GDV III - Geometric Computing
Flow Analysis and Feature Extraction with Clifford Fourier Transform

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Importance of flow feature extraction
- Simulation overview

- In FEM and FVM numeric simulation, the computation domain is divided into several smaller elements.

- Often several thousand nodes.

- Result data, i.e. velocity vectors, temperature, pressure, etc., are generated for each element.
  - A substantial amount of data to be analysed.
  - Direct analysis of these data is practically impossible or ineffective.
Importance of flow feature extraction
- Simulation post-processing

- Robust methods to:
  - analysis
  - feature extraction
  - derivative computation

are necessary for the investigation and analysis of simulation result data, i.e. flow structure.

- Post-processing tools.

- The aim is to detect specific flow features in vector fields, i.e. vortices and other swirling flows, shear flow, reversed flow, areas with convergent or divergent behavior, streamline patterns, and regions with laminar flow.
Importance of flow feature extraction
- Methods for flow analysis

- There are several methods for flow analysis, for instance:
  - Hedgehog
  - Color-coding of vorticity.
  - Line Integral Convolution (LIC)
  - Streamlines
- Most feature detection models follow an analytic model and then employ a suitable algorithm [1].
- Others are based on convolution.

Some examples of basic visualization techniques.  
**Top left:** Hedgehog.  **Top right:** Hedgehog (normalized).  **Middle left:** Color-coding of vorticity.  
**Middle right:** LIC.  **Bottom left:** Streamlines.  
**Bottom right:** Topology.

Source: Ebling [2]
Importance of flow feature extraction
- Methods for flow analysis

- However, analytic feature detection models are in general **not robust enough**!
  - Limitations detecting features that are not well defined, i.e. vortices or swirling in turbulent flows.

- An alternative robust approach:

  **Convolution-based methods**
  - A powerful tool, together with the Fourier transform, convolution theorem, shift theorem, derivation theorem and Parseval’s theorem.
  - Some image processing methods based on classical convolution are well defined on 2D vector fields.
    - But not effective in 3D vector fields.
Mathematics overview

- What is convolution?

- Defined by:

\[(h * f)(x) = \int_{E^d} h(x') \cdot f(x - x') dx'\]

- It is the amount of overlap of a function as it is shifted over another function [4].

- In other words: it filters a function \(f(x)\) through a filter \(h(x)\).

- Every linear and shift-invariant (LSI) filter can be described as a convolution with filter. Several image processing filters are LSI [3].

Source: http://mathworld.wolfram.com/Convolution.html
Derivatives can be expressed in terms of convolution, based on the Riemann–Liouville differentiation/integration operator:

\[ a \, D_{x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x - \tilde{x})^{\alpha - 1} f(\tilde{x}) \, d\tilde{x} \]

\[ = \int_{a}^{x} \left( -\frac{1}{\Gamma(\alpha)} (x')^{\alpha - 1} \right) \cdot f(x - x') \, dx' , \quad \Gamma(\alpha) = (\alpha - 1)! \]

\[ a \, D_{x}^{\alpha} = \begin{cases} \frac{d^{\alpha}}{dx^{\alpha}}, & \alpha > 0 \\ 1, & \alpha = 0 \\ \int_{a}^{x} (dx)^{-\alpha}, & \alpha < 0 \end{cases} \]
Mathematics overview
- Convolution extension to vector fields

- To compute the convolution for each component of the vector field independently.
  - Scalar fields of the components are not independent,
  - A vector has more information than its separated components [1]

- Through the scalar product of 2 vectors:

  \[(h \ast f)(x) = \int_{E^d} \langle h(x'), f(x - x') \rangle dx'\]

  then, to combine with different filter directions and an orientation tensor. According to [1], this approach was introduced by Heiberg et al., who did not formulate a Fourier Transform in their method.
  - Creates a scalar field. Not a unified operator.
Mathematics overview
- Correlation

- Feature extraction and flow analysis through template matching.
- Templates represent certain flow features.
- Similarity between a vector field and a template by computing the correlation between both.
- Templates are LSI filters.

Template examples.
**Top:** Clockwise rotation.
**Middle left:** Saddle. **Middle right:** Shear flow.
**Bottom left:** Convergence line. **Bottom right:** Convergence.

*Source: Ebling [2].*
Mathematics overview
- Correlation

- A correlation measures the similarity between 2 signals $h$ and $f$ in a certain point.

- Every correlation can be computed as a convolution with a suitably adjusted filter [1].

- Spatial correlation: $(h \ast f)(x) = \int_{E^d} h(x') \cdot f(x + x') dx'$

  $= \int_{E^d} -h(-\bar{x}) \cdot f(x - \bar{x}) d\bar{x}$

  $= \int_{E^d} \tilde{h}(x') \cdot f(x - x') dx'$

$\Rightarrow$ Correlation is a convolution with a filter reflected at its center.
Mathematics overview
- Fourier transform

- Basis transform from image space into frequency space.
- The Fourier transform of a scalar field $f$ is defined as:

$$\mathcal{F}\{f\}(u) = \int_{E^d} f(x) e^{-2\pi i \langle x, u \rangle} \, dx$$

for continuous $f: E^d \rightarrow C$, with $i^2 = -1$, provided the integral exists.
- The inverse is defined as:

$$\mathcal{F}^{-1}\{f\}(u) = \int_{E^d} f(x) e^{2\pi i \langle x, u \rangle} \, dx$$

- In general, $\mathcal{F}$ exists if [1]:

$$\int_{E^d} |f(x)| e^{-2\pi i \langle x, u \rangle} \, dx < \infty$$
Mathematics overview
- Convolution and derivative theorems

- **Convolution theorem**: Let $f, h: E^d \rightarrow C$ be continuous, and $\mathcal{F\{h\}}$ and $\mathcal{F}\{f\}$ exist, then

$$\mathcal{F}\{h \ast f\}(u) = \mathcal{F}\{h\}(u) \mathcal{F}\{f\}(u)$$

- **Derivative theorems**: Let $f: E^d \rightarrow C$ be continuous and $\mathcal{F}\{f\}$ exist, then

$$\mathcal{F}\{\nabla f\}(x) = 2\pi i u \mathcal{F}\{f\}(u)$$

$$\mathcal{F}\{\Delta f\}(x) = -4\pi^2 u^2 \mathcal{F}\{f\}(u)$$
Clifford Algebra in 3D

- Introduction

- Clifford algebra (also called geometric algebra) is an extension to the classical linear algebra.

- A vector in 3D Euclidean space ($E^3$) is spanned by:
  \[ \{ e_1, e_2, e_3 \} \]

- An element of 3D Clifford Algebra ($G^3$) is an 8-dimensional multivector, spanned by:
  \[ \{ 1, e_1, e_2, e_3, e_{12}, e_{23}, e_{31}, e_{123} \} \]

Source: Ebling [1]
Clifford Algebra in 3D

- Introduction

- The multiplication of base elements in $G^3$ is associative, bilinear, and the following holds:

\[ l e_j = e_j \quad j = 1, 2, 3 \]
\[ e_j e_j = 1 \quad j = 1, 2, 3 \]
\[ e_j e_k = -e_k e_j \quad j, k = 1, 2, 3, \ j \neq k \]

- The simplified notation is used:

\[ e_{jk} = e_j e_k \quad j, k = 1, 2, 3, \ j \neq k \]
Clifford Algebra in 3D
- Definition of Multivectors

- A multivector $\mathbf{A}$ is defined as:

$$
\mathbf{A} = \alpha + \mathbf{a} + i_3(\mathbf{b} + \beta)
$$

with $\alpha, \beta \in \mathbb{R}$, $\mathbf{a}, \mathbf{b} \in \mathbb{E}^3$, $i_3 = e_{123}$ and $(i_3)^2 = -1$.

- Note that $\alpha + i_3\beta$ is isomorphic to the complex numbers!

- The grade projectors $\langle \rangle_j : \mathbb{G}^3 \rightarrow \mathbb{G}^3$ are defined as:

  $$
  \langle \mathbf{A} \rangle_0 = \alpha \\
  \langle \mathbf{A} \rangle_1 = \mathbf{a} \\
  \langle \mathbf{A} \rangle_2 = i_3\mathbf{b} \\
  \langle \mathbf{A} \rangle_3 = i_3\beta
  $$

Clifford Algebra in 3D
- Geometric product

- The Clifford multiplication of 2 vectors $a, b \in E^3$ is defined as:

$$ab = \langle a, b \rangle + a \wedge b$$

**Inner product**  
**Outer product**

- For the angle $\omega$ between $a$ and $b$, holds:

$$\langle ab \rangle_0 = \langle a, b \rangle = \|a\|\|b\| \cos \omega$$

$$\|\langle ab \rangle_2\| = \|a \wedge b\| = \|a\|\|b\| \sin \omega$$

- $\langle ab \rangle_2$ is the plane through $a$ and $b$. 
Clifford Algebra in 3D

-Blades

- A $k$-blade (grade $k$) is defined as:

$$\mathbf{A}_{(k)} = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_k$$

$k \in [1,3]$

- 0-blade (scalar): $\mathbf{a}$

- 1-bade (vector): $\mathbf{A}_{(1)} = \mathbf{a}_1$

- 2-blaide (bivector): $\mathbf{A}_{(2)} = \mathbf{a}_1 \wedge \mathbf{a}_2$

- 3-blade (trivector): $\mathbf{A}_{(3)} = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3$

- A multivector in $G^3$ is thus the sum of $k$-blades, $k \in [0,3]$:
Clifford Algebra in 3D
-Geometric product of Multivectors

- Multiplication in $G^3$ is called geometric product. It has the following properties:
  1. Closure
  2. Distributivity: $A(B + C) = AB + AC$
     $(A + B)C = AC + BC$
  3. Associativity: $(AB)C = A(BC)$
  4. Unit (scalar element): $1A = A$
  5. Tensor contraction
  6. Commutativity with scalar: $\alpha A = A \alpha$

- Multiplication in general **not commutative**!
Clifford Algebra in 3D
-Integral and derivatives

- Riemann integral of a multivector valued function $\mathbf{F}$:
  \[ \int_{E^3} \mathbf{F}(\mathbf{x}) \, d\mathbf{x} = \lim_{\Delta x_j \to 0} \sum_{j=1}^{n} \mathbf{F}(x_j \mathbf{e}_j) \Delta x_j \]

- Derivative of $\mathbf{F}$ in direction $\mathbf{r}$:
  \[ \mathbf{F}_r(\mathbf{x}) = \lim_{h \to 0} \frac{\mathbf{F}(\mathbf{x} + h \mathbf{r}) - \mathbf{F}(\mathbf{x})}{h} \]

- Gradient of $\mathbf{F}$ from left and right (multiplication in $\mathbb{G}^3$ is not commutative):
  \[ \nabla \mathbf{F}(\mathbf{x}) = \sum_{j=1}^{d} \mathbf{e}_j \mathbf{F}_{e_j}(\mathbf{x}) \quad \mathbf{F}(\mathbf{x}) \nabla = \sum_{j=1}^{d} \mathbf{F}_{e_j}(\mathbf{x}) \mathbf{e}_j \]

- Curl and divergence of a vector valued function $\mathbf{f}$:
  \[ \text{curl } \mathbf{f} = \nabla \wedge \mathbf{f} = \frac{(\nabla \mathbf{f} - \mathbf{f} \nabla)}{2} \quad \text{div } \mathbf{f} = \langle \nabla, \mathbf{f} \rangle = \frac{(\nabla \mathbf{f} + \mathbf{f} \nabla)}{2} \]
Clifford Algebra in 3D
-Dual of a multivector

- The dual $A^*$ of a multivector $A$ is defined as:

\[ A^* = -A i_3 \]

- There are 4 dual pairs in $\mathbb{G}^3$:

\[ 1 \leftrightarrow i_3 \]
\[ e_1 \leftrightarrow e_{23} \]
\[ e_2 \leftrightarrow e_{31} \]
\[ e_3 \leftrightarrow e_{12} \]
Clifford Fourier Transform
- Clifford convolution

- The Clifford convolution is defined as:

\[
(\mathbf{H} \star_l \mathbf{F})(\mathbf{x}) = \int_{E^d} \mathbf{H}(\mathbf{x}') \mathbf{F}(\mathbf{x} - \mathbf{x}') d\mathbf{x}'
\]

\[
(\mathbf{F} \star_r \mathbf{H})(\mathbf{x}) = \int_{E^d} \mathbf{F}(\mathbf{x} - \mathbf{x}') \mathbf{H}(\mathbf{x}') d\mathbf{x}'
\]

for a multivector valued field \( \mathbf{F} \) and a multivector valued filter \( \mathbf{H} \).

- Clifford convolution is **not commutative**!

- Discretization of the Clifford convolution for 3D uniform grids:

\[
(\mathbf{H} \star_l \mathbf{F})_{j,k,l} = \sum_{s=-r}^{r} \sum_{t=-r}^{r} \sum_{u=-r}^{r} \mathbf{H}_{s,t,u} \mathbf{F}_{j-s,k-t,l-u}
\]

\[
(\mathbf{F} \star_r \mathbf{H})_{j,k,l} = \sum_{s=-r}^{r} \sum_{t=-r}^{r} \sum_{u=-r}^{r} \mathbf{F}_{j-s,k-t,l-u} \mathbf{H}_{s,t,u}
\]
Clifford Fourier Transform
- Clifford Fourier Transform (CFT)

- The CFT in 3D of a multivector valued function $F$ is defined as:

$$\mathcal{F}\{F\}(u) = \int_{E^3} F(x) e^{(-2\pi i \langle x, u \rangle)} |dx|$$

- $F$ can be regarded as a sum of four complex signals:

$$F = F_0 + F_1 e_1 + F_2 e_2 + F_3 e_3 + F_{23} e_{23} + F_{31} e_{31} + F_{12} e_{12} + F_{123} e_{123}$$

$$= F_0 + F_1 e_1 + F_2 e_2 + F_3 e_3 + F_3 i_3 e_1 + F_1 i_3 e_2 + F_2 i_3 e_3 + F_{123} i_3$$

$$= [F_0 + F_{123} i_3] e_1 + [F_1 + F_{23} i_3] e_2 + [F_2 + F_{31} i_3] e_2 + [F_3 + F_{12} i_3] e_3$$

- Then, with linearity of the CFT:

$$\mathcal{F}\{F\}(u) = [\mathcal{F}\{F_0 + F_{123} i_3\}(u)] e_1 + [\mathcal{F}\{F_1 + F_{23} i_3\}(u)] e_1 + [\mathcal{F}\{F_2 + F_{31} i_3\}(u)] e_2 + [\mathcal{F}\{F_3 + F_{12} i_3\}(u)] e_3$$
Clifford Fourier Transform
- Clifford Fourier Transform (CFT)

- The inverse CFT of a multivector valued function $F$ is defined as:

$$\mathcal{F}^{-1}\{F\}(x) = \int_{E^3} F(u) e^{2\pi i_3 \langle x, u \rangle} |du|$$

- $e^{2\pi i_3 \langle x, u \rangle}$ is multivector valued, since:

$$e^{2\pi i_3 \langle x, u \rangle} = \cos(2\pi \langle x, u \rangle) + i_3 \sin(2\pi \langle x, u \rangle)$$
Clifford Fourier Transform

- Theorems

**Shift theorem.** Let $F : E^3 \to G_3$ be multivector valued and $\mathcal{F}\{F\}$ exist, then

$$\mathcal{F}\{F(x - x')\}(u) = \mathcal{F}\{F\}(u) e^{-2\pi i \langle x', u \rangle}$$

**Convolution theorem.** Let $F, H : E^3 \to G_3$ be multivector valued and $\mathcal{F}\{F\}$ and $\mathcal{F}\{H\}$ exist, then

$$\mathcal{F}\{H \ast_{l} F\}(u) = \mathcal{F}\{H\}(u) \mathcal{F}\{F\}(u)$$

$$\mathcal{F}\{F \ast_{r} H\}(u) = \mathcal{F}\{F\}(u) \mathcal{F}\{H\}(u)$$
Clifford Fourier Transform
- Theorems

- **Derivative theorem.** Let $F: \mathbb{E}^3 \rightarrow \mathbb{G}_3$ be multivector valued and $\mathcal{F}\{F\}$ exist, then

\[
\mathcal{F}\{\nabla F\}(u) = 2\pi i_3 u \mathcal{F}\{F\}(u)
\]

\[
\mathcal{F}\{\Delta F\}(u) = -4\pi^2 u^2 \mathcal{F}\{F\}(u)
\]

\[\mathcal{F}\{F \nabla\}(u) = \mathcal{F}\{F\}(u) 2\pi i_3 u\]

\[\mathcal{F}\{F \Delta\}(u) = -4\pi^2 u^2 \mathcal{F}\{F\}(u)\]

- **Parseval’s theorem.** Let $F: \mathbb{E}^3 \rightarrow \mathbb{G}_3$ be multivector valued and $\mathcal{F}\{F\}$ exist, then

\[
\|F\|_2 = \|\mathcal{F}\{F\}\|_2
\]
Feature analysis and extraction
- Overview

- Feature extraction through correlation of vector valued templates and vector field.
- Correlation measures the similarity between template and vector field. It can be represented as a convolution with a suitably transformed filter.

3D Template representing convergence.

ICE. Color coding from dark blue (convergence) to dark red (divergence).

Source: Ebling [1]
Feature analysis and extraction
- Template matching

- Feature analysis and visualization can be achieved by color coding of similarity results.
  ⇒ Correlation between template and vector field.

- Example:

Flow over the surface of a delta wind: regions of convergence (red) and divergence (blue) on the wing. The grid is irregular, with local resampling of the field.

Source: Ebling [2]
Feature analysis and extraction
- Vortex detection

- Center of the vortex is the point of highest similarity.
- The position and size of vortex can be computed by successively enlarging the template until the similarity becomes insignificant.

Similarity for the $5^2$ rotational template (red), for the template with maximal similarity within the size computation (green), and for the template with maximal size within the region (blue).

Source: Ebling [2]
**Feature analysis and extraction**

- Clifford Fourier transform

- The CFT of a template $H$ is represented by the CFT of 3 complex signals:

$$\mathcal{F}\{H\}(u) = \left[\mathcal{F}\{H_0 + H_{123}i_3\}(u)\right]_1 + \left[\mathcal{F}\{H_1 + H_{23}i_3\}(u)\right]_e_1 +$$

$$\left[\mathcal{F}\{H_2 + H_{31}i_3\}(u)\right]_e_2 + \left[\mathcal{F}\{H_3 + H_{12}i_3\}(u)\right]_e_3$$

$\Rightarrow$ The CFT of $H$ contains vector and bivector components only.
Feature analysis and extraction
- Clifford Fourier transform

⇒ Less multivectors are necessary to represent a multivector valued template in frequency space.

- Vector (real) valued parts of the DCFT of $H$ are symmetric.
- Bivector (imaginary) valued parts of the DCFT of $H$ are anti-symmetric.

Top: 3D patterns.
Middle: The vector part of their DCFT.
Bottom: The bivector part of their DCFT, displayed as normal vector of the plane. All vectors are normalized.

Source: Ebling [2]
Feature analysis and extraction
- Rotation invariance

- Features may be positioned in different directions.
  - A rotation invariant algorithm for template matching is of great importance.

- Basic idea:
  - The angle between the directions of the template and vector field can be computed by Clifford correlation. Then, the template can be rotated into the direction of the vector field [2].
  - Additional templates with different directions are needed to compute stable results. In 3D, the directions of the principal axes can be used [2].
Feature analysis and extraction
- Template superposition

- Template matching is a linear and shift invariant (LSI) operation.
- **Linearity** \( \Rightarrow \) complex features can be represented by combination of simpler features.

\[
((ah_1 + bh_2) \ast f) = (ah_1 \ast f) + (bh_2 \ast f)
\]

Source: Ebling [2]
Conclusion
- Advantages

+ Unified notation for convolution of scalar, vector and multivector fields.
+ Convolution may be easily applied several times.
+ Discretized derivative can be computed by a Clifford convolution using a vector valued filter [1].
+ The convolution theorem, shift theorem, derivative theorem and Parseval’s theorem, along with the CFT, enable the computation of a Clifford convolution in the frequency space.
+ Properties of the classical FT can be extended to the CFT [1].
+ The existence of FFT algorithm speeds up the convolution computation [2].
Conclusion

- Speed-up through FFT

- Ebling [1] discusses the results of convolution computations in spatial domain and through CFT in her dissertation. DNS simulation results of a simple turbulent flow are used. The grid contains $2^8 \times 2^7$ nodes.

<table>
<thead>
<tr>
<th>computation</th>
<th>size of pattern</th>
<th>time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>spatial domain</td>
<td>$5 \times 5$</td>
<td>13</td>
</tr>
<tr>
<td>spatial domain</td>
<td>$15 \times 15$</td>
<td>55</td>
</tr>
<tr>
<td>using fast CFT</td>
<td>$5 \times 5$</td>
<td>10</td>
</tr>
<tr>
<td>using fast CFT</td>
<td>$15 \times 15$</td>
<td>10</td>
</tr>
</tbody>
</table>

- Maximal numerical error $\rightarrow 4\times10^{-15}$!

  **Top:** convolution computed in the frequency domain. **Bottom:** difference between results on top image and convolution computed in the spatial domain (due to numerical error). Colors are scaled from $-4\times10^{-15}$ (blue) to $4\times10^{-15}$ (red).

Source: Ebling et al. [1]
Conclusion
- Current disadvantages

- The application of the CFT in data set defined on irregular grids still needs further research.

- One approach is to resampled the data set of either field or template. However, resampling the data set can substantially increase the amount of data and computation time [2].

*Left*: Local resampling of the field. *Right*: Local resampling of the template.

Source: Ebling [2]
Potential application of the CFT and GA in CFD

- GA – a universal language for physics and mathematics

- GA provides an immensely powerful, compact and intuitive description of concepts of classical and quantum mechanics, relativity, electromagnetism, etc. [8,10]

- Useful in physics problems that involve rotations, wave phases or imaginary numbers.

- In recent years, GA methods have been successfully employed to deal with a range of problems in different scientific fields, including:
  - Computer vision
  - Robotics
  - Relativistic electro-magnetic field theory
  - FEM based simulation of elastic rods
**Potential application of the CFT and GA in CFD**

- GA – a universal language for physics and mathematics


- *David Hestenes* have successfully applied GA to a variety of physical problems [5]. He calls GA:

  “*a unified language for mathematics and physics*”
Potential application of the CFT and GA in CFD
- GA – advantage for CFD?

Can GA (and CFT) be successfully applied to CFD as well?

Would it be advantageous?
Potential application of the CFT and GA in CFD
- GA – advantage for CFD?

- Carsten Cibura [5] has extended the conservation of mass and Navier-Stokes equations to multivectors:

\[
0 = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U})
\]

\[
\rho \frac{D\mathbf{U}}{Dt} = -\nabla p - \left( \lambda - \frac{3-n}{2} \mu \right) \nabla (\nabla \cdot \mathbf{U}) - \frac{n-1}{2} \mu \nabla^2 \mathbf{U}
\]

\[
\frac{D\mathbf{U}}{Dt} = \frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U}
\]

- The overdot notation is introduced because of non-commutativity, for example:

\[
\nabla (\mathbf{A} \mathbf{B}) = \nabla \mathbf{A} \mathbf{B} + \hat{\nabla} \mathbf{A} \dot{\mathbf{B}} \quad \mathbf{\hat{\nabla} \mathbf{A} \dot{\mathbf{B}}} = \sum_{k=1}^{n} e_k \mathbf{A} \frac{\partial}{\partial x_k} \mathbf{B}
\]
Potential application of the CFT and GA in CFD
- GA – advantage for CFD?

- Vorticity (curl) is defined as:

\[ \mathbf{\omega} = \nabla \times \mathbf{U} \]

- The stream function \( \psi \) defined through:

\[ \nabla \times \psi = -\mathbf{U} \mathbf{i}_n^{-1} \]

- Vorticity equations for incompressible flows:

\[
\frac{D\mathbf{\omega}}{Dt} = (\mathbf{\omega} \times \dot{\nabla}) \cdot \dot{\mathbf{U}}, \quad \mathbf{U} = - (\nabla \times \psi) \mathbf{i}_n, \\
\nabla \cdot \psi = 0, \quad \mathbf{\omega} \mathbf{i}_n = \left[ \nabla \cdot (\nabla \times \psi) \right]
\]
Potential application of the CFT and GA in CFD
- GA – advantage for CFD?

- GA could provide a whole new approach to the modeling of turbulent flows.
  - A more intuitive description of the problem geometrically.
  - Vortices can be better represented in different dimensions.
  - There is an unified notation for the flow governing equations, convolution and Fourier transform in different dimensions.
Potential application of the CFT and GA in CFD
- Turbulence problematic

- Turbulent flows: have a random nature
  are transient
  are 3-dimensional
  contains vortices
  are dissipative

- In order to direct simulate turbulent flows, the grid has to be finer than the smallest vortex.

- A direct numeric simulation (DNS) of turbulent flows is not practicable in the industry because of its extremely high computation costs.

- Alternatives are: RANS methods
  LES methods
The idea of LES is to compute the larger structures while modeling the small structures.

Large structures are decomposed from small structures by convolution based filtering:

\[ f = \bar{f} + f' \]

where \( \bar{f} \) represents the filtered large structures and \( f' \) the small structures.

\[ \bar{f}(x_j, t) = \int_{E_d} h(x_j - x_j', t - t') \cdot f(x_j', t) \, dx' \, dy' \, dz' \, dt' \]

Source: Janicka [6]
## Potential application of the CFT and GA in CFD
- Large Eddy Simulation (LES)

- Filtering is advantageous because small and large structures have different properties.
- Mean flow described by large structures.
- Small structures are easier to model.

<table>
<thead>
<tr>
<th>Large structures:</th>
<th>Small structures:</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Originate from mean flow</td>
<td>• Originate from large structures</td>
</tr>
<tr>
<td>• Depends on problem geometry</td>
<td>• Universal</td>
</tr>
<tr>
<td>• Non-homogeneous and anisotropic</td>
<td>• Homogeneous and isotropic</td>
</tr>
<tr>
<td>• Longer life-time</td>
<td>• shorter life-time</td>
</tr>
<tr>
<td>• Energy rich</td>
<td>• Energy poor</td>
</tr>
<tr>
<td>• Diffusive</td>
<td>• Dissipative</td>
</tr>
</tbody>
</table>
Potential application of the CFT and GA in CFD
- Large Eddy Simulation (LES)

- The filtered conservation of mass and Navier-Stokes equations for incompressible Newtonian fluids are:

\[
\sum_{i=1}^{3} \frac{\partial \bar{u}_i}{\partial x_i} = 0
\]

\[
\sum_{j=1}^{3} \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial (\bar{u}_i \bar{u}_j)}{\partial x_j} + \frac{\partial \bar{p}}{\partial x_i} = \sum_{j=1}^{3} \frac{\partial (2 \nu \bar{S}_{ij})}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}
\]

\[
\bar{S}_{ij} = \frac{1}{2} \left( \frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_j} \right)
\]

where the deformation tensor \( \tau_{ij} \) is defined as:

\[
\tau_{ij} = \bar{u}_i u_j - \bar{u}_i \bar{u}_j
\]
Potential application of the CFT and GA in CFD
- Large Eddy Simulation (LES)

- Clifford convolution enables the development of LES based methods using GA, since the filtering operation can be applied to multivector fields.

- Some LES methods model the turbulence and solve the flow in frequency space.

  With the Clifford Fourier transform, new methods using GA could be developed based on the same idea.

- Further research is necessary to conclude whether or not the use of GA in LES based methods for flow simulation is advantageous.
Potential application of the CFT and GA in CFD
- Vortex method


- Major characteristic and advantages of this method are:
  - Computes the evolution of vorticity using a Lagrangian approach.
  - Computation points follow the motion of the fluid.
  - No grid required.
  - Computation of the velocity field from vorticity satisfies the mass conservation exactly.
  - Avoids numerical dissipation since vortices follow the motion of the fluid.
Potential application of the CFT and GA in CFD
- Vortex method

- Solve the vorticity $\omega$ through a Lagrangian approach (for incompressible flows):
  \[
  \frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \nabla^2 \omega, \quad \omega \hat{z} = \nabla \times \mathbf{u}
  \]

- Closure by solving the stream function $\psi$:
  \[
  \nabla \times \psi \hat{z} = \mathbf{u}, \quad u_1 = \frac{\partial \psi}{\partial y}, \quad -u_2 = \frac{\partial \psi}{\partial x}
  \]
  satisfies the mass conservation equation $\nabla \cdot \mathbf{u} = 0$

- The following Poisson equation is obtained:
  \[- \omega = \nabla^2 \psi\]
Potential application of the CFT and GA in CFD
- Vortex method

- Velocity field computed by solving:
  \[ -\omega = \nabla^2 \psi \]

- Poisson equation can be solved with the Green’s function method:
  \[
  \psi(x,t) = \int \int G(x,x') \cdot \omega(x',t) dx' dy' \\
  = G * \omega
  \]

- Then, the velocity field follows:
  \[
  u(x,t) = \int \int K(x,x') \cdot \omega(x',t) dx' dy' \\
  = K * \omega
  \]

  \[ K(x,x') = \nabla_x \times G(x,x') \hat{z} \]
Potential application of the CFT and GA in CFD
- Vortex method

- According to Subramaniam [11], the method is robust, since it does not depend on grid quality.

- However, computation time is still a concern.

- Extending the vortex method to GA could:
  - Reduce computation costs
  - Enable a model improvement through a compact and more intuitive geometric description of the problem

- Employing the CFT to solve convolutions could contribute to the computation cost reduction.
References


