|\xi|^3), \\ \mu_3(|\xi|) &= m - \frac{\alpha + \gamma_1\gamma_2}{m}\,|\xi|^2 + \mathcal{O}(|\xi|^3), \end{aligned}$$

where the positive numbers $\lambda_{\mp}$ are taken from (3.24).

(ii) The characteristic roots behave for $|\xi| \ge N \gg 1$ as

$$\begin{aligned} \mu_{1,2}(|\xi|) &= \mp i\sqrt{\alpha}\,|\xi| + \frac{\gamma_1\gamma_2 + m\kappa}{2\kappa} + \mathcal{O}(|\xi|^{-1}), \\ \mu_3(|\xi|) &= \kappa\,|\xi|^2 - \frac{\gamma_1\gamma_2}{\kappa} + \mathcal{O}(|\xi|^{-1}). \end{aligned}$$

(iii) The assumption (C) is satisfied.

With the above information and an analogous proof we obtain the same results as in Theorem 5.15 concerning L^p-L^q decay estimates. Combining that with the estimate (5.25) for the solenoidal part U^{so} of the displacement, we observe that the dissipation term is strong enough to improve the hyperbolic decay rate that is obtained when considering the Cauchy problem for classical thermoelasticity without any lower order terms to a parabolic one.
For a result on a diffusion phenomenon we define the reference system (cf. (3.23)):

$$\begin{cases} \partial_t W_2 + (M_0 - M_2\Delta)W = 0, \\ W_2(0,x) = \tilde{L}^{(2)}\tilde{L}^{(0)}\mathcal{F}^{-1}\left(V_{0,2}(\xi)\right) \end{cases} \tag{5.27}$$

with $M_0 = \operatorname{diag}(0,0,m)$, $M_2 = \operatorname{diag}(\lambda_-, \lambda_+, 0)$ and $\tilde{L}^{(i)}$ denoting constant 3×3 matrices coming from the diagonalization procedure for (5.26).
With $W(t,x) = \begin{pmatrix} 0_{(4\times 1)} \\ W_2(t,x) \end{pmatrix}$ we obtain:

Theorem 5.19. *Assume $U_0^{po}, U_1^{po}, \theta_0 \in \mathcal{S}$. Then the estimate*

$$\left\| \mathcal{F}^{-1}\left(\tilde{V} - R(\eta)\tilde{R}\hat{W}\right)\right\|_{L^q} \lesssim (1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \left\|(\langle D\rangle U_0^{po}, U_1^{po}, \theta_0)\right\|_{L^{p,r_p}}$$

holds for dual values q, $1 < p \leq 2$ and $r_p = 3\left(\frac{1}{p} - \frac{1}{q}\right)$. The matrix $\tilde{R}$ is given by $\tilde{R} = \mathrm{diag}(I_4, \tilde{R}^{(0)}) \cdot \mathrm{diag}(I_4, \tilde{R}^{(2)})$, where the $\tilde{R}^{(i)}$ are the inverse matrices to $\tilde{L}^{(i)}$.

Hence, the asymptotic profiles (from the viewpoint of decay estimates) of solutions to the Cauchy problem for (5.24) are given by solutions to (5.27).

5.2.3. Classical thermoelasticity with mass

At last we will consider the Cauchy problem for classical thermoelasticity with an additional mass term, that is,

$$\begin{cases} U_{tt} - \mu\Delta U - (\mu+\lambda)\nabla\nabla^T U + \gamma_1\nabla\theta + m^2 U = 0, \\ \theta_t - \kappa\Delta\theta + \gamma_2\nabla^T U_t = 0, \\ U(0,x) = U_0(x), U_t(0,x) = U_1(x), \theta(0,x) = \theta_0(x). \end{cases}$$

We decouple the above problem via the Helmholtz decomposition into initial value problems for

$$U_{tt}^{so} - \mu\Delta U^{so} + m^2 U^{so} = 0$$

and

$$\begin{cases} U_{tt}^{po} - (\lambda+2\mu)\Delta U^{po} + \gamma_1\nabla\theta + m^2 U^{po} = 0, \\ \theta_t - \kappa\Delta\theta + \gamma_2\nabla^T U_t^{po} = 0, \end{cases} \tag{5.28}$$

and note that the solenoidal part U^{so} thus satisfies a Cauchy problem for a Klein-Gordon equation. Hence, we obtain for U^{so} a decay estimate as in (1.11) and no diffusion phenomena, since the characteristic roots are purely imaginary. Now we devote our attention solely to the Cauchy problem for (5.28).
This can equivalently be transformed into

$$\begin{cases} \tilde{V}_t + (i|\xi|B_1^{(1)} + i\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}} B_1^{(2)} + |\xi|^2 B_2)\tilde{V} = 0, \\ \tilde{V}(0,\xi) = \tilde{V}_0(\xi) \end{cases} \tag{5.29}$$

for $\tilde{V} = \left((\hat{U}_+^{po})^T, (\hat{U}_-^{po})^T, \hat{\theta}\right)^T$, $\hat{U}_\pm^{po} = \hat{U}_t^{po} \pm i\sqrt{\alpha}\,\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}} \hat{U}^{po}$, $\alpha := \lambda + 2\mu > 0$, and matrices $B_1^{(2)} = \sqrt{\alpha}\,\mathrm{diag}(-I_3, I_3, 0)$,

$$B_1^{(1)} = \begin{pmatrix} \multicolumn{2}{c}{0_{(6\times 6)}} & \begin{matrix}\gamma_1\eta \\ \gamma_1\eta\end{matrix} \\ \frac{\gamma_2}{2}\eta^T & \frac{\gamma_2}{2}\eta^T & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \mathrm{diag}(0,0,0,0,0,0,\kappa).$$

After applying the transformation matrix L from Section 5.2.1.1, we observe that the component V_1 of $V = (V_1^T, V_2^T)^T := L\tilde{V}$ vanishes, and V_2 satisfies

$$\begin{cases} \partial_t V_2 + (i|\xi|\tilde{B}_1^{(1)} + i\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}} \tilde{B}_1^{(2)} + |\xi|^2\tilde{B}_2)V_2 = 0, \\ V_2(0,\xi) = V_{0,2}(\xi) \end{cases} \tag{5.30}$$

with $V_{0,2}$ denoting the vector given by the last three components of $L\tilde{V}_0$,

$$\begin{aligned}
\tilde{B}_1^{(1)} &= \frac{1}{2\sqrt{\alpha+\gamma_1\gamma_2}} \begin{pmatrix} -2\gamma_1\gamma_2 & -\gamma_1\gamma_2\sqrt{\alpha} & 0 \\ -2\sqrt{\alpha} & 0 & 2\sqrt{\alpha} \\ 0 & \gamma_1\gamma_2\sqrt{\alpha} & 2\gamma_1\gamma_2 \end{pmatrix}, \\
\tilde{B}_1^{(2)} &= \frac{1}{2\sqrt{\alpha+\gamma_1\gamma_2}} \begin{pmatrix} -2\alpha & \gamma_1\gamma_2\sqrt{\alpha} & 0 \\ 2\sqrt{\alpha} & 0 & -2\sqrt{\alpha} \\ 0 & -\gamma_1\gamma_2\sqrt{\alpha} & 2\alpha \end{pmatrix}
\end{aligned}$$

and $\tilde{B}_2$ being the matrix from (5.20).
We thus have

$$\tilde{V}(t,\xi) = R(\eta) \begin{pmatrix} 0_{(4\times 1)} \\ V_2(t,\xi) \end{pmatrix}$$

and for finding solution representations for V_2, we use asymptotic expansions of $\langle\xi\rangle_{\frac{m}{\sqrt{\alpha}}}$ for small and large frequencies and apply our diagonalization procedure. In particular, we obtain for the behavior of the characteristic roots to the coefficient matrix from (5.30) the same results as in Lemma 3.19, when replacing ξ by $|\xi|$.
With an analogous proof as for Theorem 5.15 and in particular making use of the estimate derived in Proposition 5.3, we thus obtain:

Theorem 5.20. *We assume $U_0^{po}, U_1^{po}, \theta_0 \in \mathcal{S}$. Then the following decay estimate holds for solutions to the Cauchy problem of (5.28):*

$$\|(U^{po}, U_t^{po}, \nabla_x U^{po}, \theta)\|_{L^q} \le C_\varepsilon (1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|(\langle D\rangle\, U_0^{po}, U_1^{po}, \theta_0)\|_{L^{p,r_p}\cap L^{p_\varepsilon}},$$

where ε is an arbitrary but fixed positive number, $1+\varepsilon \le p \le 2$, $\frac{1}{p}+\frac{1}{q}=1$, C_ε is an ε-dependent constant, $r_p = 3\left(\frac{1}{p}-\frac{1}{q}\right)$ and $p_\varepsilon = p\left(1-\varepsilon\frac{p}{1+(1+p)\varepsilon}\right)$.

Remark 5.5. We obtain the same decay rate for the solenoidal part U^{so} and thus see that the mass term has, just like the additional dissipation term did, a helping influence with respect to decay, when comparing with the results for classical 3D-thermoelasticity without any lower order terms.

With the help of the reference system (cf. (3.28))

$$\begin{cases} \partial_t W_2 + (M_0 - M_2\Delta + M_4\Delta^2)W = 0, \\ W_2(0,x) = \tilde{L}^{(0)}\mathcal{F}^{-1}\left(V_{0,2}(\xi)\right) \end{cases} \tag{5.31}$$

with

$$\begin{aligned}
M_0 &= i\,\mathrm{diag}\left(-m, m, 0\right), \\
M_2 &= \mathrm{diag}\left(-i\frac{\alpha+\gamma_1\gamma_2}{2m}, i\frac{\alpha+\gamma_1\gamma_2}{2m}, \kappa\right), \\
M_4 &= \mathrm{diag}\left(\frac{\kappa\gamma_1\gamma_2}{2m^2} + i\frac{(\alpha+\gamma_1\gamma_2)^2}{8m^3}, \frac{\kappa\gamma_1\gamma_2}{2m^2} - i\frac{(\alpha+\gamma_1\gamma_2)^2}{8m^3}, 0\right),
\end{aligned}$$

$\tilde{L}^{(0)}$ denoting a constant 3×3 matrix appearing in the diagonalization procedure for V_2 and $W(t,x) = \begin{pmatrix} 0_{(4\times 1)} \\ W_2(t,x) \end{pmatrix}$ we obtain:

Theorem 5.21. *Assume $U_0^{po}, U_1^{po}, \theta_0 \in \mathcal{S}$. Then we have the estimate*

$$\left\| \mathcal{F}^{-1}\left(\tilde{V} - R(\eta)\tilde{R}\hat{W}\right)\right\|_{L^q} \leq C_\varepsilon (1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{4}} \left\| \left(\langle D\rangle U_0^{po}, U_1^{po}, \theta_0\right)\right\|_{L^{p,r_p}\cap L^{p_\varepsilon}},$$

where ε, p, q, p_ε and r_p are as in Theorem 5.20, and the matrix $\tilde{R}$ is given by $\tilde{R} = \operatorname{diag}(I_4, \tilde{R}^{(0)})$, where $\tilde{R}^{(0)}$ is the inverse to $\tilde{L}^{(0)}$.

Hence, the asymptotic profiles (from the viewpoint of decay estimates) of solutions to the Cauchy problem for (5.28) are given by solutions to (5.31).

6. Summary and concluding remarks

6.1. Summary

In the following we will collect results for the Cauchy problems of all in Section 1.1 discussed systems of thermoelasticity.
These are on the one hand L^p-L^q decay estimates of the form

$$\|W\|_{L^q} \le C(1+t)^{-K\left(\frac{1}{p}-\frac{1}{q}\right)}\|W_0\| \tag{6.1}$$

for dual indices q, $1 < p \le 2$, a constant C possibly depending on p and some number K (specified in the tables below), $\|W\|_{L^q}$ denotes an appropriate energy measured in L^q and $\|W_0\|$ corresponding (smooth) initial data measured in an appropriate norm (most of the time in the L^{p,r_p}-norm with a sufficiently large number r_p). On the other hand we have derived results on diffusion phenomena for appropriate reference systems and results on the propagation of singularities.
Without referring to the precise results or stating them (if they can not be found here or elsewhere), we will mention in the following tables whether they can be derived with the methods provided in this thesis. For L^p-L^q estimates we will specify the decay-related number K from (6.1) and will set results into (simple) quotation marks if they have not been stated in the thesis.
We divide into collections of the results in 1D and 3D.

Summary in 1D

	lower order terms		
	none	dissipation	mass
classical thermoelasticity	$K = 1/2$, diffusion phen., prop. of sing.		
thermoelasticity with second sound	$K = 1/2$, diffusion phen., 'prop. of sing.'		
thermoelasticity of type 3	$K = 1/2$, diffusion phen., prop. of sing.		
thermoelasticity of type 2	$K = 0$	$K = 1/2$	$K = 1/3$
	–	diff. phen.	–
	'prop. of sing.'		

Summary in 3D

	lower order terms		
	none	dissipation	mass
classical thermoelasticity	$K = 1$	$K = 3/2$	
	diff. phen. for U^{po}, θ	diff. phen.	diff. phen. for U^{po}, θ
	prop. of sing.		
thermoelasticity with second sound	'$K = 1$'	'$K = 3/2$'	
	'diff. phen. for U^{po}, θ, q'	'diff. phen.'	'diff. phen. for U^{po}, θ, q'
	'prop. of sing.'		
thermoelasticity of type 3	'$K = 1$'	'$K = 3/2$'	
	'diff. phen. for U^{po}, θ'	'diff. phen.'	'diff. phen. for U^{po}, θ'
	'prop. of sing.'		
thermoelasticity of type 2	'$K = 1$'	'$K = 3/2$'	'$K = 1$' **?**
	–	'diff. phen.'	–
	'prop. of sing.'		

We should point out that the assertion concerning the decay estimate for the thermoelasticity model of type 2 in 3D with an additional mass term is a hypothesis (and therefore tagged with a question mark). It does not entirely follow from the methods and results provided in this thesis, but involves some deeper thoughts using the stationary phase method. For more detailed discussions we would like the reader to refer to the next section.

Reconsidering the above two tables, the methods provided in the thesis have proved to be strong enough to generate quite a number of results for solutions to Cauchy problems of well-recognized linear thermoelasticity models with and without additional dissipation or mass terms from a unified approach. Moreover, applications to other linearized systems are certainly possible, such as, to name only two examples, to thermoviscoelastic systems (cf. e.g. [DH82]) or to thermoelastic systems with microtemperatures (cf. [IQ00]).

Both, the additional dissipation and the additional mass term have, from the viewpoint of decay estimates, a positive influence on the discussed systems of thermoelasticity, with the dissipation term having a 'stronger' one than the mass term.

6.2. Open problems and concluding remarks

In this final section we want to remark on three problems arising in connection with the contents of this thesis. This is certainly not a complete list, but only a (subjective) choice of possible generalizations, applications and noteworthy parallel developments.

L^p-L^q decay estimate for the thermoelasticity model of type 2 in 3D with mass. When considering the Cauchy problem for (1.6b) with an additional mass term (as discussed in Section 1.1), then we should (after applying the Helmholtz decomposition and noting that the solenoidal part of the displacement satisfies a Cauchy problem for a Klein-Gordon equation and thus in particular a decay estimate as in (1.11)) study the initial value problem

$$\begin{cases} U^{po}_{tt} - \alpha\Delta U^{po} + \gamma_1 \nabla\theta + m^2 U^{po} = 0, \\ \theta_{tt} - \kappa\Delta\theta + \gamma_2 \nabla^T U^{po}_{tt} = 0, \\ U^{po}(0,x) = U^{po}_0(x), U^{po}_t(0,x) = U^{po}_1(x), \theta(0,x) = \theta_0(x), \theta_t(0,x) = \theta_1(x), \end{cases} \tag{6.2}$$

$\alpha = \lambda + 2\mu$, in detail.

It turns out that the solutions to (6.2) behave quite similar to the ones of the Cauchy problem (4.1) for the thermoelasticity model of type 2 with an additional mass term in 1D from Chapter 4. In particular, the components of U^{po} and θ satisfy a partial differential equation analogous to the one in (4.2), whose always purely imaginary characteristic roots are given by (4.3), when replacing ξ by $|\xi|$ there.

For the regions of small and large frequencies we can apply our diagonalization procedure and determine the asymptotic expansions of the characteristic roots, which are in fact given by Lemma 3.23, *(i)*, when setting $\delta = 0$ and replacing ξ by $|\xi|$, and Lemma 4.1. Again, since the characteristic roots are given by (4.3) with ξ being replaced by $|\xi|$, we obtain moreover a lot of information from Lemma 4.2. In particular, we should postulate a condition (D) as in Chapter 4. Then, as in Remark 4.1, we come to the conclusion that two of the roots behave like roots to the classical Klein-Gordon equation in the whole phase space, while the other two roots are in their behavior in some sense in between that of the roots to the classical Klein-Gordon and the classical wave equation with exceptional behaviors in $\xi = 0$ and for $|\xi| = r_0$.

Assuming that the small frequencies, as in the 1D-considerations, indicate the correct decay rate and that the same terms (as in 1D) in the asymptotic expansions of the roots are the decay-determining ones, then, with the help of Lemma B.5, the guess from the foregoing table, and thus that we obtain a hyperbolic decay only, should be the right one.

However, the proof of this assertion, in particular the considerations in the region of bounded frequencies with a positive distance to zero, involve, due to the fact that we do not have such powerful tools at hand as in 1D (i.e., the Lemma B.3 of van der Corput), some deeper thoughts using the stationary phase method.

Global, small, smooth solutions to Cauchy problems for nonlinear thermoelasticity models. L^p-L^q decay estimates are known to be important tools for continuing local solutions of initial value problems for nonlinear evolution equations to global ones. For classical thermoelasticity such global existence results for small

data solutions may be found in [Rac92] (and references therein). There the Cauchy problem for the nonlinear system arising for a homogeneous, initially isotropic medium is discussed, and there occurs a significant difference between the one-dimensional and the three-dimensional case. For the one-dimensional problem global solutions always exist if the data are sufficiently small without any restrictions on the nonlinearity (other than the ones coming from the derivation), while for the model in 3D purely quadratic nonlinearities in the displacement have to be excluded. This difference corresponds with the observed one in the decay rates, cf. Remark 5.4.
The application of other decay estimates derived in this thesis to answer questions on the existence of global solutions to corresponding nonlinear problems maybe even via a unified approach is desirable.

Anisotropic media. Recently, the authors Reissig and Wirth introduced in [RW08] a unified approach for deriving L^p-L^q decay estimates for solutions to the Cauchy problems of classical thermoelasticity in 2D for homogeneous but anisotropic media. The approach is based on the partial Fourier transformation and a refined diagonalization procedure (similar to the one discussed here) applied to a corresponding first order system in phase space that allows to locate *hyperbolic* and *parabolic* microlocal directions and yields sufficient information for the treatment of those, i.e., in particular of the hyperbolic ones.
Applications to this unified approach may be found in [Wir08], and the analytical tools generalize to higher dimensions and allow to treat models in 3D (outside degenerate directions). The transference of these considerations not only to 3D but moreover to alternative models of thermoelasticity, such as the ones discussed here, should be of interest.

Appendices

A. Notations - Guide to the reader

A.1. Preliminaries

We use C and c to denote arbitrary constants throughout the thesis. They may differ at each occurance, unless explicitly stated otherwise. The symbol D is used to denote $D = -i\nabla$, $\nabla^T = (\partial_{x_1}, \ldots, \partial_{x_n})$, $i = \sqrt{-1}$, and we do not consider zero to be a natural number, i.e., $\mathbb{N} = \{1, 2, 3, \ldots\}$. We use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a complex number $z \in \mathbb{C}$ we denote by $\operatorname{Re} z$ its real and by $\operatorname{Im} z$ its imaginary part.
Bracket symbols with a special meaning are:

$\lvert\cdot\rvert$	denotes the absolute value of a scalar expression, for a vector the Euclidean and for a matrix the Frobenius norm,
$\langle\cdot\rangle_m$	stands for $\langle x\rangle_m = \sqrt{\lvert x\rvert^2 + m^2}$, $m \neq 0$, $\langle x\rangle := \langle x\rangle_1$,
$\lfloor\cdot\rfloor$	denotes the largest integer smaller/ equal to a given number, $\lfloor x\rfloor = \max\{m \in \mathbb{Z} : x \geq m\}$,
$\lceil\cdot\rceil$	stands for $\lceil x\rceil = \min\{m \in \mathbb{Z} : x \leq m\}$.

We frequently use the asymptotic relations

$f \lesssim g$	if there exists a constant $C > 0$ such that for all arguments we have $f \leq Cg$ and
$f \gtrsim g$	if $g \lesssim f$

for nonnegative functions f and g.
The Landau symbol $\mathcal{O}$ will as usual be used for describing an asymptotic upper bound of a function in terms of another, simpler function, that is, we say $f(x) = \mathcal{O}(g(x))$ as $x \to x_0$ for functions $f, g : X \subset \mathbb{R}^n \to \mathbb{C}$, $x_0 \in X'$ being from the derived set of X, if there exists a $\delta > 0$ such that

$$|f(x)| \lesssim |g(x)| \qquad \text{for } x \in X \text{ with } |x - x_0| < \delta.$$

Analogous definitions are employed for $|x| \to \infty$ and $x \to \pm\infty$ if $n = 1$. Moreover, we will use the $\mathcal{O}$ notation for matrix-valued functions f as well and by this mean that each component is of the order of g.
For $f \in L^1(\mathbb{R}^n)$ the Fourier transform $\mathcal{F}(f) = \hat{f} = \mathcal{F}^-(f)$ and the inverse Fourier transform $\mathcal{F}^{-1}(f) = \mathcal{F}^+(f)$ shall be defined by

$$\mathcal{F}^{\pm}(f)(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\pm ix\cdot\xi} f(x)dx.$$

For more general f, such as $f \in L^2(\mathbb{R}^n)$ or even $f \in \mathcal{D}'(\mathbb{R}^n)$ the corresponding natural definitions are employed. The symbols $\mathcal{F}$ and $\mathcal{F}^{-1}$ are further often used to denote the partial Fourier transforms $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ with respect to the space variable x.
For matrices we use the notations

I_n or I	for the identity matrix of dimension $n \times n$, I is used if the dimension is clear,
$0_{(k\times l)}$	denotes a zero matrix of dimension $k \times l$,
$\operatorname{diag}(A_1, \ldots, A_n)$	denotes a diagonal/ block-diagonal matrix with the scalars/ blocks $A_1, \ldots, A_n$ on the diagonal.

In the diagonalization procedure described in Section 2.2.2.1 we make use of some special notations. We denote by

$A_k^{(l)}$	a matrix valued in $\mathbb{C}^{d\times d}$ that appears in step l of the procedure with a corresponding summand of the order $\mathcal{O}(\xi^k)$,
Λ_k	a constant diagonal matrix from $\mathbb{C}^{d\times d}$, whose corresponding summand is of the order $\mathcal{O}(\xi^k)$,
$K_{(l)}$	a constant matrix from $\mathbb{C}^{d\times d}$ appearing in step l,
$\tilde{L}^{(l)}, \tilde{R}^{(l)}$	constant matrices from $\mathbb{C}^{d\times d}$ of left/ right eigenvectors appearing in step l,
$V^{(l)}$	the vector of unknowns in step l.

To the above terms (except for Λ_k) a '$\tilde{}$' is attached if the corresponding step involved eigenvalue theory. The same notations with an additional '$\hat{}$' are used in the diagonalization procedure for large frequencies.

A.2. Function spaces

In the following we have collected function spaces frequently used in the thesis together with a short definition:

$L^p(\mathbb{R}^n)$	Lebesgue spaces, $1 \le p \le \infty$,
$L^{p,r}(\mathbb{R}^n)$	Bessel potential spaces (cf. Section B.2), $L^{p,r}(\mathbb{R}^n) = \langle D\rangle^{-r} L^p(\mathbb{R}^n)$, $1 \le p < \infty$, $r \in \mathbb{R}$,
$H^s(\mathbb{R}^n)$	Sobolev space based on $L^2(\mathbb{R}^n)$, $H^s(\mathbb{R}^n) = L^{2,s}(\mathbb{R}^n)$,
$H^s_{\text{loc}}(\{x_0\})$	$f \in H^s_{\text{loc}}(\{x_0\})$ means $f \in H^s_{\text{loc}}(U_\varepsilon(x_0))$ for an $\varepsilon > 0$, $U_\varepsilon(x_0) = \{x \in \mathbb{R}^n : \lvert x - x_0\rvert < \varepsilon\}$, hence, $\langle\xi\rangle^s \lvert\mathcal{F}(\psi f)(\xi)\rvert \in L^2(\mathbb{R}^n)$ for all $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subset U_\varepsilon(x_0)$; $H^s_{\text{loc}}(\Omega)$, $\Omega \subset \mathbb{R}^n$, are defined analogously,
$C^k(\mathbb{R}^n)$	space of k-times continuously differentiable functions,
$C^\infty(\mathbb{R}^n)$	space of infinitely often differentiable functions,
$C_0^\infty(\mathbb{R}^n)$	space of $C^\infty(\mathbb{R}^n)$-functions with compact support,

$\mathcal{S}(\mathbb{R}^n)$	Schwartz space of rapidly decreasing functions, $\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : x^\alpha \partial_x^\beta f(x) \in L^\infty(\mathbb{R}^n), \forall \alpha, \beta \in \mathbb{N}_0^n\}$,
$\mathcal{D}'(\mathbb{R}^n)$	space of distributions, continuous dual space to $\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$,
$\mathcal{S}'(\mathbb{R}^n)$	space of tempered distributions, dual space to $\mathcal{S}(\mathbb{R}^n)$,
$B^s_{pq}(\mathbb{R}^n)$	Besov space $B^s_{pq}(\mathbb{R}^n) = \{f \in \mathcal{S}' : \|f\|_{B^s_{pq}} < \infty\}$, $s \in \mathbb{R}$, $1 \le p, q \le \infty$, where $\|f\|_{B^s_{pq}} = \|\varphi_0(D)f\|_{L^p} + \left(\sum_{m=1}^\infty (2^{sm}\|\varphi_m(D)f\|_{L^p})^q\right)^{1/q}$ and $\{\varphi_m(\xi)\}_{m\in\mathbb{N}_0}$ is a dyadic partition of unity, [BL76],
$C([0,T], H^s(\mathbb{R}^n))$	mixed space of all $u = u(t,x)$ with $u(t,\cdot) \in H^s(\mathbb{R}^n)$ for all $t \in [0,T]$ and $\lim_{t_1\to t_2} \|u(t_1,\cdot) - u(t_2,\cdot)\|_{H^s} = 0$ (continuity in t), normed by $\|u\|_{C([0,T],H^s)} = \max_{t\in[0,T]} \|u(t,\cdot)\|_{H^s}$,
$C^k([0,T], H^s(\mathbb{R}^n))$	space of all $u = u(t,x)$ with $\partial_t^j u \in C([0,T], H^s(\mathbb{R}^n))$ for $0 \le j \le k$.

We often neglect to explicitly state that a function space is considered on $\mathbb{R}^n$, that is, we write for example L^p instead of $L^p(\mathbb{R}^n)$. We further note that Besov spaces B^s_{pq} are independent of the chosen dyadic decomposition together with an equivalence of resulting norms and that other mixed spaces are defined in an equivalent way.

For completeness we mention the symbol classes of Hörmander. A function $P = P(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ belongs to $S^m_{1,0}(\mathbb{R}^n)$, $m \in \mathbb{R}$, if it satisfies

$$|\partial_x^\beta \partial_\xi^\alpha P(x,\xi)| \le C_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|}$$

for all $x, \xi \in \mathbb{R}^n$ and all multiindices $\alpha, \beta \in \mathbb{N}_0^n$. The corresponding operator $P(x,D)$ is called pseudodifferential operator of order m and belongs to the class $\Psi^m_{1,0}(\mathbb{R}^n)$.

B. Basic tools

B.1. The class of diagonalizable matrices

The following facts may be found in many books about linear algebra or matrix computations (e.g. [ZF84]).

Definition B.1. *A matrix $A \in \mathbb{C}^{d\times d}$ is called diagonalizable if it is similar to a diagonal matrix, i.e., if there exists a regular matrix T so that $T^{-1}AT$ is diagonal.*

Diagonalizable matrices could also be characterized as the ones, where the algebraic and geometric multiplicities of all eigenvalues are equal.

Lemma B.1. *Right and left eigenvectors r_k and l_j (corresponding to the eigenvalues λ_k and λ_j) of a diagonalizable matrix $A \in \mathbb{C}^{d\times d}$ can always be chosen such that*

$$l_j r_k = \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k \end{cases}$$

for all $k, j = 1, \ldots, d$ or $LR = I_d$ with the notations $R = (r_1, \ldots, r_d)$ and $L^T = (l_1^T, \ldots, l_d^T)$, i.e., the vectors r_k and l_j^T form a biorthonormal system.

With the above notations we conclude

$$LAR = \operatorname{diag}(\lambda_1, \ldots, \lambda_d).$$

The (for us) most important subset of the (in $\mathbb{C}^{d\times d}$ dense) set of diagonalizable matrices is the class of symmetrizable matrices. Symmetrizable matrices are similar to hermitian and thus to real diagonal matrices. Hence, they have real eigenvalues.

B.2. Bessel potential spaces

The Bessel potential spaces (or generalized Sobolev spaces) are defined by

$$L^{p,r}(\mathbb{R}^n) = \langle D\rangle^{-r} L^p(\mathbb{R}^n) = \{f = \mathcal{F}^{-1}\left(\langle\xi\rangle^{-r}\mathcal{F}(g)\right) : g \in L^p(\mathbb{R}^n)\}$$

(cf. [AH96, BL76]). They may be written in the form

$$L^{p,r}(\mathbb{R}^n) = \{f = G_r * g : g \in L^p(\mathbb{R}^n)\},$$

where $G_r = (2\pi)^{-\frac{n}{2}}\mathcal{F}^{-1}(\langle\xi\rangle^{-r})$ denotes the Bessel kernel. Thus, the Bessel potential spaces are certainly well-defined for $1 \le p < \infty$ and $r \in \mathbb{R}$. They are normed by

$$\|f\|_{L^{p,r}(\mathbb{R}^n)} = \|G_{-r} * f\|_{L^p(\mathbb{R}^n)},$$

complete, and we have the theorem of Calderón (cf. [AH96]):

Theorem B.2. *(A. P. Calderón)*
For $r \in \mathbb{N}$, $1 < p < \infty$ and $W_p^r(\mathbb{R}^n)$ denoting the usual Sobolev space based on $L^p(\mathbb{R}^n)$ we have $L^{p,r}(\mathbb{R}^n) = W_p^r(\mathbb{R}^n)$ with the equivalence of the involved norms, i.e., there is a constant A such that for all f

$$A^{-1}\|f\|_{L^{p,r}(\mathbb{R}^n)} \le \|f\|_{W_p^r(\mathbb{R}^n)} \le A\|f\|_{L^{p,r}(\mathbb{R}^n)}.$$

B.3. Oscillatory integrals

As major tools for deriving decay estimates in 1D theory we need the following lemma and its corollary (cf. [Ste93]):

Lemma B.3. *(van der Corput)*
Suppose ϕ is real-valued and smooth in (a,b) and that $|\phi^{(k)}(x)| \ge \delta > 0$ for all $x \in (a,b)$. Then

$$\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \le c_k\lambda^{-1/k}$$

holds when:

(i) $k \ge 2$ or

(ii) $k = 1$ and $\phi'(x)$ is monotonic.

The bound c_k is independent of ϕ and λ.

Corollary B.4. *Suppose that the assumptions on ϕ from Lemma B.3 hold and that ψ is complex-valued and smooth. Then*

$$\left|\int_a^b e^{i\lambda\phi(x)}\psi(x)dx\right| \le c_k\lambda^{-1/k}\left[|\psi(b)| + \int_a^b |\psi'(x)|dx\right],$$

where the bound c_k is independent of ϕ, ψ and λ.

In 3D theory we will make use of a result from Pecher (cf. [Pec76]):

Lemma B.5. *Let $\delta \ge 0, m \in \mathbb{N}$ and*

$$\rho := \begin{cases} n-1 & \text{if } m = 1, \\ n & \text{if } m \ge 2. \end{cases}$$

Then we have for all $v \in \mathcal{S}$ and all $t > 0$ the estimate

$$\left\| \mathcal{F}^{-1} \left(\frac{\exp(i\, t\, |\xi|^m)}{|\xi|^{2m\delta}}\, \hat{v}(\xi) \right) \right\|_{L^q(\mathbb{R}^n)} \lesssim t^{-\frac{n}{m}\left(\frac{1}{p}-\frac{1}{q}\right)+2\delta} \|v\|_{L^p(\mathbb{R}^n)}, \tag{B.1}$$

provided the following conditions are satisfied:

$$1 < p \le 2, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \frac{1}{p} - \frac{1}{q} \ge \frac{2m\delta}{n} \quad \text{and} \quad \left(\frac{1}{p} - \frac{1}{2} \right)(2n - m\rho) \le 2m\delta.$$

We need a slight generalization of the above lemma for t-dependent data v.

Corollary B.6. *Assume $v = v(t,x)$ to be an element of $B([0,\infty), L^p(\mathbb{R}^n))$, that is, we assume $v \in B([0,T], L^p(\mathbb{R}^n))$ for every $T < \infty$. Then we have the same statement as in Lemma B.5, when replacing (B.1) by*

$$\left\| \mathcal{F}^{-1} \left(\frac{\exp(i\, t\, |\xi|^m)}{|\xi|^{2m\delta}} \mathcal{F}_{x\to\xi}(v(t,\cdot))(\xi) \right) \right\|_{L^q(\mathbb{R}^n)} \le C t^{-\frac{n}{m}\left(\frac{1}{p}-\frac{1}{q}\right)+2\delta} \|v(t,\cdot)\|_{L^p(\mathbb{R}^n)},$$

where the constant C depends (in general) on p and q.

Proof. We sketch the proof to convince the reader of the fact that the above constant $C = C(p,q)$ is independent of t.
Pecher obtained in a first step of the proof to Lemma B.5 in [Pec76] the boundedness of the mapping

$$L^p \ni w \mapsto \mathcal{F}^{-1} \left(\frac{\exp(i|\eta|^m)}{|\eta|^{2m\delta}}\, \hat{w}(\eta) \right) \in L^q$$

for $1 < p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p} - \frac{1}{q} \ge \frac{2m\delta}{n}$ and $\left(\frac{1}{p} - \frac{1}{2}\right)(2n - m\rho) \le 2m\delta$. Hence, we have with a constant $C = C(p,q)$ being independent of t

$$\left\| \mathcal{F}^{-1} \left(\frac{\exp(i|\eta|^m)}{|\eta|^{2m\delta}} \mathcal{F}_{x\to\eta}(v(t,\cdot))(\eta) \right) \right\|_{L^q} \le C \|v(t,\cdot)\|_{L^p}$$

for any fixed $t > 0$.
With the help of the substitutions $\eta = t^{\frac{1}{m}}\xi$ and $x = t^{\frac{1}{m}} z$ we rewrite

$$\begin{aligned} &\mathcal{F}^{-1} \left(\frac{\exp(it|\xi|^m)}{|\xi|^{2m\delta}}\, \mathcal{F}_{x\to\xi}(v(t,\cdot))(\xi) \right)(y) \\ &\quad = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{iyt^{-\frac{1}{m}}\cdot\eta} e^{-iz\cdot\eta} \frac{\exp(i|\eta|^m)}{|\eta|^{2m\delta}}\, t^{2\delta} v(t, t^{\frac{1}{m}} z)\, dz\, d\eta \\ &\quad = t^{2\delta} g(t^{-\frac{1}{m}} y) \end{aligned}$$

with

$$g(x) = \mathcal{F}^{-1} \left(\frac{\exp(i|\eta|^m)}{|\eta|^{2m\delta}} \mathcal{F}_{z\to\eta}(v(t, t^{\frac{1}{m}} z))(\eta) \right)(x)$$

and estimate

$$\begin{aligned}
&\left\| \mathcal{F}^{-1}\left(\frac{\exp(i\,t\,|\xi|^m)}{|\xi|^{2m\delta}} \mathcal{F}_{x\to\xi}(v(t,\cdot))(\xi) \right) \right\|_{L^q} \\
&\qquad = \; t^{2\delta} \left(\int_{\mathbb{R}^n} \left| g(t^{-\frac{1}{m}} y) \right|^q dy \right)^{\frac{1}{q}} = t^{2\delta + \frac{n}{mq}} \|g\|_{L^q} \\
&\qquad \leq \; C t^{2\delta + \frac{n}{mq}} \left(\int_{\mathbb{R}^n} \left| v(t, t^{\frac{1}{m}} z) \right|^p dz \right)^{\frac{1}{p}} = C t^{2\delta + \frac{n}{mq} - \frac{n}{mp}} \|v(t,\cdot)\|_{L^p}.
\end{aligned}$$

□

At last we want to provide the Mikhlin-Hörmander multiplier theorem (cf. [Ste70], Chapter IV, §3) and one important corollary of Lemma B.5.

Lemma B.7. *(Mikhlin-Hörmander multiplier theorem)*
Assume that $m \in C^k(\mathbb{R}^n \backslash \{0\})$ for $k = \lceil \frac{n+1}{2} \rceil$ satisfies for any $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ the estimate

$$\left| \partial_\xi^\alpha m(\xi) \right| \leq M |\xi|^{-|\alpha|} \qquad \forall \xi \neq 0.$$

Then the operator

$$L^p \ni f \mapsto \mathcal{F}^{-1}\left(m(\xi) \mathcal{F}(f) \right) \in L^p$$

is bounded for all $1 < p < \infty$, that is, m is a multiplier in L^p.

An immediate consequence of Lemma B.5 is:

Corollary B.8. *For dual values q, $1 < p \leq 2$, $r_p = n\left(\frac{1}{p} - \frac{1}{q}\right)$, $n \geq 2$ and data $v \in \mathcal{S}$ we have the estimate*

$$\left\| \mathcal{F}^{-1}\left(e^{i|\xi|t} \hat{v}(\xi) \right) \right\|_{L^q(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|v\|_{L^{p,r_p}(\mathbb{R}^n)}.$$

Proof. Note with the help of Lemma B.7 that $(|\xi| / \langle\xi\rangle)^{2\delta}$ is a multiplier in L^q for $1 < q < \infty$ and $\delta \geq 0$ and hence

$$\left\| \mathcal{F}^{-1}\left(e^{i|\xi|t} \hat{v}(\xi) \right) \right\|_{L^q} \lesssim \left\| \mathcal{F}^{-1}\left(\frac{e^{i|\xi|t}}{|\xi|^{2\delta}} \langle\xi\rangle^{2\delta} \hat{v}(\xi) \right) \right\|_{L^q} \lesssim t^{-n\left(\frac{1}{p}-\frac{1}{q}\right)+2\delta} \|v\|_{L^{p,2\delta}}$$

for $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{n+1}{2}\left(\frac{1}{p} - \frac{1}{q}\right) \leq 2\delta \leq n\left(\frac{1}{p} - \frac{1}{q}\right)$.
We choose for large times $t \geq 1$ the smallest possible δ, i.e., $2\delta = \frac{n+1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$, and for small times $0 < t \leq 1$ the largest possible one, i.e., $2\delta = n\left(\frac{1}{p} - \frac{1}{q}\right)$.

□

Bibliography

[AH96] ADAMS, D. R. & HEDBERG, L. I.: *Function Spaces and Potential Theory.* Grundlehren der mathematischen Wissenschaften, Vol. 314, Springer, Berlin, 1996.

[Bea89] BEALS, M.: *Propagation and Interaction of Singularities in Nonlinear Hyperbolic Problems.* Birkhäuser, Boston, 1989.

[BL76] BERGH, J. & LÖFSTRÖM, J.: *Interpolation Spaces. An Introduction.* Springer, Berlin, 1976.

[Bre75] BRENNER, P.: On $L_p - L_{p'}$ Estimates for the Wave Equation, *Math. Z.*, 145(1975), 251-254.

[Bro79] BRONSHTEIN, M. D.: Smoothness of roots of polynomials depending on parameters, *Sibirsk. Math. Zh.*, 20(1979), 493-501.

[Cat48] CATTANEO, C.: Sulla conduzione de calore, *Atti del Semin. Mat. e Fis. Univ. Modena*, 3(1948), 83-101.

[CP82] CHAZARAIN, J. & PIRIOU, A.: *Introduction to the Theory of Linear Partial Differential Equations.* North-Holland, Amsterdam, 1982.

[DH82] DAFERMOS, C. M. & HSIAO, L.: Global smooth solutions to the initial-boundary value problem for the equations of 1-d nonlinear thermoviscoelasticity, *SIAM J. Math. Anal.*, 13(1982), 397-408.

[Duh38] DUHAMEL, J. M. C.: Mémoire sur le calcul des actions moléculaires développées par les changements de température dans les corps solides, *Mémoirs par Divers Savans (Acad. Sci. Paris)*, 5(1838), 440-498.

[Gar64] GARABEDIAN, P. R.: *Partial Differential Equations.* John Wiley & Sons, New York, 1964.

[GN91] GREEN, A. E. & NAGHDI, P. M.: A re-examination of the basic postulates of thermomechanics, *Proc. R. Soc. Lond. A*, 432(1991), 171-194.

[GN92] GREEN, A. E. & NAGHDI, P. M.: On undamped heat waves in an elastic solid, *J. Thermal Stresses*, 15(1992), 253-264.

[GN93] GREEN, A. E. & NAGHDI, P. M.: Thermoelasticity without energy dissipation, *J. Elasticity*, 31(1993), 189-208.

[Hör97] HÖRMANDER, L.: *Lectures on Nonlinear Hyperbolic Differential Equations.* Mathématiques & Applications, Vol. 26, Springer, Berlin, 1997.

[HL92] HSIAO, L. & LIU, T. P.: Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, *Commun. Math. Phys.*, 143(1992), 599-605.

[IQ00] IESAN, D. & QUINTANILLA, R.: On a theory of thermoelasticity with microtemperatures, *J. Thermal Stresses*, 23(2000), 199-215.

[JR00] JIANG, S. & RACKE, R.: *Evolution Equations in Thermoelasticity.* Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, Vol. 112, Chapman & Hall/CRC, Boca Raton, 2000.

[JP89] JOSEPH, D. D. & PREZIOSI, L.: Heat waves, *Rev. mod. Phys.*, 61(1989), 41-73.

[Kat80] KATO, T.: *Perturbation Theory for Linear Operators.* Springer, New York, 1980.

[Kno96] KNOPP, K.: *Theory of Functions, Parts I and II, Two Volumes Bound as One.* Dover Publ., New York, 1996.

[Kov70] KOVALENKO, A. D.: The current theory of thermoelasticity, *Int. Appl. Mech.*, 6, No. 4(1970), 355-360.

[KL89] KREISS, H. O. & LORENZ, J.: *Initial-Boundary Value Problems and the Navier-Stokes Equations.* Academic Press, New York, 1989.

[Lan41] LANDAU, L.: The theory of superfluidity of helium II, *J. Phys.*, 5(1941), 71-90.

[LL53] LANDAU, L. D. & LIFSHITZ, E. M.: *Mechanics of Continuous Media* (2nd Russian edition), Gostekhisdat, Moscow, 1953.

[Lei86] LEIS, R.: *Initial Boundary Value Problems in Mathematical Physics.* Teubner, Stuttgart, 1986.

[Mat76] MATSUMURA, A.: On the asymptotic behavior of solutions of semi-linear wave equations, *Publ. Res. Inst. Math. Sci.*, 12(1976), 169-189.

[Neu41] NEUMANN, K. E.: Die Gesetze der Doppelbrechung des Lichts in comprimirten oder ungleichförmig erwärmten unkrystallinischen Körpern, *Pogg. Ann. Phys. Chem.*, 54(1841), 449-476.

[Pec76] PECHER, H.: L^p-Abschätzungen und klassische Lösungen für nichtlineare Wellengleichungen. I, *Math. Z.*, 150(1976), 159-183.

[Rac92] RACKE, R.: *Lectures on Nonlinear Evolution Equations: Initial Value Problems.* Aspects of Mathematics, Vol. E19, Vieweg, Braunschweig, 1992.

[RW98] RACKE, R. & WANG, Y. G.: Propagation of singularities in one-dimensional thermoelasticity, *J. Math. Anal. Appl.*, 223(1998), 216-247.

[RW05] RACKE, R. & WANG, Y. G.: Asymptotic behavior of discontinuous solutions to thermoelastic systems with second sound, *J. Anal. Appl.*, 24 (2005), 117-135.

[Rau91] RAUCH, J.: *Partial Differential Equations.* Springer, New York, 1991.

[RW99a] REISSIG, M. & WANG, Y. G.: Analysis of Weak Singularities of Solutions to 1-D Thermoelasticity, *Proceedings of the International Conference on PDE and their Applications (Wuhan 1999)*, 272-282, World Scientific, Singapore, 1999.

[RW99b] REISSIG, M. & WANG, Y. G.: Propagation of mild singularities in higher dimensional thermoelasticity, *J. Math. Anal. Appl.*, 240(1999), 398-415.

[RW05] REISSIG, M. & WANG, Y. G.: Cauchy problems for linear thermoelastic systems of type III in one space variable, *Math. Methods Appl. Sci.*, 28(2005), 1359-1381.

[RW08] REISSIG, M. & WIRTH, J.: Anisotropic thermo-elasticity in 2D. Part I: A unified treatment, *Asymp. Anal.*, 57(2008), 1-27.

[RS05] RUZHANSKY, M. & SMITH, J.: Global time estimates for solutions to equations of dissipative type, *Journées équations aux dérivées partielles*, 12(2005), 1-29.

[Ste70] STEIN, E. M.: *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, Princeton, NJ, 1970.

[Ste93] STEIN, E. M.: *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals.* Princeton University Press, Princeton, NJ, 1993.

[Str70a] STRICHARTZ, R. S.: Convolutions with kernels having singularities on a sphere, *Trans. Amer. Math. Soc.*, 148(1970), 461-471.

[Str70b] STRICHARTZ, R. S.: A priori estimates for the wave equation and some applications, *J. Funct. Analysis*, 5(1970), 218-235.

[Tho57] THOMSON, W.: On the Thermo-Elastic and Thermo-Magnetic Properties of Matter, *Quart. J. Math.*, 1(1857), 57-77.

[Tis38] TISZA, L.: Sur la supraconductibilité thermique de l'helium II liquide et la statistique de Bose-Einstein, *C. R. Acad. Sci.*, 207(1938), 1035-1037.

[vW71] VON WAHL, W.: L^p-decay rates for homogeneous wave-equations, *Math. Z.*, 120(1971), 93-106.

[Wan02] WANG, Y. G.: Remarks on Propagation of Singularities in Thermoelasticity, *J. Math. Anal. Appl.*, 266(2002), 169-185.

[Wan03a] WANG, Y. G.: A new approach to study hyperbolic-parabolic coupled systems, *Banach Center Publications*, 60(2003), 227-236.

[Wan03b] WANG, Y. G.: Microlocal analysis in nonlinear thermoelasticity, *Nonlinear Anal.*, 54(2003), 683-705.

[WY06] WANG, Y. G. & YANG, L.: L^p-L^q decay estimates for Cauchy problems of linear thermoelastic systems with second sound in three dimensions, *Proc. Edinburgh Math. Soc. A*, 136(2006), 189-207.

[Wir05] WIRTH, J.: Asymptotic properties of solutions to wave equations with time-dependent dissipation, Dissertation, TU Bergakademie Freiberg, 2005.

[Wir08] WIRTH, J.: Anisotropic thermo-elasticity in 2D. Part II: Applications, *Asymp. Anal.*, 57(2008), 29-40.

[YM00] YANG, H. & MILANI, A.: On the diffusion phenomenon of quasilinear hyperbolic waves, *Bull. Sci. Math.*, 124(2000), 415-433.

[YW06a] YANG, L. & WANG, Y. G.: L^p-L^q decay estimates for the Cauchy problem of linear thermoelastic systems with second sound in one space variable, *Quart. Appl. Math.*, 64(2006), 1-15.

[YW06b] YANG, L. & WANG, Y. G.: Well-posedness and Decay Estimates for Cauchy Problems of Linear Thermoelastic Systems of Type III in 3-D, *Indiana Univ. Math. J.*, 55(2006), 1333-1362.

[ZZ03] ZHANG, X. & ZUAZUA, E.: Decay of solutions of the system of thermoelasticity of type III, *Comm. Contemp. Math.*, 5(2003), 25-83.

[ZF84] ZURMÜHL, R. & FALK, S.: *Matrizen und ihre Anwendungen, Teil 1: Grundlagen.* Springer, Berlin, 1984.