Non exponential quasiparticle decay and phase relaxation in low dimensional conductors

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We show that in low dimensional disordered conductors, the quasiparticle decay and the relaxation of the phase are not exponential processes. In the quasi-one dimensional case, both behave at small time as $e^{-(t/\tau_{in})^{3/2}}$ where the inelastic time τ_{in}, identical for both processes, is a power $T^{2/3}$ of the temperature. The non exponential quasiparticle decay results from a modified derivation of the Fermi golden rule. This result implies the existence of an unusual distribution of relaxation times.

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The issue of dephasing in the presence of electron-electron interactions in disordered conductors is of great importance in mesoscopic physics. This problem, first addressed by Altshuler, Aronov and Khmelnitskii (AAK) [1], has been recently revisited in the light of a new set of experiments [2–4]. A related problem is the understanding of the time evolution of a quasiparticle state due to electron-electron interaction, which governs the relaxation towards thermal equilibrium [5]. Since coherent effects in disordered systems result from the coherent pairing of two scattering amplitudes defined for a given quasiparticle state, the coherence is lost once this state has relaxed. Thus it seems natural to assume that quasiparticle and phase relaxations are of the same nature [6] and are driven by the same time scale. In this letter, we show that in low dimensional disordered conductors and in particular for quasi-one-dimensional (quasi-$1d$) wires, both relaxations are faster than exponential and are driven by the same characteristic time [7]. This non exponential behavior reflects the existence of a distribution of relaxation times. Such a non exponential decay is unusual in quantum condensed matter physics but more frequent in the context of molecular relaxation processes and in glassy systems. Stretched and compressed exponentials are mostly used as a way to fit unusual relaxations but no microscopic basis can be assigned to account for this behavior [8]. Here, we derive it from a new treatment of the Fermi golden rule.

We shall first consider the quasiparticle decay, using the Fermi golden rule [9] . We show that, due to screened Coulomb interactions with small energy transfer, the relaxation rate is not constant, implying a non exponential decay. This results from a key step in the Fermi golden rule which stems that the transitions conserve energy within $\hbar/t$ where t is the duration of the perturbation. Usually this constraint is of no practical importance and energy conservation is described by a delta function. Here we show that this is no longer possible. As a result, we find that for quasi-$1d$ wires, the probability for a quasiparticle to stay in its initial state behaves, at small times, as $\mathcal{P}(t) = e^{-\beta(t/\tau_{in})^{3/2}}$. The temperature dependence of the inelastic time is $\tau_{in}(T) \propto T^{-2/3}$ and β is a numerical constant.

Then, we shall come to the relaxation of phase coherence. It is given by the average $\langle e^{i\Phi(t)}\rangle$ of the relative phase $\Phi(t)$ between time reversed trajectories. Starting from the AAK calculation, we show that the phase relaxation is also non-exponential and that, at small times, it behaves like $\langle e^{i\Phi(t)}\rangle \simeq e^{-(t/\tau_\phi)^{3/2}}$ where the phase coherence time τ_ϕ is proportional to the quasiparticle decay time τ_{in}.

We start by considering the decay of a quasiparticle, recalling first some known features of the derivation. Using the Fermi golden rule, the quasiparticle lifetime can be written in terms of a kernel $K(\omega)$ which is the average over disorder of the squared matrix element of the screened Coulomb interaction [7, 10]:

$$\frac{\hbar}{\tau_{in}(T)} = 8\pi\nu_0^3 \int_0^\infty d\omega K(\omega) \frac{\omega}{\sinh \beta\hbar\omega} \quad , \tag{1}$$

where ν_0 is the total density of states per spin direction. The temperature dependence results from the occupancy of the initial and final quasiparticle states. Upon disorder averaging, the kernel $K(\omega)$ is obtained as the squared product of the dynamically screened interaction and of a long range contribution called diffuson. As a result, we have [5] :

$$K(\omega) = \frac{1}{4\pi^2\nu_0^4} \sum_{\mathbf{q}} \frac{1}{\omega^2 + D^2q^4} \quad . \tag{2}$$

The diffusive nature of the electronic motion implies a strong dependence of the transition probability upon the space dimensionality d that appears in the sum over the modes $\mathbf{q}$. The kernel $K(\omega)$ then depends on the space dimensionality d and it is given by [5] $K(\omega) = (\alpha_d/16\nu_0^4\omega^2)\,(\omega/E_c)^{d/2}$ where $\alpha_1 = \sqrt{2}/\pi^2$, $\alpha_2 = 1/2\pi^2$, $\alpha_3 = \sqrt{2}/2\pi^3$ and $E_c = D/L^2$ is the Thouless frequency. For $d = 3$, the integral in (1) is convergent so that $\tau_{in}(T)$ is well defined and behaves like $T^{-3/2}$. However for $d \le 2$, the integral in (1) diverges at low energy transfer ω. To cure this divergence, it is commonly argued that the low frequency cut-off needed is $1/\tau_{in}$ itself, since no energy transfer can be smaller than $\hbar/\tau_{in}$. Consequently, the lifetime is solution of a self-consistent equation whose solution in $d = 1$ is :

$$\frac{1}{\tau_{in}(T)} = \left(\frac{Te^2\sqrt{D}}{S\sigma\hbar^2}\right)^{2/3} \quad . \tag{3}$$

where the conductivity is $\sigma = 2e^2\nu_0 D/(LS)$. This temperature dependence has been first obtained by Altshuler and Aronov [5].

We argue here that this divergence is indeed the signature of a *new behavior for the quasiparticle decay*. We prove that this decay is actually *non exponential*. The crucial point in our argument is that it is not correct to cut-off the integral at $1/\tau_{in}$. To grasp the relevance of this statement, it is important to recall the Fermi golden rule prescription namely that, after a time t, the range of accessible states is limited to energies larger than $\hbar/t$, *not* $\hbar/\tau_{in}$. This leads to the replacement of Eq.(1) by the following expression for the disorder averaged transition probability $\mathcal{P}^{(2)}(t)$ towards final states, calculated up to second-order in perturbation :

$$\mathcal{P}^{(2)}(t) = \frac{\pi\alpha_d T}{2\nu_0\hbar^2}\; t\int_{1/t}^{T/\hbar}\frac{d\omega}{\omega^2}\left(\frac{\omega}{E_c}\right)^{d/2} \tag{4}$$

where, for simplicity, the thermal factor has been replaced by an upper cut-off at $\hbar\omega \sim T$. We have used the above expressions for the kernel in d dimensions. In one dimension, this leads immediately to a $t^{3/2}$ power law : $\mathcal{P}^{(2)}(t) = \frac{\sqrt{2}T}{\pi\nu_0\hbar^2\sqrt{E_c}}t^{3/2}$ so that the quasiparticle relaxation is not exponential.

Let us prove now this qualitative statement, coming back to the derivation of the Fermi golden rule. A given initial quasiparticle state α interacts with a quasiparticle of energy ϵ_γ, leading to two quasiparticles of final energies ϵ_β and ϵ_δ. As known from quantum mechanics textbooks [11], the transition probability towards final states is :

$$\mathcal{P}^{(2)}_\alpha(t) = 2\sum_{\beta\gamma\delta}|U_{\alpha\gamma,\beta\delta}|^2 f_t\left(\frac{\epsilon_\alpha+\epsilon_\gamma-\epsilon_\beta-\epsilon_\delta}{\hbar}\right) \tag{5}$$

where $U_{\alpha\gamma,\beta\delta}$ is the matrix element of the interaction. The function $f_t(\Delta\omega)$ of width π/t is given by [11]

$$f_t(\Delta\omega) = \left(\frac{\sin\Delta\omega t/2}{\Delta\omega/2}\right)^2 \quad . \tag{6}$$

Its maximum is equal to t^2 and its integral is $2\pi t$. Usually, this function can be approximated by $2\pi t\ \delta(\Delta\omega)$, so that the decay is linear in t, and the prefactor is the inverse quasiparticle time.

The main idea here is that this approximation is not always valid. To see this, we first calculate the disorder average of Eq. (5) using standard methods [7]. The new input is that the energy of the initial and final states may differ by a small amount $\Delta\omega$ of order $1/t$, as explicitly seen on Eqs.(5) and (6). As a result, the diffusons and dynamically screened interactions which enter in the kernel K have to be taken at different frequencies $\omega_\pm = \omega \pm \Delta\omega/2$. This immediately leads to a kernel $K_{\Delta\omega}(\omega)$ which now depends on the energy difference $\Delta\omega$:

$$K_{\Delta\omega}(\omega) = \frac{1}{4\pi^2\nu_0^4}\sum_{\mathbf{q}\neq 0}\frac{1}{\sqrt{(\omega_+^2 + D^2q^4)(\omega_-^2 + D^2q^4)}} \tag{7}$$

instead of (2). This kernel yields to a transition probability of the form :

$$\mathcal{P}^{(2)}(t) = \frac{4\nu_0^3 T}{\hbar^2}\int_0^{T/\hbar} d\omega g_t(\omega) \quad , \tag{8}$$

$$\text{where} \qquad g_t(\omega) = \int_{-\infty}^{\infty} d\Delta\omega\ f_t(\Delta\omega)K_{\Delta\omega}(\omega) \quad . \tag{9}$$

Letting $f_t(\Delta\omega) = 2\pi t\delta(\Delta\omega)$ leads immediately to the usual result, namely a behavior of $\mathcal{P}^{(2)}(t)$ linear in t and defines $1/\tau_{in}$ provided the integral is convergent.

However in $1d$, when $\Delta\omega = 0$, $K_{\Delta\omega=0}(\omega) = K(\omega) \propto 1/\omega^{3/2}$ and the integral on ω becomes divergent. It is thus crucial to keep the full expression of $f_t(\Delta\omega)$. Doing this, we find that for $\omega \gg 1/t$, we still have $g_t(\omega) = 2\pi t K(\omega) \propto t/\omega^{3/2}$, while for $\omega \ll 1/t$, it is easy to check, since $f_t(0) = t^2$, that the function $g_t(\omega) \propto t^2/\sqrt{\omega}$, so that its integral near zero frequency is indeed convergent. More precisely, we can show that $g_t(\omega)$ is of the form $g_t(\omega) = 2\pi t K(\omega)h(\omega t)$ where the function $h(\omega t)$ is linear for small argument and tends to 1 for large argument. This function is calculated numerically and is shown on Fig. 1. The quasiparticle decay probability takes the form

$$\mathcal{P}^{(2)}(t) = \frac{Tt}{\pi\sqrt{2}\hbar^2\nu_0\sqrt{E_c}}\int_0^\infty\frac{d\omega}{\omega^{3/2}}h(\omega t) \ . \tag{10}$$

We have thus proven that the cut-off in (4) appears as a natural consequence of a proper use of the Fermi

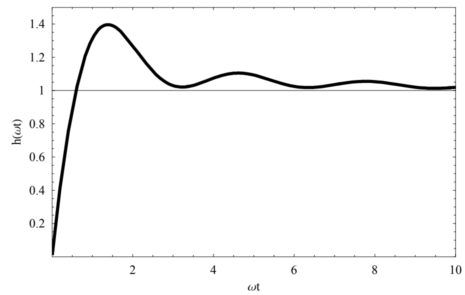

FIG. 1: *Plot of the function $h(\omega t)$. It vanishes for small argument, justifying the cut-off of order $1/t$ in Eq. (4).*

golden rule. More precisely, because of the function h which naturally provides a lower cut-off of order $1/t$, the integral now converges and

$$\mathcal{P}^{(2)}(t) = \frac{\sqrt{2}e^2\sqrt{D}T}{\pi\sigma S\hbar^2}\, t^{3/2} \int_0^\infty \frac{dx}{x^{3/2}} h(x) \ . \qquad (11)$$

At small times, the survival probability $\mathcal{P}(t)$, *i.e.* the probability that a quasiparticle stays in its original state, is given by $\mathcal{P}(t) = 1 - \mathcal{P}^{(2)}(t)$. By exponentiating this relation, we obtain :

$$\mathcal{P}(t) = e^{-\beta\left(\frac{t}{\tau_{in}}\right)^{3/2}} \ . \qquad (12)$$

The survival probability is thus given by a compressed exponential characterized by the inelastic time Eq.(3) and where $\beta \simeq 5.83\sqrt{2}/\pi$. Similarly, in two dimensions for a film of thickness a, using (4), one finds a logarithmic correction $\mathcal{P}(t) \sim e^{-\frac{t}{\tau_{in}}\frac{1}{\ln Tt}}$ with $\tau_{in}^{-1}(T) \propto \frac{e^2 T}{\hbar^2\sigma a}$. These temporal behaviors constitute one of the main results of this paper. We emphasize again that there are direct consequences of the Fermi golden rule prescription according to which the energy is conserved, *not within the decay rate $\hbar/\tau_{in}$, but rather within the inverse time $\hbar/t$* [11].

Now, one can wonder whether this peculiar behavior of the energy relaxation has its signature in the time dependence of the phase relaxation of coherent effects in weakly disordered systems. These effects result from the coherent pairing of two scattering amplitudes defined for a given quasiparticle state. In particular, we consider pairs of time reversed trajectories (the cooperon) as it appears in the weak-localization correction to the conductivity. The relaxation of the cooperon is driven by a characteristic time τ_ϕ called the phase coherence time. It seems quite intuitive that, as far as Coulomb interactions are involved, the quasiparticle decay and the cooperon relaxation are related. Thus the question arises of the relation between the inelastic time τ_{in} and the phase coherence time τ_ϕ. We shall now show that these two relaxation processes are indeed identical and characterized by the same time scale.

To that purpose, we consider the time relaxation of the cooperon by replacing the Coulomb interaction by a classical fluctuating potential $V(\mathbf{r},\tau)$ whose characteristics are determined by the fluctuation-dissipation theorem [1]. The cooperon contribution to the return probability can be written under the form :

$$P_c(\mathbf{r},\mathbf{r},t) = P_c^{(0)}(\mathbf{r},\mathbf{r},t)\left\langle e^{i\Phi(\mathbf{r},t)}\right\rangle_{T,\mathcal{C}} \qquad (13)$$

where $P_c^{(0)}$ is the cooperon in the absence of the fluctuating potential and $\Phi = \Phi(\mathbf{r},t)$ is the relative phase of a pair of time reversed trajectories at time t :

$$\Phi = \frac{1}{\hbar}\int_0^t [V(\mathbf{r}(\tau),\tau) - V(\mathbf{r}(\tau),\overline{\tau})]d\tau \qquad (14)$$

This expression is valid in the eikonal approximation *i.e.* for a slowly varying potential whose effect is to multiply the disorder averaged Green function by a phase term proportional to the circulation of $V(\mathbf{r}(\tau),\tau)$ along the trajectory between the times 0 and t. We define $\overline{\tau} = t-\tau$.

The dephasing Φ is accumulated along the diffusive electronic trajectories paired in the cooperon. One of them propagates in the time interval $0 \le \tau \le t$ whereas its time reversed counterpart propagates from $\tau = t$ to $\tau = 0$. We denote by $\langle\cdots\rangle_{T,\mathcal{C}}$ the average taken both over the distribution of the diffusive trajectories ($\langle\cdots\rangle_{\mathcal{C}}$) and over the thermal fluctuations ($\langle\cdots\rangle_T$) of the electric potential. The latter are Gaussian so that the thermal average $\left\langle e^{i\Phi}\right\rangle_T = e^{-\frac{1}{2}\langle\Phi^2\rangle_T}$. Using (14) and the fluctuation-dissipation theorem in the classical limit ($\beta\hbar\omega \ll 1$), namely, $\langle V(\mathbf{q},\omega)V(-\mathbf{q},\omega)\rangle_T = \frac{2e^2T}{\sigma q^2}$, we obtain for the dephasing the following expression

$$\left\langle\Phi^2\right\rangle_T = \frac{4e^2T}{\sigma\hbar^2}\int_0^t d\tau \int \frac{d\mathbf{q}}{(2\pi)^d}\frac{1-\cos\mathbf{q}.(\mathbf{r}(\tau)-\mathbf{r}(\overline{\tau}))}{q^2} \qquad (15)$$

The average $\langle\cos\mathbf{q}.(\mathbf{r}(\tau)-\mathbf{r}(\overline{\tau}))\rangle$ over the diffusive trajectories of time t is $e^{-2Dq^2\tau|1-2\tau/t|}$. For a quasi-1d wire, the integrations over q and τ lead to

$$\left\langle\Phi^2\right\rangle_{T,\mathcal{C}} = \frac{\sqrt{\pi}e^2T}{2\hbar^2\sigma S}\sqrt{D}\, t^{3/2} = \frac{\sqrt{\pi}}{2}\left(\frac{t}{\tau_{in}}\right)^{3/2} \qquad (16)$$

Assuming first that $\left\langle e^{-\frac{1}{2}\langle\Phi^2\rangle_T}\right\rangle_{\mathcal{C}} \simeq e^{-\frac{1}{2}\langle\Phi^2\rangle_{T,\mathcal{C}}}$, we obtain for the cooperon, at small time, the compressed exponential behavior

$$\left\langle e^{i\Phi}\right\rangle_{T,\mathcal{C}} \simeq e^{-\sqrt{\pi}/4(t/\tau_{in})^{3/2}} \qquad (17)$$

identical to the energy relaxation (12) and with the same characteristic time τ_{in} given by (3). A similar behavior for the phase relaxation has been also found in [12]. It is interesting at this stage to compare (15) with (4) obtained for the transition probability of a quasiparticle state. Although these expressions behave similarly, the convergence in Eq.(4) results from a cut-off at small ω whose origin is in the Fermi golden rule prescription namely, that among the large number of accessible states in $d = 1$, only those with energy transfer larger than $\hbar/t$ are accessible after a time t. This low energy cutoff does not exist in (15) and the convergence results from the compensation between the two terms in the bracket that describe respectively the contributions of the correlations $\langle V(\mathbf{r}(\tau),\tau)V(\mathbf{r}(\tau'),\tau')\rangle_T$ and $\langle V(\mathbf{r}(\tau),\tau)V(\mathbf{r}(\tau'),\overline{\tau}')\rangle_T$ to the cooperon.

The result (17) is not fully correct since we have approximated the average $\langle\cdots\rangle_{\mathcal{C}}$ of the exponential by the exponential of the average. Using the functional integral approach presented in [1], there is a way to derive an expression for the phase relaxation valid at all

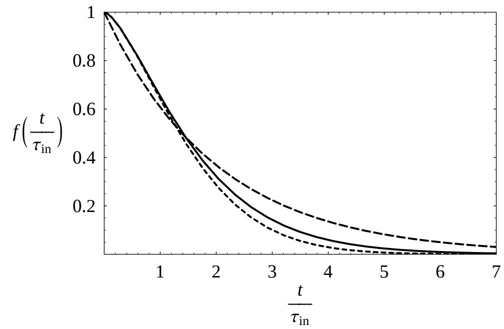

FIG. 2: *Behaviour of* $\langle e^{i\Phi(t)}\rangle_{T,\mathcal{C}}$. *The continuous line is the exact result (20). The dotted line is obtained from the small time expansion (17). The dashed line shows the exponential fit* $e^{-t/2\tau_{in}}$.

times by considering the Laplace transform $P_\gamma(r,r) = \int dt P_c(r,r,t)e^{-\gamma t}$ of the cooperon. In quasi-1d, one has :

$$P_\gamma(r,r) = -\frac{1}{2}\sqrt{\frac{\tau_{in}}{D}}\frac{\mathrm{Ai}(\tau_{in}/\tau_\gamma)}{\mathrm{Ai}'(\tau_{in}/\tau_\gamma)} \tag{18}$$

where Ai et Ai′ are respectively the Airy function and its derivative [13] and $\tau_\gamma = 1/\gamma$. The probability $P_c(r,r,t)$ in (13) can thus be obtained from the inverse Laplace transform of (18). Since, in the quasi one-dimensional limit, one has $P_c^{(0)}(r,r,t) = 1/\sqrt{4\pi Dt}$, the dephasing term $\left\langle e^{i\Phi}\right\rangle_{T,\mathcal{C}}$ is a function of t/τ_{in} that satisfies

$$\int_0^\infty \frac{dt}{\sqrt{t}}\left\langle e^{i\Phi}\right\rangle_{T,\mathcal{C}} e^{-t/\tau_\gamma} = -\sqrt{\pi\tau_{in}}\frac{\mathrm{Ai}(\tau_{in}/\tau_\gamma)}{\mathrm{Ai}'(\tau_{in}/\tau_\gamma)} \tag{19}$$

The inverse Laplace transform is obtained by noticing that the Airy function and its derivative are analytic and non meromorphic functions whose zeroes lie on the negative real axis. Then, by performing the integral in the complex plane with the residues $\mathrm{Res}(e^{st}\mathrm{Ai}(s)/\mathrm{Ai}'(s)) = e^{-|u_n|t}/|u_n|$ where the u_n are the zeros of $\mathrm{Ai}'(s)$ given at a very good approximation by $|u_n| = \left(\frac{3\pi}{2}(n-\frac{3}{4})\right)^{2/3}$ [13], we obtain the analytic function

$$\left\langle e^{i\Phi}\right\rangle_{T,\mathcal{C}} = \sqrt{\frac{\pi t}{\tau_{in}}}\sum_{n=1}^{\infty}\frac{e^{-|u_n|t/\tau_{in}}}{|u_n|} \tag{20}$$

At small times $t < \tau_{in}$, it behaves like Eq.(17). At large time, the relaxation is driven by the first zero of the Ai′ function, namely $\left\langle e^{i\Phi}\right\rangle_{T,\mathcal{C}} \simeq \sqrt{\pi t/\tau_{in}}e^{-|u_1|t/\tau_{in}}/|u_1|$ with $|u_1| \simeq 1.019$. Clearly, the relaxation (20) is never exponential. It appears as a distribution of relaxation times $\tau_{in}/|u_n|$ which is at the origin of the rather unexpected compressed exponential behavior of the quasiparticle decay and of the cooperon phase relaxation. The expression (20) constitutes one of the main results of this paper. The question arises of how this behavior could show up experimentally. It has been stressed by Pierre *et al.* [4], that the Laplace transform (18) is well approximated by the relation $P_\gamma(r,r) = (1/2\sqrt{D})\left(\frac{1}{\tau_\gamma}+\frac{a}{\tau_{in}}\right)^{1/2}$ where $a \simeq 1/2$ is an adjustable numerical constant. This approximation corresponds to an exponential relaxation $\langle e^{i\Phi(t)}\rangle_{T,\mathcal{C}} \simeq e^{-t/2\tau_{in}}$ that is clearly at odd with the behavior (20) (see Fig. 2). However, the difference between the exact relation and the exponential approximation is difficult to see experimentally. The time τ_γ accounts for other processes such as the decay rate in a magnetic field which, for a wire of section S, is given by $\tau_\gamma^{-1} = \tau_B^{-1} = DS^2e^2B^2/3\hbar^2$ [14]. A possibility to probe the $t^{3/2}$ behavior at small time is to study the limit $\tau_B \ll \tau_{in}$ where the weak localization correction to the conductivity is

$$\Delta\sigma = -\frac{e^2}{\pi\hbar}\sqrt{D\tau_B}\left[1-\frac{1}{4}\left(\frac{\tau_B}{\tau_{in}}\right)^{3/2}\right] \tag{21}$$

The power-law dependence of the correction term in (21) is a direct signature of the $t^{3/2}$ behavior of the relaxation at small time. The asymptotic behavior $\Delta\sigma(B) - \Delta\sigma(B\to\infty) \propto T/B^4$ reflects both the $T^{-2/3}$ dependence of τ_{in} and the non exponential $t^{3/2}$ phase relaxation. Another direct probe of the non exponential relaxation of the phase should may also be provided by the behavior of the *ac* conductivity $\sigma(\omega)$.

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