A Fixed Point Theorem for Expansive Type Mappings in Cone Metric Spaces

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Abstract

The objective of this paper is to obtain a fixed point theorem for expansive type mappings in cone metric space.

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1 Introduction

In 2007, Huang and Zhang [5] introduced the notion of cone metric spaces (CMSs) by replacing real numbers with an ordering Banach space. The authors there gave an example of a function which is a contraction in the category of cone metric space but not contraction if considered over metric space and hence, by proving a fixed point theorem in cone metric spaces, ensured that this map must have a unique fixed point. After that series of articles about cone metric spaces started to appear. Some of those articles dealt with the extension of certain fixed point theorems to cone metric spaces [see, e.g.,[1,2,4,6]], and some other with the structure of the spaces themselves (see, e.g.[1,7]).

Very recently, Rezapour and Hambarani [6] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point
theory in cone metric space.

In this manuscript, the known result [10] are extended to cone metric spaces where the existence of fixed points for expansive type mappings on cone metric spaces is investigated.

2 Preliminaries

Definition 2.1.[5]: Let $B$ be a real Banach space and $P$ be a subset of $B$. $P$ is called a cone if.

(i) $P$ is a closed, nonempty and $P \neq \{0\}$

(ii) $a, b \in R, a, b \geq 0, x, y \in P$ implies $ax + by \in P$.

(iii) $x \in P$ and $-x \in P$ imply $x = 0$.

Given a cone $P \subseteq B$, we define a partial ordering "$\leq$" in $B$ by $x \leq y$ if $y - x \in P$. We write $x < y$ to denote $x \leq y$ but $x \neq y$ and $x \ll y$ to denote $y - x \in P^0$, where $P^0$ stands for the interior of $P$.

Proposition 2.2[8]: Let $P$ be a cone in a real Banach space $B$. If $a \in P$ and $a \leq ka$, for some $k \in [0,1)$ then $a = 0$.

Proposition 2.3[8]: Let $P$ be a cone in a real Banach space $B$. If for $a \in B$ and $a \ll c$, for all $c \in P^0$, then $a = 0$.

Remark 2.4[6]: $\lambda P^0 \subseteq P^0$, for $\lambda > 0$ and $P^0 + P^0 \subseteq P^0$.

Definition 2.5[5]: Let $X$ be a nonempty set. Suppose the mapping $d : X \times X \rightarrow B$ satisfies:

(a) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space (CMS).
Example 2.6([3][10]): Let $B = \mathbb{R}^3$, $P = \{(x,y,z) \in B : x,y,z \geq 0\}$ and $X = \mathbb{R}$. Define $d : X \times X \to B$ by $d(x,y) = (\alpha |x-y|, \beta |x-y|, \gamma |x-y|)$ where $\alpha, \beta, \gamma$ are positive constants. Then $(X,d)$ is a cone metric space.

Definition 2.7[5]: Let $(X,d)$ be a cone metric space. Let $\{x_n\}$ be a sequence in $X$ and $x \in X$. If for every $c \in B$ with $0 \ll c$ there is a positive integer $N$ such that for all $n > N$, $d(x_n,x) \ll c$, then the sequence $\{x_n\}$ is said to converge to $x$ and $x$ is called limit of $\{x_n\}$. We write $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 2.8[5]: Let $(X,d)$ be a cone metric space. Let $\{x_n\}$ be a sequence in $X$. If for any $c \in B$ with $0 \ll c$ there is a $N$ such that for all $n, m > N$, $d(x_n,x_m) \ll c$, then the sequence $\{x_n\}$ is said to be a Cauchy sequence in $X$.

Definition 2.9[5]: Let $(X,d)$ be a cone metric space. If every Cauchy sequence in $X$ is convergent in $X$, then $X$ is called a complete cone metric space.

Proposition 2.10[8]: Let $(X,d)$ be a cone metric space and $P$ be a cone in a real Banach space $B$. If $u \leq v$, $v \ll w$ then $u \ll w$.

Lemma 2.11[8]: Let $(X,d)$ be a cone metric space and $P$ be a cone in a real Banach space $B$ and $k_1, k_2, k_3, k_4, k > 0$. If $x_n \to x$, $y_n \to y$, $z_n \to z$, and $p_n \to p$ in $X$ and

(i) $ka \leq k_1 d(x_n,x) + k_2 d(y_n,y) + k_3 d(z_n,z) + k_4 d(p_n,p)$ then $a = 0$.

3 Main Result

Theorem 3.1. Let $(X,d)$ be a complete cone metric space with respect to a cone $P$ contained in a real Banach space $B$. Let $T$ be a surjective self map of $X$ satisfying.

$$d(Tx,Ty)+k[d(x,Ty)+d(y,Tx)] \geq ad(x,Tx)+bd(y,Ty)+cd(x,y) \quad \ldots (3.1.1)$$

for each $x, y \in X$, $x \neq y$ where $a, b, c, k \geq 0$, $a + b + c > 1 + 2k$, $b + c > k$ and $c > 2k$. Then $T$ has a unique fixed point.

Proof: Let $x_0$ be an arbitrary point in $X$, there is $x_1$ in $X$ such that $T(x_1) = x_0$. In this way we define a sequence $\{x_n\}$ as follows.

$$x_n = Tx_{n+1} \text{ for } n=0,1,2, \ldots \ldots \ldots \quad (3.1.2)$$
If \( x_n = x_{n+1} \) for some \( n \) then we see that \( x_n \) is a fixed point of \( T \), therefore we suppose that no two consecutive terms of sequence \( \{x_n\} \) are equal. We claim that the inequality (3.1.1) for \( x = x_{n+1} \) and \( y = x_{n+2} \) implies that

\[
d(Tx_{n+1}, Tx_{n+2}) + k [d(x_{n+1}, Tx_{n+2}) + d(x_{n+2}, Tx_{n+1})] \\
\geq ad(x_{n+1}, Tx_{n+1}) + bd(x_{n+2}, Tx_{n+2}) + c d(x_{n+1}, x_{n+2}) \\
\Rightarrow d(x_n, x_{n+1}) + k [d(x_{n+1}, x_{n+1}) + d(x_{n+2}, x_{n})] \geq ad(x_{n+1}, x_n) + bd(x_{n+2}, x_{n+1}) + cd(x_{n+1}, x_{n+2}) \\
\Rightarrow (1 + k - a)d(x_{n+1}, x_n) \geq (b + c - k)d(x_{n+1}, x_{n+2}) \\
\Rightarrow d(x_{n+1}, x_{n+2}) \leq \left( \frac{1+k-a}{b+c-k} \right) d(x_n, x_{n+1}) \\
\Rightarrow d(x_{n+1}, x_{n+2}) \leq K d(x_n, x_{n+1})
\]

Where \( K = \left[ \frac{1+k-a}{b+c-k} \right] < 1 \) \((As \ a + b + c > 1 + 2k)\)

In general

\[
\Rightarrow d(x_n, x_{n+1}) \leq K d(x_{n-1}, x_n) \\
\Rightarrow d(x_n, x_{n+1}) \leq K^n d(x_0, x_1) \ldots \ldots . (3.1.3)
\]

Now we shall prove that \( \{x_n\} \) is a Cauchy sequence. For this for every positive integer \( p \), we have

\[
d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \\
\leq (K^n + K^{n+1} + K^{n+2} + \ldots + K^{n+p-1}) d(x_0, x_1) \\
= K^n(1 + K + K^2 + \ldots + K^{p-1}) d(x_0, x_1) \\
< \frac{K^n}{(1-K)} d(x_0, x_1)
\]

Therefore \( \{x_n\} \) is a Cauchy sequence in \( X \), which is complete space, so \( \{x_n\} \rightarrow x \in X \).
Existence of fixed point: Since $T$ is a surjective map. So there exist a point $y$ in $X$, such that $x = Ty$ ........ (3.1.4)

Consider

$$d(x_n, x) = d(Tx_{n+1}, Ty)$$

$$\geq -k \left[ d(x_{n+1}, Ty) + d(y, Tx_{n+1}) \right] + a \ d(x_{n+1}, Tx_{n+1}) + b \ d(y, Ty) + c \ d(x_{n+1}, y)$$

$$\Rightarrow d(x, x) \geq -k \left[ d(x, x) + d(y, x) \right] + a \ d(x, x) + b \ d(y, x) + c \ d(x, y)$$

$$\Rightarrow 0 \geq (b + c - k) d(x, y)$$

$$\Rightarrow d(x, y) = 0 \ [As \ b + c - k > 0 \ and \ by \ Lemma \ 2.10]$$

$$\Rightarrow x = y \ ........(3.1.5)$$

The fact (3.1.4) along with (3.1.5) shows that $x$ is a fixed point of $T$.

Uniqueness Let $z$ be another fixed point of $T$, that is $Tz = z$

$$d(x, z) = d(Tx, Tz)$$

$$\geq -k[d(x, Tz) + d(z, Tx)] + ad(x, Tx) + bd(z, Tz) + cd(x, z)$$

$$= -k[d(x, z) + d(z, x)] + cd(x, z)$$

$$= (-2k + c)d(x, z)$$

$$\Rightarrow d(x, z) \leq \frac{1}{(c - 2k)} d(x, z)$$

$$\Rightarrow d(x, z) = 0 \ [As \ c > 2k \ and \ by \ Prop \ 2.2]$$

$$\Rightarrow x = z \ ............(3.1.6)$$

This completes the proof of the theorems 3.1.

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