

On $\mathcal{N} = 1$ Mirror Symmetry for Open Type II Strings

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Abstract

We study the open string extension of the mirror map for $\mathcal{N} = 1$ supersymmetric type II vacua with D-branes on non-compact Calabi-Yau manifolds. Its definition is given in terms of a system of differential equations that annihilate certain period and chain integrals. The solutions describe the flat coordinates on the $\mathcal{N} = 1$ parameter space, and the exact, disc instanton corrected superpotential on the D-brane world-volume. A gauged linear sigma model for the combined open-closed string system is also given. It allows to use methods of toric geometry to describe D-brane phase transitions and the $\mathcal{N} = 1$ Kähler cone. Applications to a variety of D-brane geometries are described in some detail.

1. Introduction

Closed string mirror symmetry [1] has been very effective in exactly computing instanton effects in $\mathcal{N} = 2$ supersymmetric type II strings. An important aspect is that certain holomorphic quantities in the physical type II string theory are computed by the topologically twisted theory [2], which provides a simplified structure essential for the application of mirror symmetry.

There is also an open string version of the topologically twisted string [3], which computes amplitudes of physical type II open-closed string systems with $\mathcal{N} = 1$ supersymmetry in four dimensions [4]-[6]. Specifically, one version of it, the A-model, is related to the type IIA string compactified on a Calabi–Yau (CY) 3-fold Y^* , with D6-branes partially wrapping 3-cycles in Y^* and filling space-time. Mirror symmetry relates the A-model to the B-model, which computes amplitudes of the type IIB string with odd-dimensional branes wrapping holomorphic sub-manifolds in the mirror manifold Y of Y^* . An extension of mirror symmetry that relates the topological amplitudes of a mirror pair of D-brane geometries has been pioneered [7,8] and leads to an expression for the exact instanton corrected superpotential for a class of non-compact D6-branes [8]. See also [9]-[17] for other papers on this subject.

Based on the developments of [8], a definition of $\mathcal{N} = 1$ open string mirror symmetry very similar to that for closed strings has been given in [13]. A system of differential equations has been derived, whose solutions around a limit point of maximal unipotent monodromy describe the mirror map for the flat coordinates of the combined open-closed string moduli space¹. The remaining solutions represent the exact instanton sum for the non-perturbatively generated $\mathcal{N} = 1$ superpotential. It was also argued that this differential structure describes the restricted geometry of the holomorphic F-terms, referred to as $\mathcal{N} = 1$ special geometry in the following. The wording reflects the fact that this geometry is a close relative of $\mathcal{N} = 2$ special geometry, however defined by a set of *several* independent holomorphic functions, as opposed to the single holomorphic prepotential for $\mathcal{N} = 2$.

In this note we develop these ideas further and generalize them in various ways. In particular the arguments in [13] have been based on a new duality between the open-closed type II strings in four dimensions and a closed string background without D-branes in two dimensions. Instead we derive in section 2 the differential system for

¹ We will loosely refer to the complex scalar manifold of vev’s of the massless chiral $\mathcal{N} = 1$ multiplets as the moduli space, although there is a superpotential for them which will fix some of these vev’s. The notation “moduli” is partially justified by the fact that in the appropriate regime the perturbative superpotential is zero.

the holomorphic $\mathcal{N} = 1$ amplitudes directly in the B-model for the four-dimensional open-closed string theory. Although the open/closed string duality is an interesting subject in itself and seems to be quite a general phenomenon, our derivation here provides the alluded to differential system for $\mathcal{N} = 1$ mirror symmetry independently of the existence of a closed string dual. We proceed with a study of the general properties of the solutions. In section 3 we describe the construction of a gauged linear sigma model (LSM) for the A-model that represents the mirror D-brane geometry. We use the techniques of toric geometry to define the Kähler cones of the $\mathcal{N} = 1$ moduli space and study phase transitions between different classical vacua. In section 4 we discuss some aspects of $\mathcal{N} = 1$ special geometry and the so-called framing ambiguity. In the Appendix we included detailed computations for D-branes on a collection of non-compact CY 3-folds.

2. B-model: Picard-Fuchs equations for open strings

Our first aim will be to study the $\mathcal{N} = 1$ moduli space of the B-model with D-branes and to derive a system of differential equations for it. The solutions of this generalized Picard-Fuchs (PF) system describes the flat coordinates on this space, as well as the holomorphic topological correlation functions on it. The differential system will agree with that derived in [13] for the cases that have closed string duals. A differential constraint on the superpotential, which is however not equivalent to the PF system described below, has recently been proposed in [14].

2.1. Generalized Picard-Fuchs equations for open strings

The ordinary Picard-Fuchs equation is a differential equation that expresses the linear dependence of $k + 1$ p -forms in the cohomology group $H^p(Y, \mathbf{C})$ of dimension $h^p(Y) = k$. E.g. for a CY 3-fold Y with² $h^{1,2}(Y) = 1$, there is a differential equation for the holomorphic 3-form $\Omega(z)$

$$\mathcal{D}\Omega(z) = \sum_{i=0}^4 f_i(z) \frac{d^i}{dz^i} \Omega(z) = d\eta(z).$$

Here z denotes the single complex structure modulus. The above equation expresses the fact that the sum of the five 3-forms $\frac{d^i}{dz^i} \Omega$ for $i = 0, \dots, 4$ must be linearly dependent in the cohomology group $H^3(Y, \mathbf{C})$ and thus is proportional to an exact form $d\eta$. It

² $h^{p,q}(Y) = \dim H^{p,q}(Y)$.

follows that the differential operator \mathcal{D} annihilates the period integrals, $\mathcal{D} \int_{\gamma} \Omega = 0$ where γ is a topological (z -independent) 3-cycle in $H_3(Y, \mathbf{Z})$.

The open-string sector of the B-model consists of D5-branes wrapped on a 2-cycle C in Y . The superpotential for the D5-brane on the curve C is [8,18]

$$W(z_0) = \int_{\Gamma(z_0)} \Omega = \int_{C(z_0) - C_*} \omega, \quad (2.1)$$

where Γ is a 3-chain with boundary $C - C_*$ and Ω may be locally written as $\Omega = d\omega$ on Γ . Moreover z_0 is an open string modulus that moves the position of C in Y and C_* is a fixed reference curve homologous to C . The Picard-Fuchs operator applied to the above chain integral gives a non-zero result from the boundary

$$\mathcal{D} \int_{\Gamma} \Omega(z) = \int_{\Gamma} d\eta(z) = \int_{\partial\Gamma} \eta(z). \quad (2.2)$$

In fact the non-vanishing term on the r.h.s. may be interpreted as a new observable in the open string sector for each boundary component $c_i \subset \partial\Gamma$. This is closely related to the fact that in 2d terms the BRST variation of a closed string bulk observable is zero up to a boundary term proportional to an open string observable.

Note that, contrary to the closed string periods, the chain integral (2.1) depends in particular on the choice of a representative for the cohomology class $\Omega \in H^3(X, \mathbf{C})$. In other words, *the exact piece of Ω is observable in the open string sector*. Specifically, the exact piece of Ω corresponds to a quantum number of the open-closed string vacuum which needs to be fixed to define the system. As will be discussed somewhere else it is integrally quantized and related to the framing ambiguity discussed in refs.[10,19].

The equation (2.2) suggests that an appropriate modification of the Picard-Fuchs operator \mathcal{D} may annihilate the non-vanishing boundary term. To describe the necessary generalization, we recall briefly the construction of the complex structure moduli space of Y . Concretely we study the topologically twisted B-model on a toric CY manifold Y defined as the vacuum geometry of a 2d linear sigma model [20] with superpotential

$$\mathcal{W}_{D=2} = \sum_{i=1}^{N+3} a_i \tilde{y}_i, \quad \prod_i \tilde{y}_i^{l_i^{(a)}} = 1, \quad a = 1, \dots, N. \quad (2.3)$$

Here the $N + 3$ variables \tilde{y}_i take value in \mathbf{C}^* and the a_i are constants that parametrize the complex structure. A concrete CY is specified by the integral charge vectors $l^{(a)}$ that define the set of $N = h^{1,2}(Y)$ relations on the r.h.s of (2.3). After solving the

relations for a choice of three coordinates $\tilde{y}_{i_1}, \tilde{y}_{i_2}, \tilde{y}_{i_3}$, one obtains the superpotential $\mathcal{W}_{D=2}(\tilde{y}_{i_1}, \tilde{y}_{i_2}, \tilde{y}_{i_3}; a_i)$. An equivalent definition [21] is in terms of the hypersurface

$$P = F(y_{i_1}, y_{i_2}; a_i) + xz = 0, \quad y_{i_k} = \frac{\tilde{y}_{i_k}}{\tilde{y}_{i_3}}, \quad (2.4)$$

where $F = \mathcal{W}/\tilde{y}_{i_3}$.

In the following we will often set the indices $\{i_n\}$ to the specific values $(1, 2, 3)$ to simplify notation. This choice of coordinates is appropriate to describe D-branes classically located in a patch where \tilde{y}_3 is “large”. D-branes in other patches may be described similarly after an appropriate relabeling of the coordinates. We will switch back to a global notation with general i_n where it is useful.

The naive moduli space \mathcal{M}_0 is

$$\mathcal{M}_0 = (\mathbf{C}^*)^M / (\mathbf{C}^*)^m,$$

where $(\mathbf{C}^*)^M$ is parametrized by the a_i and the quotient by $(\mathbf{C}^*)^m$ is generated by the reparametrizations

$$\tilde{y}_i \rightarrow \lambda_i \tilde{y}_i, \quad \lambda_i \in \mathbf{C}^*, \quad (2.5)$$

of the independent coordinates \tilde{y}_i . In the present case, $M = N + 3$ and $m = 3$. The true moduli space is obtained from \mathcal{M}_0 by an appropriate compactification that adds some limit points and subsequently removing the discriminant locus where the surface Y is singular [22]. The first step is accomplished by passing to the $(\mathbf{C}^*)^m$ invariant coordinates

$$z_a = \prod_i a_i^{l_i^{(a)}}, \quad a = 1, \dots, N, \quad (2.6)$$

which provide good local coordinates for the complex structure moduli space M of Y in a neighborhood of $z_a = 0$.

The complex structure of Y is locally also parametrized by its periods $\int_{\gamma_\alpha} \Omega$, where $\gamma_\alpha \in H_3(Y, \mathbf{Z})$. The periods satisfy a system of differential equations of generalized hyper-geometric, so-called GKZ type [23]. It is defined by two sets of differential operators. The first set is of the form

$$\tilde{\mathcal{D}}_j = \sum_i \nu_{i,j} a_i \frac{\partial}{\partial a_i} - \beta_j, \quad (2.7)$$

where β_j is a vector of exponents which is identically zero in the non-compact case. Moreover the ν_i are $N + 3$ vertices of the polyhedron Δ defining the toric variety Y ; its construction will be described in the next section. For the moment it suffices to

know that the differential operators $\tilde{\mathcal{D}}_j$ express the invariance of the period integrals under the \mathbf{C}^* scalings (2.5). One may therefore solve the equations (2.7) by using the \mathbf{C}^* scalings to write the 3-form $\Omega(a_i)$ as a function $\Omega(z_a)$ of the z_a , only.³

On the surface $P = 0$, the 3-form Ω is given by the residuum formula

$$\Omega = \frac{dy_1}{y_1} \frac{dy_2}{y_2} \frac{dx dz}{P} \xrightarrow{res} \frac{dy_1}{y_1} \frac{dy_2}{y_2} \frac{dz}{z}. \quad (2.8)$$

From this explicit form it is easy to see that Ω is annihilated by a second set of differential operators of the form

$$\mathcal{D}_a = \prod_{l_i^{(a)} > 0} \left(\frac{\partial}{\partial a_i} \right)^{l_i^{(a)}} - \prod_{l_i^{(a)} < 0} \left(\frac{\partial}{\partial a_i} \right)^{-l_i^{(a)}}. \quad (2.9)$$

These operators reflect the relations between the monomials \tilde{y}_i in (2.3). They also annihilate the period integrals of Ω on a basis of topological cycles γ_a spanning $H_3(Y, \mathbf{Z})$.

To describe the open string modifications to the above picture we consider from now on a specific class of D5-branes on a non-compact 2-cycle C as in [8]. These are mirror to D6-branes on the mirror manifold Y^* with a non-zero, however entirely non-perturbative superpotential W .⁴ Specifically, the curve C is defined by the equations

$$x = F = 0, \quad y_i = y_i(z), \quad y_i(\infty) = y_i^*. \quad (2.10)$$

The reference curve C_* may be chosen to be the holomorphic cycle $y_i(z) = y_i^*$. The rational equivalence class of C is parametrized holomorphically by the quantity

$$y_1(0) = z_0, \quad (2.11)$$

which measures the deformation from the configuration C_* . Note that the value of y_1 fixes the value of y_2 by the constraint $F = 0$. With an appropriate labeling of coordinates, any family of 2-cycles C may be written as (2.11); alternatively one may write the general boundary condition as

$$\tilde{y}_i(0) = z_0 \tilde{y}_j(0), \quad (2.12)$$

which will be also be needed below and reduces to (2.11) for the coordinates $(i, j) = (1, 3)$ used in the definition (2.4).

³ In the compact case there is also a normalization factor corresponding to non-zero exponents β_j .

⁴ Supersymmetric D-branes and their mirrors have been described in [24,21,8,25].

An important fact is that the open string sector – the location of the D-brane on the curve C – breaks the \mathbf{C}^* scaling symmetry (2.5) of the closed string sector. Specifically, the \mathbf{C}^* symmetry $y_1 \rightarrow \lambda y_1$ does not leave invariant the boundary condition (2.11). Instead one may use this \mathbf{C}^* action to move the moduli dependence from the integration contour to the integrand

$$\int_{\Gamma(z_0)} \Omega(z_a) = \int_{\Gamma} \Omega(z_a; z_0). \quad (2.13)$$

Here $\Gamma = \Gamma(1)$ is a *topological* 3-chain⁵ independent of the open string modulus. Moreover $\Omega(z_a, z_0)$ is obtained from $\Omega(z_a)$ by rescaling of y_1 , which replaces $P(y_1, y_2)$ with $P(z_0 y_1, y_2)$ in the definition (2.8). The 3-form $\Omega(z_a; z_0)$ and its integrals over topological 3-cycles *and* the 3-chain Γ is annihilated by the GKZ operators \mathcal{D}_a , rewritten in terms of z_a and the new modulus z_0 .

The same integrals are in addition annihilated by a further “boundary differential operator” \mathcal{D}_0 , as we will show now. The combined system of differential operators $\{D_0, D_a\}$ will be complete in the sense that the integrals $\int_{\gamma_a} \Omega$ and $\int_{\Gamma} \Omega$ provide a basis of solutions.

To derive the additional differential operator satisfied by the relevant integrals of Ω , consider a hyperplane $H \in Y$ that intersects $\partial\Gamma$ in the point (2.11), or more generally (2.12)

$$H : z_0^{-1} \prod_i \tilde{y}_i^{l_i^{(0)}} = 1, \quad l^{(0)} = (0, \dots, \underset{i\text{-th}}{1}, \dots, \underset{j\text{-th position}}{-1}, \dots, 0). \quad (2.14)$$

We define a 2-form Ω_0 by $\Omega = \Omega_0 \frac{dy_1}{y_1}$. The restriction of Ω_0 to H is annihilated by the differential operator

$$\mathcal{L} \Omega_0(z_a, z_0)|_H = 0 \quad \mathcal{L} = \partial_{a_i} - \partial_{a_j}.$$

Note that the origin of \mathcal{L} is very similar to that of the operators D_a in (2.9); in fact the differential operators $\{\mathcal{D}_a, \mathcal{L}\}$ correspond to the GKZ operators (2.9) derived from the “boundary superpotential”

$$\mathcal{W}_{\text{bound}} = \sum_{i=1}^{N+3} a_i \tilde{y}_i, \quad \prod_i \tilde{y}_i^{l_i^{(\alpha)}} = 1, \quad \alpha = 0, \dots, N, \quad (2.15)$$

⁵ A rigorous mathematical definition exists.

which depends, as compared to the closed string ‘‘bulk’’ superpotential (2.3), also on the additional D-brane modulus $z_0 = \prod a_i^{l_i^{(0)}}$ ⁶.

The operator \mathcal{L} does neither annihilate the periods nor the chain integrals. To derive a differential operator \mathcal{D}_0 from it that does so, consider an infinitesimal variation of the chain integral under a shift $z_0 \rightarrow z_0 + \delta z_0$

$$\begin{aligned} \int_{\Gamma(z_0+\delta z_0)} \Omega(z_a) - \int_{\Gamma(z_0)} \Omega(z_a) &= \int_{z_0}^{z_0+\delta z_0} \left(\int \Omega_0(z_a) \right) \frac{dy_1}{y_1} = \\ \int_1^{1+\delta z_0/z_0} \left(\int \Omega_0(z_a; z_0) \right) \frac{dy_1}{y_1} &= \left(\int \Omega_0(z_a; z_0) \right) |_H \cdot \delta z_0/z_0. \end{aligned}$$

Thus the searched for differential operator is

$$\mathcal{D}_0 \int_{\Gamma(z_0)} \Omega(z_a) = 0, \quad \mathcal{D}_0 = \mathcal{L} z_0 \frac{d}{dz_0},$$

which annihilates trivially also the period integrals $\int_{\gamma_a} \Omega$ since they do not depend on z_0 . The above derivation may also be generalized to yield a complete system for several independent chain integrals parametrized by several open string moduli z_0^μ , $\mu = 1, \dots, l$.

It remains to rewrite the system of differential operators $\{\mathcal{D}_\alpha\} = \{\mathcal{D}_a, \mathcal{D}_0\}$ in terms of derivatives of the good local coordinates $(z_a; z_0)$. The result can be given in a simple closed form as follows. Define an extended set of charge vectors $l^{(\alpha)}$, $\alpha = 0, \dots, N = h^{1,2}$, with

$$\begin{aligned} \tilde{l}^{(a)} &= (\quad \quad \quad l^{(a)}(Y) \quad \quad \quad ; 0 \quad 0), \quad a = 1, \dots, N, \\ \tilde{l}^{(0)} &= (0, \quad \dots, 0, 1, 0, \dots \quad \dots, 0, -1, 0, \dots \quad \dots, 0; 1 \quad -1). \\ &\quad \quad \quad i\text{-th position} \quad \quad \quad j\text{-th position} \end{aligned} \tag{2.16}$$

With these definitions, the extended system of differential operators can be written in the closed form

$$\mathcal{D}_\alpha = \prod_{\tilde{l}_i^{(\alpha)} > 0} \prod_{j=0}^{\tilde{l}_i^{(\alpha)} - 1} \left(\sum_{\beta} \tilde{l}_i^{(\beta)} \theta_\beta - j \right) - z_\alpha \prod_{\tilde{l}_i^{(\alpha)} < 0} \prod_{j=0}^{-\tilde{l}_i^{(\alpha)} - 1} \left(\sum_{\beta} \tilde{l}_i^{(\beta)} \theta_\beta - j \right). \tag{2.17}$$

A system of differential equations of the type (2.16), (2.17) has been derived in [13] using a duality of the open-closed string background on Y to a closed string background without branes on a CY 4-fold. Here we have obtained the differential equations

⁶ A CFT interpretation of \mathcal{W}_{bound} exists, as will be discussed elsewhere.

that govern the open-closed holomorphic data purely from open string arguments. In particular the above line of arguments is applicable no matter whether a closed string dual exists. Note that a given set of charge vectors $\tilde{l}^{(a)}$ is in general not equivalent to specifying a toric manifold X . It would be interesting to know under which conditions the open-closed string data define a toric manifold for a closed string background.

2.2. Open string mirror map and $N = 1$ superpotentials

We proceed with a discussion of some general properties of the solutions to the open-closed string GKZ system (2.17). As for notation, we drop the tildes on $\tilde{l}^{(\alpha)}$ and take the following convention in this section⁷:

$$\begin{aligned} l^{(a)} &= (l^{(a)}(Y) \quad ; 0, 0), \quad a = 1, \dots, N = h^{1,2}, \\ l^{(0)} &= (1, -1, 0, \dots; 1, -1). \end{aligned} \tag{2.18}$$

The first entry in each row corresponds to the non-compact direction and is labelled by $l_0^{(\alpha)}$.

For the above choice of $l^{(\alpha)}$, the algebraic coordinates (2.6) will be good coordinates near the point $z_\alpha = 0$ of so-called maximally unipotent monodromy. It is quite non-trivial that this concept generalizes to include the open string moduli and this is in fact a consequence of the underlying $\mathcal{N} = 1$ special geometry⁸. A concrete justification can be given by constructing the appropriate gauged linear sigma model for the A-model on the mirror manifold, as will be done in the next section.

The behavior of the solutions ω_i near the limit point $z_\alpha = 0$ is of the form $\omega_i \sim (1; \ln(z); \ln(z)^2, \dots)$, reflecting the distinct monodromy of the solution vector ω_i on a circle around $z_\alpha = 0$. The solutions may be defined in terms of the power series $\omega = \sum_{n_\alpha} c(n_\alpha; \rho_\alpha) \prod_\alpha z_\alpha^{n_\alpha + \rho_\alpha}$, with

$$c(n_\alpha; \rho_\alpha) = \frac{1}{\prod_i \Gamma(1 + \sum_\alpha l_i^{(a)}(n_\alpha + \rho_\alpha))},$$

and derivatives of ω with respect to the indices ρ_a . The series solution $\omega|_{\rho_\alpha=0}$ is in fact a constant as a consequence of the non-compactness of the 3-fold. The solutions

⁷ In general there will be several patches in the moduli that include different large complex structure limit points. For the present section, the notation is adapted to an “outer” phase as defined in the next section.

⁸ See [26][27] for the selection of the limit point in the closed string case.

$\frac{\partial}{\partial \rho_\alpha} \omega|_{\rho_\alpha=0}$ with single logarithmic behavior define the flat coordinates of the open-closed topological string theory [13]. They have the form

$$t_\alpha(z_\beta) = \frac{1}{2\pi i} \log(z_\alpha) + S_\alpha(z_\beta) , \quad (2.19)$$

where t_α, t_0 are the closed and open string flat coordinates. A general property is that the series

$$S_\alpha(z_b) = \sum_{n_b \geq 0} c_\alpha(n_b) z_1^{n_1} \dots z_N^{n_N} \quad (2.20)$$

depend only on the bulk moduli and expressly not on the boundary modulus z_0 [10,13]. The coefficients can be written as

$$c_a(n_b) = (-)^{m_k+1} l_k^{(a)} (m_k - 1)! \left(\prod_{i \neq k} \left(\sum_b l_i^{(b)} n_b \right)! \right)^{-1} , \quad (2.21)$$

where m_k is such that $\sum_b l_i^{(b)} n_b \equiv -m_k < 0$. It can be shown that $c_a(n_b)$ is non-zero only when there exists a single such positive m_k . Moreover, the series for the D-brane modulus mirror map is a simple linear combination of the bulk moduli series [13]:

$$S_0(z_a) = \sum (A^{-1})_b^a A_0^a S_b(z_a) , \quad (2.22)$$

where A is the integer coefficient matrix characterizing the linear part of the PF system, ie., $\mathcal{D}_a = z_a \sum A_b^a \theta_b + \dots$ (when A is degenerate one needs to restrict to linearly independent $S_a(z_b)$).

The superpotential corresponds to the z_0 dependent double-logarithmic solution corresponding to a double derivative of ω w.r.t. the indices. More precisely, it is obtained by dropping the logarithmic pieces, because the classical superpotential for the D-brane is zero by construction. We find:

$$W(z_a, z_0) = \sum_{\substack{n_a \geq 0, n_0 > 0 \\ n_0 - \sum l_1^{(a)} n_a > 0}} \frac{(-)^{\sum l_1^{(a)} n_a} (n_0 - \sum l_1^{(a)} n_a - 1)!}{n_0 (n_0 + \sum l_0^{(a)} n_a)! \prod_{i=2}^{N+2} (\sum l_i^{(a)} n_a)!} z_0^{n_0} \prod_a z_a^{n_a} . \quad (2.23)$$

This result was obtained by applying the Frobenius method (as nicely explained in [28]) and noting that there can be a contribution if and only if two m_k are positive. Moreover it turns out that these must be m_1 and m_{N+4} in order to give actual solutions of the Picard-Fuchs system (2.17). Note that the purely z_0 dependent terms give the dilogarithm function,

$$W(0, z_0) = \sum_{n_0 > 0} \frac{1}{n_0^2} z_0^{n_0} \equiv \text{Li}_2(z_0) , \quad (2.24)$$

which is consistent in that the large radius limit reproduces the known result for \mathbf{C}^3 [8].

Inserting the inverse mirror map $z_\alpha(t_\beta)$ (2.19) into $W(z_\alpha)$ one obtains the expansion of the superpotential in terms of the exponentiated flat coordinates $q_\alpha \equiv e^{2\pi i t_\alpha}$. It is predicted to be of the form [5]:

$$W(q_a, q_0) = \sum_{n_a, n_0} N_{n_a, n_0} \text{Li}_2(q_0^{n_0} \prod_a q_a^{n_a}). \quad (2.25)$$

Here the *integral* coefficients N_{n_a, n_0} count the numbers⁹ of disc instantons of the corresponding degrees.

We can improve the formula (2.23) by noting that it can be conveniently summed over z_0 . Defining $\xi(n) = \max(\sum l_1^{(a)} n_a, 0)$, we can rewrite the superpotential as

$$\begin{aligned} W(z_a, z_0) &= \sum_{n_a \geq 0} \frac{(-)^{\sum l_1^{(a)} n_a} (\xi(n) - \sum l_1^{(a)} n_a)!}{(1 + \xi(n)) \Gamma(2 + \xi(n) + \sum l_0^{(a)} n_a) \prod_{i=2}^{N+2} (\sum l_i^{(a)} n_a)!} \\ &\times {}_3F_2 \left(1, \quad 1 + \xi(n), \quad 1 + \xi(n) - \sum l_1^{(a)} n_a; \quad z_0 \right) z_0^{1+\xi(n)} \prod_a z_a^{n_a}. \end{aligned} \quad (2.26)$$

This form facilitates analytic continuation in the boundary modulus z_0 , and in particular allows to determine the n_0 dependence of the instanton coefficients in a closed form. This is because the hyper-geometric function ${}_3F_2$ in (2.26) has degenerate arguments so that it takes the following generic functional form: $c \log(1 - z_0) + (1 - z_0)^{-d} Q(z_0)$, where c, d are constants and $Q(z_0)$ is a *finite* polynomial. That this polynomial is finite translates to the property of the generating function:

$$N_{n_a}(x) \equiv \sum_{n_0} N_{n_a, n_0} x^{n_0} \quad (2.27)$$

to be a ratio of two finite polynomials in x ; examples for such generating functions are presented in Appendix A. Moreover note that upon taking a derivative, (2.26) can be further condensed because the ordinary hyper-geometric function on the r.h.s. of

$$z_0 \partial_{z_0} \left[z_0^{1+\xi(n)} {}_3F_2(\dots; z_0) \right] = \frac{z_0^{1+\xi(n)}}{\xi(n)!} {}_2F_1 \left(1, 1 + \xi(n) - \sum l_1^{(a)} n_a, 2 + \xi(n) + \sum l_0^{(a)} n_a; z_0 \right) \quad (2.28)$$

⁹ The definition of the integers N_{n_a, n_0} is motivated by physics and based on appropriately counting the numbers of D4-brane domain walls wrapping the discs [8][5].

has a very simple structure. For example, for $\sum l_1^{(a)} n_a = \xi > 0$ and $\sum l_0^{(a)} n_a + \sum l_1^{(a)} n_a \equiv -k \leq 0$, it reduces to

$$\frac{z_0}{\Gamma(2-k)} {}_2F_1(1, 1, 2-k; z_0) = \begin{cases} -\log(1-z_0), & k=0 \\ (k-1)! \left(\frac{z_0}{1-z_0}\right)^k, & k \geq 1. \end{cases} \quad (2.29)$$

These simple expressions lead to a firm control over the moduli sub-space spanned by z_0 .

3. A-model: a gauged LSM for open strings and topology changing phase transitions

As promised in [13] and the previous section, we complete in the following the toric description of the D-brane geometry by constructing a gauged LSM for it. It describes the A-model for the mirror D6-branes wrapped on special Lagrangian (sL) 3-cycles on the mirror manifold Y^* . This will in particular be useful to define the phases of the $\mathcal{N} = 1$ moduli space and the Kähler cones for it. We will also consider D-brane phase transitions which change the topology of the gauged LSM. The LSM can be interpreted as describing a CY 4-fold X^* for a closed string compactification mirror to those considered in [13]. Alternatively there could be an interpretation in terms of a “boundary linear sigma model”. This point of view has been pursued in [14], although with results different from ours¹⁰.

The connection between this gauged LSM and the D-branes on Y^* has been demonstrated in [13] via an M-theory lift. A crucial ingredient has been the proposal, based on a consistency argument, that a particular $U(1)$ orbit of the gauged LSM serves as an M-theory fiber. Specifically, the D-term equations for the $U(1)$ gauge symmetry of the LSM are

$$\sum_i l_i^{(\alpha)} |x_i|^2 = r_\alpha, \quad l_i^{(\alpha)} \in \mathbf{Z}. \quad (3.1)$$

The proposal says that the corresponding $U(1)$ orbit

$$x_i \rightarrow e^{i \sum l_i^{(\alpha)} \theta_i} x_i$$

serves as the fiber of the M-theory lift for the theory with flux through the 2-cycle dual to the charge vector $l^{(\alpha)}$. This ansatz may now be understood as a consequence of the world-volume analysis of ref.[17].

¹⁰ The LSM considered in [14] does not obviously reproduce the relevant PF system (2.17).

3.1. D-branes on the mirror Y^* and a gauged LSM for them

The starting point will be the gauged LSM for the closed string background, namely the CY 3-fold Y^* mirror to Y . The manifold Y^* is the total space of the canonical bundle $K(S)$ over a compact toric variety S with $c_1 > 0$. For simplicity we will assume $\dim_{\mathbf{C}}(S) = 2$ so that there is one non-compact direction; the necessary modifications for $\dim_{\mathbf{C}}(S) = 1$ are straightforward. The gauged LSM describes the Calabi–Yau Y^* as the vacuum of a 2d (2,2) supersymmetric gauge theory with gauge group $U(1)^N$, $N + 3 = h^{1,1}(Y^*) + 3$ matter multiplets x_i with charges $l^{(a)}$ under the a -th $U(1)$ factor and zero superpotential [20]. The vacuum is the space of solutions of the D-term equations (3.1) divided by the $U(1)^N$ gauge symmetry. The constants r_a are the FI parameters which combine with the integrals of the B-fields into the complex scalar fields t_a . Alternatively, Y^* is the symplectic quotient

$$\Xi \backslash \mathbf{C}^{N+3} / (\mathbf{C}^*)^N, \quad (\mathbf{C}^*)^N : x_i \rightarrow \lambda^{l_i^{(a)}} x_i, \quad \lambda \in \mathbf{C}^*,$$

where x_i are coordinates on \mathbf{C}^{N+3} and Ξ is the fixed point set of the $(\mathbf{C}^*)^N$ action [20][29]. The toric data for this quotient are summarized in the toric polyhedron $\Delta \in \mathbf{Z}^3$, spanned by the $N + 3$ vertices ν_i that satisfy the relation $\sum_i l_i^{(a)} \nu_i = 0$.¹¹

The D-branes on sL 3-cycles on Y^* may be defined classically by the equations [8]

$$\sum_i q_i^\alpha |x_i|^2 = c_\alpha, \quad \theta^i = q_i^\alpha \phi_\alpha, \quad \alpha = 1, \dots, 3 - k,$$

where $x_i = |x_i| e^{i\theta_i}$, the c_α are complex constants and $q_i^\alpha \in \mathbf{Z}$ are integers that satisfy $\sum_i q_i^\alpha = 0$. More precisely, L is a complete 3-manifold if it ends on an edge $|x_i|^2 - |x_j|^2 = 0$ of the image of the moment map $x_i \rightarrow |x_i|^2$; otherwise one has to add another component [8]. For $k = 2$ the topology of L is $\mathbf{R}_{\geq 0} \times S^1 \times S^1$. Mirror symmetry on the sL T^3 fiber defined by the independent phases of the x_i removes the two S^1 's and adds a different S^1 , resulting in a non-compact 2-cycle C of topology $S^1 \times \mathbf{R}_{\geq 0}$ as in (2.10).

More explicitly, mirror symmetry predicts the relation $|y_i| = e^{-|x_i|^2}$ [30], and a sL cycle L mirror to the 2-cycle C (2.10) is defined by the equations

$$|x_1|^2 - |x_3|^2 = c_1, \quad |x_2|^2 - |x_3|^2 = c_2, \quad \sum_i \theta_i = \text{const.} \quad (3.2)$$

¹¹ As there is no superpotential in the A-model, the only moduli are the Kähler moduli of Y^* and the equivalent data of the complex structure of Y . Both spaces are encoded in the single polyhedron Δ .

Moreover the open string modulus z_0 in (2.11) is related to the above constants by

$$|z_0| = e^{-c_1}, \quad (3.3)$$

as follows from (2.12). For an appropriate choice of coordinate labels, these equations describe D-branes of the chosen topology in any patch of the CY manifold. It is assumed that this choice has been made such that $e^{-c_1} \sim |z_0| \leq 1$.

The toric geometry that encodes the LSM for Y^* and the above D-branes on it is defined by a polyhedron Δ_{bound} obtained from the polyhedron Δ for Y^* as follows. The new polyhedron $\Delta_{bound} \in \mathbf{Z}^4$ contains the vertices of the polyhedron Δ on a hyperplane H , say $\nu'_i = (\nu_i, 0)$, $i = 1, \dots, N + 3$. In addition Δ_{bound} has two extra vertices $\nu'_{N+4} = (\nu_i, 1)$ and $\nu'_{N+5} = (\nu_j, 1)$ above H :

$$\Delta_{bound} = \text{convex hull} \left(\begin{array}{c} (\nu_1, 0) \\ \vdots \\ (\nu_{N+3}, 0) \\ (\nu_i, 1) \\ (\nu_j, 1) \end{array} \right). \quad (3.4)$$

As before, the integers (i, j) may be chosen general, and Δ_{bound} describes the mirror of the D-brane on (2.10) in a phase parametrized as in (2.4) for $(i, j) = (1, 3)$. In the above we have used the standard correspondence of toric geometry that assigns a specific vertex ν_i to the coordinate x_i (and a monomial \tilde{y}_i in the mirror Y) [29].

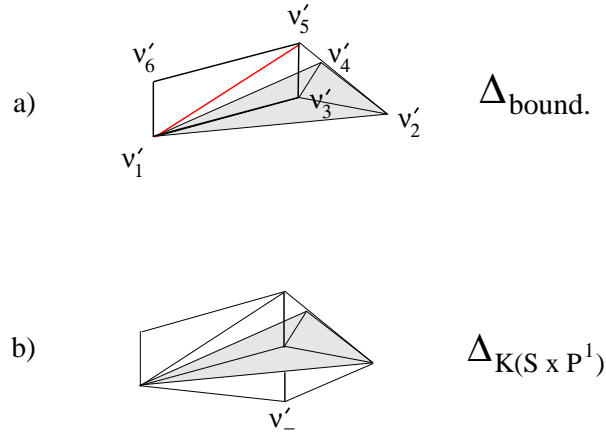


Fig. 1: a) The toric polyhedron Δ_{bound} that defines the LSM for the CY $Y = K(\mathbf{P}^2)$ with D-branes; b) The enlarged polyhedron Δ' that describes a fibration with base \mathbf{P}^1 .

In LSM terms we have added two new matter fields uncharged under the $U(1)^N$ gauge group and one new $U(1)$ gauge symmetry that acts as

$$(x_i, x_j, x_{N+4}, x_{N+5}) \rightarrow (\lambda x_i, \lambda^{-1} x_j, \lambda x_{N+4}, \lambda^{-1} x_{N+5}),$$

with the other fields invariant. To understand better the geometry defined by Δ_{bound} we may add a further vertex ν'_- below the hyperplane H to obtain an extended polyhedron $\Delta' \in \mathbf{Z}^4$. The toric geometry X' associated to Δ' is the canonical bundle of a fibration of S over \mathbf{P}^1 , with the position of the point ν'_- specifying the fibration. Moreover one of the two extra vertices above H represents a blow up of this fibration above a point p on the base \mathbf{P}^1 . Removing ν'_- again corresponds to taking the large volume limit of the base \mathbf{P}^1 of the fibration and concentrating on a neighborhood of the special point p isomorphic to \mathbf{C} .

In fact geometries of this type have already played a role in the context of F-theory and Abel-Jacobi maps in [31]. For instance, for $S = \mathbf{P}^2$ and choosing a non-trivial fibration, the hypersurface $f = 0$ in the toric variety X' defined by Δ' is an elliptically fibered K3 surface with Picard lattice of rank three. The two points ν'_k , $k > N + 3$ correspond to the blow up of a singular fiber of “ A_1 type” in the elliptic fibration¹². Finally removing ν'_- corresponds to the large base limit and concentrating on a neighborhood of the A_1 fiber in the fibration. The moduli of this local manifold is the same as that of a flat $SU(2)$ bundle on the torus [31], which in turn is the same as a point on the dual torus in which the Abel-Jacobi map takes value. In fact such a relation holds for any simply laced group G instead of A_1 , which may be realized [32] by $\text{rank}(G) + 1$ vertices above the hyperplane H .

A similar comment applies also for other choices for S . Note that the interpretation of the polyhedron Δ_{bound} as defining the ambient space for an embedded K3 surface implies that the solutions of the GKZ system (2.17), which describe the open string superpotential, are in fact related to the periods of K3 manifolds. This follows from the relation [21]

$$\Pi_{comp.} = \frac{\partial}{\partial t} \Pi_{non-comp.} \quad (3.5)$$

between the periods of Y^* and those of a compact K3 manifold embedded in it. It would be interesting to study the implications of this further.

3.2. Kähler cones, (D-brane) flops and existence of vacua

We will now use the LSM to study the Kähler cones of the $\mathcal{N} = 1$ moduli space and phase transitions between them. The moduli space of the gauged LSM will have in general several limit points of maximally unipotent monodromy which, in the present context, correspond to the classical vacua of the D-brane geometry. More precisely

¹² Specifically, the resolution replaces the singular fiber by two spheres intersecting according to the affine Dynkin diagram of A_1 .

the superpotential at these points is entirely non-perturbative. This is reflected in the relation $z_\alpha \sim e^{2\pi i t_\alpha}$ and the polynomial dependence (2.23) of the superpotential $W(z_\alpha)$ near these limit points.

Different limit points are distinguished by different choices for a basis $l^{(\alpha)}$ of the integral charge lattice. The correct basis is distinguished by the property that the moduli t_α measure the sizes of the fundamental holomorphic world-sheet maps of minimal volume. That is, the minimal volume of a holomorphic world-sheet in any homology class must be a *positive* linear combination of the t_a . The existing limit points and the associated basis for the $l^{(\alpha)}$ may be determined by studying the triangulations of the toric polyhedron Δ_{bound} [29]. A change between different bases amounts to taking integral linear combinations $l^{(\alpha)} \rightarrow M_{\alpha\beta} l^{(\beta)}$ and is accompanied by a redefinition of the flat coordinates $t_\alpha \rightarrow M_{\alpha\beta} t_\beta$.

In general there are two, physically very different types of classical phases for the D-branes on L in (3.2), which are reflected in two different types of triangulations of Δ_{bound} . In the first case, referred to as an *inner phase* in the following, the D-brane ends on the compact zero section $S : x_0 = 0$ of the Calabi–Yau Y^* . Note that the closed string degrees of freedom coming with Y^* are localized on S and thus the position of L relative to S determines the open-closed string interactions. This inner phase corresponds to a so-called star triangulation of Δ_{bound} , where all vertices are connected to the distinguished vertex ν_0 . In contrast, an *outer phase* is defined by the D-brane ending on a vertex $x_i = x_j = 0$, $i, j \neq 0$ away from the zero section. In this case one of the extra vertices representing the D-brane does not lie in a cone together with ν_0 .

As will be argued below, the different open-closed string interactions in these two types of phases lead to quite different vacuum structures. Whereas the D-brane in the outer phase may be treated as a perturbation of the CY background and has always a vacuum solution, the nearly classical D-brane in the inner phase destabilizes the CY geometry Y^* .

To avoid inflated notation, we will describe some general features of these limit points, the associated triangulations and how they encode the physics of the superpotential mainly at the hand of a simple example. The adaptation to other situations is straightforward; see also Appendix A for other examples. We consider again the non-compact Calabi–Yau $Y^* = K(\mathbf{P}^2) = \mathcal{O}(-3)_{\mathbf{P}^2}$. The vertices for the toric polyhedron Δ and the single charge vector $l^{(1)}$ are

$$\begin{aligned} \nu_0 &= (0, 0), \quad \nu_1 = (1, 1), \quad \nu_2 = (-1, 0), \quad \nu_3 = (0, -1), \\ l^{(1)} &= (-3, 1, 1, 1), \end{aligned} \tag{3.6}$$

where here and in the following we distinguish the non-compact direction of Y^* by the label “0” as in (2.18). The LSM for the D-branes on Y^* will be defined by the toric polyhedron Δ_{bound} with vertices

$$\begin{aligned} \nu_0 &= (0, 0, 0), \quad \nu_1 = (1, 1, 0), \quad \nu_2 = (-1, 0, 0), \\ \nu_3 &= (0, -1, 0), \quad \nu_4 = (1, 1, 1), \quad \nu_5 = (0, 0, 1). \end{aligned} \tag{3.7}$$

As alluded to above, the D-brane in an outer phase may be treated as a perturbation, and as a consequence the appropriate coordinates on the open-closed string moduli space should agree with the coordinates on the closed string moduli space, plus an extra modulus for the D-brane. This is confirmed by the existence of a non-star triangulation of Δ_{bound} leading to a basis of charge vectors of the form (2.18)

$$\begin{aligned} l^{(a)} &= (l^{(a)}(Y^*) \quad ; 0, 0), \quad a = 1, \dots, h^{1,1}(Y^*), \\ l^{(0)} &= (1, -1, 0, \dots; 0; 1, -1). \end{aligned}$$

Here the first entry corresponds to the non-compact direction and the last two entries to the new vertices not contained in the hyperplane $\nu_{i,3} = 0$. The point of maximal unipotent monodromy is specified by $z_\alpha = 0$. This translates by (3.3) to the condition $|x_0|^2 \gg |x_1|^2$, which indeed identifies this triangulation as the outer phase. This phase is sketched on the l.h.s. of Fig.2. In this non-star triangulation, the point ν'_6 is not connected to the distinguished vertex ν_0 and can be decoupled (decompactified). Physicswise the semi-classical D-brane does not destabilize the CY geometry Y^* , and has a runaway vacuum where the D-brane moves to an infinite distance away from the compact divisor x_0 that supports the closed string excitations.

For increasing z_0 , the perturbative expansion (2.23) fails to converge near the wall $\text{Re}(z_0) \sim 1$ or $t_0 \sim 0$, which is a highly non-classical regime from the point of string world-sheet expansion. Continuing further to large z_0 one finds a new point of maximal unipotent monodromy corresponding to a new classical limit. In the classical D-brane picture, the end of the brane on $x_i = x_j = 0$ has moved to an inner vertex $x_0 = x_i = 0$ which intersects the zero section S . In the LSM this “transition” amounts to a flop in the triangulation, as illustrated in Fig. 2¹³. A basis of charge vectors associated to the new triangulation is

¹³ Note however that the classical definition of the D-brane after the flop involves also a shift of the zero energy. This is related to the change of the classical terms and their subtraction in the A-model (or a change of the reference curve C_* in (2.1) in the B-model). The associated classical limits are indicated as dashed lines in Fig.2.

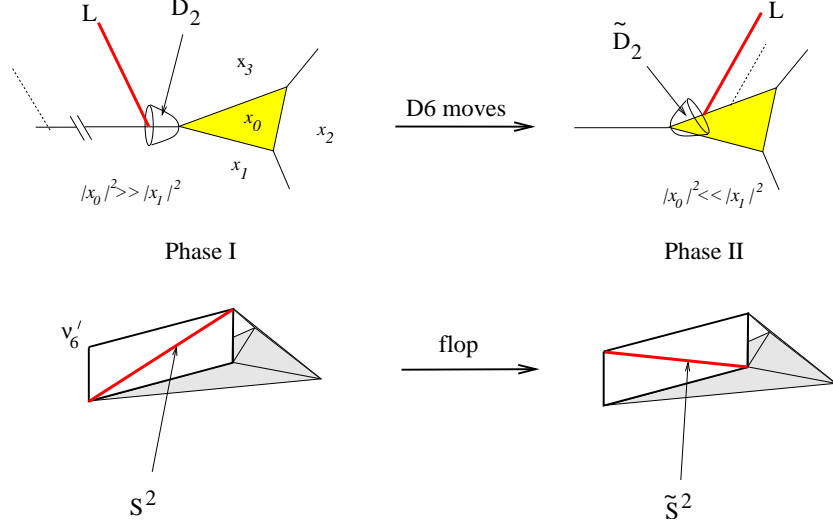


Fig. 2: The (D-brane) flop transition in the open string picture and in the associated LSM defined by Δ_{bound} . In the former, the D-brane on L moves (in the image of the moment map $x_i \rightarrow |x_i|^2$) towards a vertex of \mathbf{P}^2 , where the area of a disc instanton on D_2 vanishes, and continues to the other side where a new disc instanton appears on \tilde{D}_2 . In the LSM the same phase transition is described by the flop in the (image of the) triangulation for Δ_{bound} .

$$\begin{aligned} \tilde{l}^{(1)} &= (-2 \ 0 \ 1 \ 1 \ -1 \ 1) , \\ \tilde{l}^{(0)} &= (-1 \ 1 \ 0 \ 0 \ 1 \ -1) , \end{aligned} \quad (3.8)$$

and leads by (2.6) to new coordinates in a patch $\tilde{z}_\alpha = 0$.

Note that this choice of coordinates mixes the notion of closed and open string moduli, in contrast to the classical picture where the D-brane just perturbs the CY moduli space. In particular the non-trivial mixing has the effect that the superpotential in this phase destabilizes the CY geometry near the new expansion point $\tilde{z}_\alpha = 0$. Specifically, analogous to what we did in section 2.3, we can determine the expansion of the superpotential in the inner phase to be:

$$W(z_a, z_0) = \sum_{\substack{n_a \geq 0, n_0 \geq 0, n_0 \neq n_1 \\ n_0 - \sum \tilde{l}_0^{(a)} n_a > 0}} \frac{(-1)^{-n_1 + \sum \tilde{l}_0^{(a)} n_a} (n_0 - \sum \tilde{l}_0^{(a)} n_a - 1)!}{(n_0 - n_1) (n_0 + \sum \tilde{l}_1^{(a)} n_a)! \prod_{i=2}^{N+2} (\sum \tilde{l}_i^{(a)} n_a)!} z_0^{n_0} \prod_a z_a^{n_a}. \quad (3.9)$$

This expression is actually more general¹⁴ and applies to all flops in CY geometries with charge vectors (2.18) for which the new charge vectors can be represented by $\tilde{l}^{(1)} = l^{(1)} + l^{(0)}$, $\tilde{l}^{(0)} = -l^{(0)}$.

¹⁴ With some little more effort it is possible to write down a superpotential for any given kind of flop.

It follows straightforwardly from the expression above that, different as compared to the superpotential in the outer phase, the superpotential (3.9) has no solution $dW = 0$ at finite volume of the Calabi–Yau Y^* near the limit point $z_a = 0$. This behavior is also in agreement with the classical picture that the volumes of the discs in this phase are bounded by the volume of 2-spheres and thus the disc instantons drive the CY towards its decompactification (along its compact direction)¹⁵.

4. Discussion

As argued in [13], the existence of a Picard-Fuchs system for the $\mathcal{N} = 1$ superpotential of the open-closed string system reflects a new structure, namely the $\mathcal{N} = 1$ special geometry of the deformation space of the topologically twisted theories. This is explicit in the open/closed duals studied there, where the $\mathcal{N} = 1$ special geometry derives from the special geometry of CY 4-folds. We have not yet discussed the non-holomorphic content of this geometry of the $\mathcal{N} = 1$ “moduli space”, which is governed by the tt^* equations of [33,4]. In particular we have not considered the kinetic terms of the chiral multiplets for the “moduli” fields. This is an interesting subject for further study.

Here we restrict ourselves to point out one particularly interesting aspect of the kinetic terms, namely the relation to the theory of modular functions for the CY moduli space and open string generalizations thereof. The holomorphic superpotential W enters the effective $\mathcal{N} = 1$ four-dimensional supergravity theory via the function

$$G = K(\phi_i, \bar{\phi}_i) + \ln W(\phi_i) + \ln \bar{W}(\bar{\phi}_i),$$

where K is the Kähler potential. $\mathcal{N} = 1$ special geometry predicts a relation between K and W . On the other hand, in general supergravity theories the two functions K and W are completely independent. However an $\mathcal{N} = 1$ string effective supergravity from the type II string on a Calabi–Yau manifold will have quite generally discrete quantum symmetries which imply a strong correlation between the Kähler and the superpotential. In fact the assumption of the invariance of the function G under a given modular group $SL(2, \mathbf{Z})$ has been used in [34] to predict exact instanton corrected formulae for the Kähler potential and the superpotential starting from the classical

¹⁵ The structure of the superpotential in the inner and outer phases parallels that of $\mathcal{N} = 1$ SYM theories with few matter multiplets embedded in string theory, with masses below and above the field theory scale Λ , respectively. In view of the dualities of [6] there may well be a concrete correspondence of this kind in a certain regime of the moduli space.

expressions. The same line of thought should be quite powerful also in the open-closed string systems considered in this paper, where the exact expression for W is known and might be used to determine the kinetic terms. The open string superpotential contributes the term

$$\ln W = \ln \left(\sum_{k=1}^{\infty} q_0^k f_k(q_a) \right) = 2\pi i t_0 + F(q_a) + \sum_{k=1}^{\infty} q_0^k \tilde{f}_k(q_a)$$

to the G -function. The functions of the closed string moduli q_a in the above expression have modular properties with respect to the modular group Γ of the complex structure moduli space of the CY 3-fold. E.g. the leading term for large vev's of the open string modulus t_0 comprises the instanton corrections (2.22) to the open string mirror map and is of the form

$$F(q_a) = \sum_a \frac{b_a}{2\pi i} \ln(q_a/z_a(q_a)).$$

Here the b_a are some rational coefficients determined in (2.22) and the non-trivial functions $z_a(q_a)$ have a relatively straightforward interpretation in terms of modular functions of Γ . The higher orders in q_0 should describe the embedding of Γ into the larger modular group Γ' of the total open-closed string moduli space¹⁶. This is similar to the embedding of modular groups of K3 into modular groups of CY 3-folds studied in [35]. It would be interesting to study the quantum symmetries and their implications for the metric and extrema of the superpotential further.

Another comment concerns the fact that in the approach of [8], there is an integral ambiguity in the non-perturbative definition of the open-closed string system [10]. This is related to the framing ambiguity of knots in the Chern-Simons theory on the internal part of the D-brane. On the other hand the definition of the mirror map by the Picard-Fuchs system for the period and chain integrals does not have such an ambiguity and thus provides a preferred framing. Different framings correspond to different parametrizations of the curve $F = 0$ with F defined in (2.4). It is straightforward to see that the preferred framings are related to those parametrizations of F that are obtained from the 2d superpotential $\mathcal{W}_{D=2}$ by factoring out a coordinate \tilde{y}_i *linearly*. This is in fact a necessary condition in order that the two superpotentials (2.3) and (2.4) are equivalent on the level of period integrals [21], and thus explains the selection of framing made by the PF system.

Acknowledgments: We are grateful to I. Brunner and S. Stieberger for discussions.

¹⁶ Again it is helpful to think of the group Γ' in terms of the modular group of the complex structure moduli space of a CY 4-fold via the open/closed string dualities in [13].

Appendix A. Picard-Fuchs equations, linear sigma models and superpotentials from toric geometry

We present below a collection of toric computations for D-branes on non-compact 3-folds, given by the canonical bundles over \mathbf{P}^2 and the Hirzebruch surfaces F_0 , F_1 and F_2 . For those cases that have been discussed already in refs. [8,10]¹⁷, our results are consistent with the known results¹⁸. The closed formulas that we obtain allow to determine the disc instanton numbers N_{n_a, n_0} for all n_0 explicitly, for given definite values of n_a . This is also the natural form of the results obtained by the localization techniques of ref.[15]. From the point of view of differential equations, their origin may be traced back to recursion relations for the instanton numbers derived from the boundary differential operator \mathcal{D}_0 , which are quadratic in n_0 and determine N_{n_a, n_0} in terms of the N_{n_a, n_0-1} . We will consider two phases for each case, the first corresponding to an outer phase and the second to an inner phase obtained by a flop, as discussed in section 3.

A.1. $\text{---} Y^* = K(\mathbf{P}^2) \text{---}$

The application of the GKZ system (2.17) to this case has been already sketched in [13] and we restrict ourselves below to a complete the discussion and to add some results. The polyhedron Δ_{bound} and a basis of charge vectors for the outer phase have been defined in (3.7) and (2.18). The associated Picard-Fuchs system is

$$\begin{aligned} \mathcal{D}_1 &= \theta_1^2 (\theta_1 - \theta_0) + z_1 (3\theta_1 - \theta_0) (1 + 3\theta_1 - \theta_0) (2 + 3\theta_1 - \theta_0) , \\ \mathcal{D}_0 &= (3\theta_1 - \theta_0) \theta_0 - z_0 (\theta_1 - \theta_0) \theta_0 . \end{aligned} \tag{A.1}$$

The corrections to the mirror map (2.19) are $S_0 = A$, $S_1 = -3A$, with

$$A = - \sum_{n_1 > 0} \frac{(-)^{n_1} (3n_1 - 1)!}{n_1!^3} z_1^{n_1} . \tag{A.2}$$

The superpotential (2.23) is given by

$$W(z_1, z_0) = \sum_{n_1 \geq 0, n_0 > n_1} \frac{(-)^{n_1} (n_0 - n_1 - 1)!}{(n_0 - 3n_1)! n_1!^2 n_0} z_0^{n_0} z_1^{n_1} . \tag{A.3}$$

¹⁷ See also [16] for similar computations for other CY geometries.

¹⁸ With two exceptions: the instanton numbers given in [10] for the outer phase of F_0 and an inner phase of F_1 ; see below.

The above expression can be rewritten in terms of hyper-geometric functions as in (2.26) and (2.28), and moreover taking a derivative and using (2.29) we have

$$\begin{aligned}
z_0 \partial_{z_0} W(z_1, z_0) &= \sum_{n \geq 0} \frac{(-)^n z_0^{n+1} z_1^n}{n!^2 \Gamma(2-2n)} {}_2F_1(1, 1, 2-2n; z_0) \\
&= -\log(1-z_0) + \sum_{n \geq 0} \frac{(-)^n (2n-1)!}{n!^2} \left(\frac{z_0}{1-z_0} \right)^{2n} (z_0 z_1)^n \\
&= -\log \left(\frac{1}{2} \left[1 - z_0 + \sqrt{4z_0^3 z_1 + (z_0 - 1)^2} \right] \right).
\end{aligned}$$

The last expression is the logarithm of the solution for z_u of the equation $z_u + z_0 - z_1 z_0^3 z_u^{-1} - 1 = 0$, which reproduces the Riemann surface of ref. [10]. Note that the expression in the square root is a component of the discriminant of the PF system (A.1); this appears to be a general feature.

Our explicit formula allow to extract closed expressions for the disc instanton numbers. Concretely we find for the first generating functions $N_{n_1}(x) = \sum_{n_0} N_{n_1, n_0} x^{n_0}$:

1	$\frac{1}{x-1} x(2-x)$	(A.4)
2	$\frac{1}{(x-1)^3(1+x)} x(-5+6x+5x^2-8x^3+x^5)$	
3	$\frac{1}{(1-x)^5(1+x+x^2)} x(-32+107x-126x^2+86x^3-109x^4+125x^5-56x^6+2x^7+x^9)$	

These are consistent with the results listed in Table 6 of ref.[10] obtained by the method of [8]. Closed formulae for the invariants N_{n_1, n_0} valid for $n_0 > n_1 + 1$ are:¹⁹

$$\begin{aligned}
N_{1, n_0} &= -1 \\
N_{2, n_0} &= \frac{1}{4}(15 - 4n_0 + n_0^2) + \frac{1}{4}\epsilon(2, n_0) \\
N_{3, n_0} &= \frac{1}{36}(-976 + 348n_0 - 103n_0^2 + 12n_0^3 - n_0^4) + \frac{1}{9}\epsilon(3, n_0) \\
N_{4, n_0} &= \frac{1}{576}(147996 - 59400n_0 + 18961n_0^2 - 3072n_0^3 + 370n_0^4 \\
&\quad - 24n_0^5 + n_0^6) - \frac{1}{64}(60 - 8n_0 + n_0^2)\epsilon(2, n_0)
\end{aligned} \tag{A.5}$$

with $N_{n_1, 0} \equiv 0$.

¹⁹ $\epsilon(n, x)$ is defined to be equal to be one if $x \pmod n = 0$, and zero otherwise. The dependence of our expressions on $\epsilon(n, x)$, with a coefficient proportional to the inverse square, should reflect the open string instanton multi-covering formulae of ref. [19].

The LSM charge vectors in the flopped phase have been given in (3.8). The associated Picard-Fuchs operators are:

$$\begin{aligned}\mathcal{D}_1 &= \theta_1^2 (\theta_1 - \theta_0) + z_1 (\theta_1 - \theta_0) (2\theta_1 + \theta_0) (1 + 2\theta_1 + \theta_0) \\ \mathcal{D}_0 &= \theta_0 (-\theta_1 + \theta_0) + z_0 (\theta_1 - \theta_0) (2\theta_1 + \theta_0) .\end{aligned}\tag{A.6}$$

The corrections to the mirror map are $S_0 = -A$, $S_1 = -2A$, with

$$A(z_1, z_0) = - \sum_{n>0} \frac{(-)^n (3n-1)!}{n!^3} (z_0 z_1)^n .\tag{A.7}$$

The superpotential (3.9) becomes:

$$W(z_1, z_0) = \sum_{n_0, 1 \geq 0, n_0 \neq n_1} \frac{(-)^{n_1} (n_0 + 2n_1 - 1)!}{n_0! n_1!^2 (n_0 - n_1)} z_0^{n_0} z_1^{n_1} .\tag{A.8}$$

The first generating functions $N_{n_1}(x) = \sum_{n_0} N_{n_1, n_0} x^{n_0}$ (2.27) for instanton numbers are

1	$\frac{1}{x-1}(x-1+x^2)$	(A.9)
2	$\frac{1}{(x-1)^3(1+x)}(1-4x^2-3x^3+6x^4+3x^5-4x^6)$	
3	$\frac{1}{(x-1)^5(1+x+x^2)}(-1-2x^2+29x^3-17x^4-53x^5+49x^6-9x^7+55x^8-76x^9+27x^{10})$	

Closed formulae for some invariants N_{n_1, n_0} valid for $n_0 > n_1 + 1$ are:

$$\begin{aligned}N_{1, n_0} &= -1 \\ N_{2, n_0} &= \frac{1}{4}(11 + n_0^2) + \frac{1}{4}\epsilon(2, n_0) \\ N_{3, n_0} &= \frac{1}{36}(-616 + 54n_0 - 49n_0^2 - n_0^4) + \frac{1}{9}\epsilon(3, n_0) \\ N_{4, n_0} &= \frac{1}{576}(91404 - 14976n_0 + 6097n_0^2 - 288n_0^3 + 130n_0^4 + n_0^6) \\ &\quad - \frac{1}{64}(44 + n_0^2)\epsilon(2, n_0)\end{aligned}\tag{A.10}$$

with $N_{n, n} \equiv 0$. These expressions are again consistent with Table 5 of ref.[10] after the shift of labels $n_0 \rightarrow n_0 - n_1$. See [13] for an explanation of the origin of the shift and Table 3 therein for an explicit listing of the above disc numbers.

A.2. $Y^* = K(F_0)$

The LSM for D-branes on the canonical bundle of F_0 is defined by the toric polyhedron Δ_{bound} with vertices

$$\begin{aligned}\nu_0 &= (0, 0, 0), \nu_1 = (-1, 0, 0), \nu_2 = (1, 0, 0), \nu_3 = (0, -1, 0), \\ \nu_4 &= (0, 1, 0), \nu_5 = (0, -1, 1), \nu_6 = (0, 0, 1).\end{aligned}$$

In the outer phase the D-brane ends on the edge $|x_1|^2 = |x_3|^2 = 0$ and the charge vectors are

$$\begin{aligned} l^{(1)} &= \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\ l^{(2)} &= \begin{pmatrix} -2 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \\ l^{(0)} &= \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 & -1 \end{pmatrix} \end{aligned} \quad (\text{A.11})$$

The period and chain integrals satisfy the generalized PF system

$$\begin{aligned} \mathcal{D}_1 &= \theta_1^2 - z_1 (2\theta_1 + 2\theta_2 - \theta_0) (1 + 2\theta_1 + 2\theta_2 - \theta_0) \\ \mathcal{D}_2 &= \theta_2 (\theta_2 - \theta_0) - z_2 (2\theta_1 + 2\theta_2 - \theta_0) (1 + 2\theta_1 + 2\theta_2 - \theta_0) \\ \mathcal{D}_0 &= (2\theta_1 + 2\theta_2 - \theta_0) \theta_0 - z_0 (\theta_2 - \theta_0) \theta_0 \end{aligned} \quad (\text{A.12})$$

The corrections to the logarithmic solutions that define the open string mirror map are $S_0 = A$, $S_1 = -2A$, $S_2 = -2A$, with

$$A = - \sum_{\substack{n_1, n_2 \geq 0 \\ (n_1, n_2) \neq (0, 0)}} \frac{(2n_1 + 2n_2 - 1)!}{n_1!^2 n_2!^2} z_1^{n_1} z_2^{n_2} \quad (\text{A.13})$$

The superpotential W is given by the series expansion

$$W(z_a, z_0) = \sum_{n_i \geq 0, n_0 > n_2} \frac{(-)^{n_2} (n_0 - n_2 - 1)!}{n_1!^2 (n_0 - 2n_1 - 2n_2)! n_2! n_0} z_0^{n_0} z_1^{n_1} z_2^{n_2} \quad (\text{A.14})$$

The first generating functions $N_{n_1, n_2}(x) = \sum_{n_0} N_{n_1, n_2, n_0} x^{n_0}$ for the disc instantons are:

	0	1	2
0	0	x	0
1	$\frac{1}{1-x}x$	$\frac{1}{x-1}(x-3)x$	$\frac{1}{x-1}x(-5+x+x^2)$
2	$\frac{1}{(1-x)^3(1+x)}x^3$	$\frac{1}{(x-1)^3}x(-5+11x-9x^2+x^3)$	$\frac{1}{(x-1)^3(1+x)}x(-35+46x+23x^2-52x^3+6x^4+2x^5)$

(A.15)

The disc instanton numbers of low degree are collected in the following table, with the horizontal (vertical) direction corresponding to n_1 (n_2) and the bold face letter in the corner denoting n_0 :

1	0	1	2	3	4	2	0	1	2	3	4
0	1	1	0	0	0	0	0	0	0	0	0
1	1	3	5	7	9	1	1	2	4	6	8
2	0	5	35	135	385	2	0	4	24	96	280
3	0	7	135	1100	5772	3	0	6	96	750	3936
4	0	9	385	5772	50250	4	0	8	280	3936	33544
5	0	11	910	22950	309638	5	0	10	684	15876	206656
6	0	13	1890	75174	1495832	6	0	12	1456	52992	1007208

3	0	1	2	3	4	4	0	1	2	3	4
0	0	0	0	0	0	0	0	0	0	0	0
1	1	2	3	5	7	1	1	2	3	4	6
2	1	6	25	84	234	2	2	10	32	90	224
3	0	10	112	729	3476	3	1	20	165	896	3775
4	0	14	360	4191	31876	4	0	30	576	5650	38016
5	0	18	935	18187	210075	5	0	40	1595	26316	269843
6	0	22	2093	64395	1086215	6	0	50	3744	98588	1481984

5	0	1	2	3	4	6	0	1	2	3	4
0	0	0	0	0	0	0	0	0	0	0	0
1	1	2	3	4	5	1	1	2	3	4	5
2	4	16	45	110	245	2	6	24	62	140	288
3	4	42	265	1232	4644	3	11	86	440	1782	6096
4	1	70	1015	8472	51018	4	6	168	1866	13400	72750
5	0	98	2997	42262	388063	5	1	252	5920	71680	592287
6	0	126	7403	167440	2257665	6	0	336	15430	300672	

Table A.1: Disc instanton numbers for the D-brane ending on the outer edge $|x_1|^2 = |x_3|^2 = 0$ of F_0 .

These numbers disagree with those given in Table 3 of ref.[10], computed in the approach of [8]. A subsequent recalculation of the superpotential with the methods of [8] turns out to be consistent with our result above. Closed formulas for some invariants N_{n_1, n_2, n_0} are

$$\begin{aligned}
N_{1, n_2, n_0} &= n_2 + 1, n_0 > n_2 \\
N_{0, 1, n_0} &= 0, n_0 > 2, \quad N_{0, n_2, n_0} = 0, n_2 > 1, \forall n_0 \\
N_{2, 0, n_0} &= \frac{1}{4}(n_0 - 1)^2 - \frac{1}{4}\epsilon(2, n_0), \forall n_0 \\
N_{2, 1, n_0} &= 6 - 3n_0 + n_0^2, n_0 > 1 \\
N_{2, 2, n_0} &= \frac{5}{2}(13 - 4n_0 + n_0^2) - \frac{1}{2}\epsilon(2, n_0), n_0 > 2.
\end{aligned} \tag{A.16}$$

After the (flop) transition to the inner phase, the D-brane ends on the edge $|x_1|^2 = |x_0|^2 = 0$ and the basis of charge vectors becomes

$$\begin{aligned}
l^{(1)} &= \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
l^{(2)} &= \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix} \\
l^{(0)} &= \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix}
\end{aligned} \tag{A.17}$$

leading to the generalized PF system

$$\begin{aligned}
\mathcal{D}_1 &= \theta_1^2 - z_1 (2\theta_1 + \theta_2 + \theta_0) (1 + 2\theta_1 + \theta_2 + \theta_0) \\
\mathcal{D}_2 &= \theta_2 (\theta_2 - \theta_0) - z_2 (\theta_2 - \theta_0) (2\theta_1 + \theta_2 + \theta_0) \\
\mathcal{D}_0 &= \theta_0 (-\theta_2 + \theta_0) + z_0 (\theta_2 - \theta_0) (2\theta_1 + \theta_2 + \theta_0) .
\end{aligned} \tag{A.18}$$

The corrections to the mirror map become $S_0 = -A$, $S_1 = -2A$, $S_2 = -A$, where

$$A = - \sum_{\substack{n_1, n \geq 0 \\ (n_1, n) \neq (0, 0)}} \frac{(2n_1 + 2n - 1)!}{n_1!^2 n!^2} z_1^{n_1} (z_0 z_2)^n . \tag{A.19}$$

Moreover the superpotential is given by

$$W(z_a, z_0) = \sum_{n_i \geq 0, n_0 \neq n_2} \frac{(n_0 + 2n_1 + n_2 - 1)!}{n_0! n_1!^2 n_2! (n_0 - n_2)} z_0^{n_0} z_1^{n_1} z_2^{n_2} . \tag{A.20}$$

The first generating functions for disc instanton numbers $N_{n_1, n_2}(x) = \sum_{n_0} N_{n_1, n_2, n_0} x^{n_0}$ are

	0	1	2
0	0	-1	0
1	$\frac{1}{1-x} x$	$\frac{1}{1-x} (-1+x+2x^2)$	$\frac{1}{x-1} (1+x-2x^2-3x^3)$
2	$\frac{1}{1-x} x$	$\frac{1}{(x-1)^3} (1-3x-7x^2+13x^3-6x^4)$	$\frac{1}{(x-1)^3 (1+x)} (2+6x-20x^2-41x^3+46x^4+29x^5-32x^6)$

(A.21)

The closed formulae for some N_{n_1, n_2, n_0} are:

$$\begin{aligned}
N_{0, n_2, n_0}(x) &= 0, n_2 > 1 \\
N_{1, n_2, n_0} &= n_2 + 1, n_0 > n_2 + 1 \\
N_{2, 0, n_0} &= \frac{1}{4} (1 + n_0)^2 - \frac{1}{4} \epsilon(2, n_0), \forall n_0 \\
N_{2, 1, n_0} &= 4 + n_0 + n_0^2, n_0 > 1 \\
N_{2, 2, n_0} &= \frac{1}{2} (45 + 5n_0^2) - \frac{1}{2} \epsilon(2, n_0), n_0 > 2 \\
N_{3, 0, n_0} &= \frac{1}{36} (2 + 3n_0 + n_0^2)^2 - \frac{1}{9} \epsilon(3, n_0), \forall n_0 \\
N_{3, 1, n_0} &= \frac{1}{6} (36 + 14n_0 + 17n_0^2 + 4n_0^3 + n_0^4), n_0 > 1 .
\end{aligned} \tag{A.22}$$

These integers are consistent with results listed in Table 1 of ref. [10].

A.3. **—————** $Y^* = K(F_1)$ **—————**

The LSM for D-branes on the canonical bundle of F_1 is defined by the toric polyhedron Δ_{bound} with vertices

$$\begin{aligned} \nu_0 &= (0, 0, 0), \nu_1 = (-1, 0, 0), \nu_2 = (1, 0, 0), \nu_3 = (1, 1, 0), \\ \nu_4 &= (0, -1, 0), \nu_5 = (1, 1, 1), \nu_6 = (0, 0, 1). \end{aligned}$$

We will first consider a D-brane ending on the outer leg of the toric diagram with $|x_1|^2 - |x_3|^2 = 0$. The matrix of charge vectors for this phase is:

$$\begin{aligned} l^{(1)} &= \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\ l^{(2)} &= \begin{pmatrix} -1 & 0 & -1 & 1 & 1 & 0 & 0 \end{pmatrix} \\ l^{(0)} &= \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 & -1 \end{pmatrix} \end{aligned} \quad (\text{A.23})$$

The associated Picard-Fuchs operators look:

$$\begin{aligned} \mathcal{D}_1 &= \theta_1 (\theta_1 - \theta_2) - z_1 (2\theta_1 + \theta_2 - \theta_0) (1 + 2\theta_1 + \theta_2 - \theta_0) \\ \mathcal{D}_2 &= \theta_2 (\theta_2 - \theta_0) + z_2 (\theta_1 - \theta_2) (2\theta_1 + \theta_2 - \theta_0) \\ \mathcal{D}_0 &= (2\theta_1 + \theta_2 - \theta_0) \theta_0 - z_0 (\theta_2 - \theta_0) \theta_0 \end{aligned} \quad (\text{A.24})$$

The corrections to the mirror map are $S_0 = A$, $S_1 = -2A$, $S_2 = -A$, where

$$A = - \sum_{n_a \geq 0, n_2 \leq n_1} \frac{(-)^{n_2} (2n_1 + n_2 - 1)!}{n_1! (n_2!)^2 (n_1 - n_2)!} z_1^{n_1} z_2^{n_2} \quad (\text{A.25})$$

The superpotential (2.23) takes the form:

$$W(z_a, z_0) = \sum_{n_a \geq 0, n_0 > n_2} \frac{(-)^{n_2} (n_0 - n_2 - 1)!}{n_1! (n_0 - 2n_1 - n_2)! (n_1 - n_2)! n_2! n_0} z_0^{n_0} z_1^{n_1} z_2^{n_2} \quad (\text{A.26})$$

Some generating functions for disc instanton numbers $N_{n_1, n_2}(x) = \sum_{n_0} N_{n_1, n_2, n_0} x^{n_0}$ are:

	0	1	2
0	0	0	0
1	$\frac{1-x}{1-x} x$	$\frac{1}{1-x} x(x-2)$	0
2	$\frac{1}{(1-x)^3(1+x)} x^3$	$\frac{1}{(1-x)^3} x(-4+9x-7x^2+x^3)$	$\frac{1}{(x-1)^3(1+x)} x(-5+6x+5x^2-8x^3+x^5)$

(A.27)

With the same conventions as in Table A.1, the disc instanton numbers of low degree are collected in the following table.

1	0	1	2	3	4	2	0	1	2	3	4
0	1	0	0	0	0	0	0	0	0	0	0
1	1	-2	0	0	0	1	1	-1	0	0	0
2	0	-4	5	0	0	2	0	-3	4	0	0
3	0	-6	35	-32	0	3	0	-5	24	-21	0
4	0	-8	135	-400	286	4	0	-7	96	-261	180
5	0	-10	385	-2592	5187	5	0	-9	280	-1716	3288
6	0	-12	910	-11760	47775	6	0	-11	684	-7956	30604

3	0	1	2	3	4	4	0	1	2	3	4
0	0	0	0	0	0	0	0	0	0	0	0
1	1	-1	0	0	0	1	1	-1	0	0	0
2	1	-4	3	0	0	2	2	-6	4	0	0
3	0	-8	25	-18	0	3	1	-15	34	-20	0
4	0	-12	112	-248	153	4	0	-25	168	-301	160
5	0	-16	360	-1780	2970	5	0	-35	580	-2349	3385
6	0	-20	935	-8892	29614	6	0	-45	1600	-12584	36288

Table A.2: Disc instanton numbers for the D-brane ending on the outer edge $|x_1|^2 = |x_3|^2$ of F_1 .

Examples for closed formulas for some N_{n_1, n_2, n_0} look:

$$\begin{aligned}
N_{n_1, n_2, n_0} &= 0, n_1 < n_2 \\
N_{1, 0, n_0} &= 1, n_0 > 1 \\
N_{1, 1, n_0} &= -1, n_0 > 2 \\
N_{2, 0, n_0} &= \frac{1}{4}(n_0 - 1)^2 - \frac{1}{4}\epsilon(2, n_0), \forall n_0 \\
N_{2, 1, n_0} &= \frac{1}{2}(-8 + 3n_0 - n_0^2), n_0 > 1 \\
N_{2, 2, n_0} &= \frac{1}{4}(15 - 4n_0 + n_0^2) + \frac{1}{4}\epsilon(2, n_0), n_0 > 2 \\
N_{3, 0, n_0} &= \frac{1}{36}(2 - 3n_0 + n_0^2)^2 - \frac{1}{9}\epsilon(3, n_0), \forall n_0 \\
N_{3, 1, n_0} &= \frac{1}{12}(-108 + 82n_0 - 41n_0^2 + 8n_0^3 - n_0^4), n_0 > 1 \\
N_{3, 2, n_0} &= \frac{1}{12}(432 - 194n_0 + 71n_0^2 - 10n_0^3 + n_0^4), n_0 > 2 \\
N_{3, 3, n_0} &= \frac{1}{36}(-976 + 348n_0 - 103n_0^2 + 12n_0^3 - n_0^4) + \frac{1}{9}\epsilon(3, n_0), n_0 > 3.
\end{aligned} \tag{A.28}$$

We now consider the flopped Phase with charge vector matrix:

$$\begin{aligned} l^{(1)} &= \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\ l^{(2)} &= \begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 1 & -1 \end{pmatrix} \\ l^{(0)} &= \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix} \end{aligned} \quad (\text{A.29})$$

The corresponding Picard-Fuchs operators are:

$$\begin{aligned} \mathcal{D}_1 &= \theta_1 (\theta_1 - \theta_2) - z_1 (2\theta_1 + \theta_0) (1 + 2\theta_1 + \theta_0) \\ \mathcal{D}_2 &= \theta_2 (\theta_2 - \theta_0) + z_2 (\theta_1 - \theta_2) (\theta_2 - \theta_0) \\ \mathcal{D}_0 &= \theta_0 (\theta_0 - \theta_2) + z_0 (\theta_2 - \theta_0) (2\theta_1 + \theta_0) \end{aligned} \quad (\text{A.30})$$

The corrections to the mirror map take the form $S_0 = -A$, $S_1 = -2A$, $S_2 = 0$, with

$$A = - \sum_{n_a \geq 0, n \leq n_1} \frac{(-)^n (2n_1 + n - 1)!}{n_1! (n!)^2 (n_1 - n)!} z_1^{n_1} (z_0 z_2)^n \quad (\text{A.31})$$

The superpotential (3.9) looks:

$$W(z_a, z_0) = \sum_{\substack{n_a \geq 0, n_0 > 2n_1 \\ n_0 \neq n_2}} \frac{(-)^{n_2} (n_0 + 2n_1 - 1)!}{n_0! n_1! (n_1 - n_2)! n_2! (n_0 - n_2)} z_0^{n_0} z_1^{n_1} z_2^{n_2} \quad (\text{A.32})$$

Some generating functions for disc instanton numbers $N_{n_1, n_2}(x) = \sum_{n_0} N_{n_1, n_2, n_0} x^{n_0}$ are:

	0	1	2
0	0	0	0
1	$\frac{1}{1-x}x$	$\frac{1}{x-1}(x-1+x^2)$	0
2	$\frac{1}{(1-x)^3(1+x)}x$	$\frac{1}{(x-1)^3}(-1+3x+3x^2-8x^3+4x^4)$	$\frac{1}{(x-1)^3(1+x)}(1-4x^2-3x^3+6x^4+3x^5-4x^6)$
3	$\frac{1}{(1-x)^5(1+x+x^2)}x(1+x^2)$	$\frac{1}{(x-1)^5}(-1+5x+10x^2-47x^3+65x^4-39x^5+9x^6)$	*

(A.33)

Closed formulae for invariants N_{n_1, n_2, n_0} are:

$$\begin{aligned} N_{n_1, n_2, n_0} &= 0, n_1 < n_2 \\ N_{1, 0, n_0} &= 1, n_0 > 1 \\ N_{1, 1, n_0} &= -1, n_0 > 2 \\ N_{2, 0, n_0} &= \frac{1}{4}(1 + n_0)^2 - \frac{1}{4}\epsilon(2, n_0), \forall n_0 \\ N_{2, 1, n_0} &= -\frac{1}{2}(6 + n_0 + n_0^2), n_0 > 1 \\ N_{2, 2, n_0} &= \frac{1}{4}(11 + n_0^2) + \frac{1}{4}\epsilon(2, n_0), n_0 > 2 \\ N_{3, 0, n_0} &= \frac{1}{36}(2 + 3n_0 + n_0^2)^2 - \frac{1}{9}\epsilon(3, n_0), \forall n_0 \\ N_{3, 1, n_0} &= \frac{1}{12}(-60 - 20n_0 - 23n_0^2 - 4n_0^3 - n_0^4), n_0 > 1 \\ N_{3, 2, n_0} &= \frac{1}{12}(264 - 2n_0 + 35n_0^2 + 2n_0^3 + n_0^4), n_0 > 2 \\ N_{3, 3, n_0} &= \frac{1}{36}(-616 + 54n_0 - 49n_0^2 - n_0^4) + \frac{1}{9}\epsilon(3, n_0), n_0 > 3. \end{aligned} \quad (\text{A.34})$$

These integers are consistent with the findings of ref. [10], listed in their Table 8(I).²⁰

A.4. $Y^* = K(F_2)$

The LSM for D-branes on the canonical bundle of F_2 is defined by the toric polyhedron Δ_{bound} with vertices

$$\begin{aligned} \nu_0 &= (0, 0, 0), \nu_1 = (-1, 0, 0), \nu_2 = (1, 0, 0), \nu_3 = (0, -1, 0), \\ \nu_4 &= (2, 1, 0), \nu_5 = (0, -1, 1), \nu_6 = (0, 0, 1). \end{aligned}$$

We start with the D-brane ending on the outer leg $|x_1|^2 = |x_3|^2$ of the toric diagram, described by the LSM charge vector matrix:

$$\begin{aligned} l^{(1)} &= \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\ l^{(2)} &= \begin{pmatrix} 0 & 0 & -2 & 1 & 1 & 0 & 0 \end{pmatrix} \\ l^{(0)} &= \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{aligned} \quad (\text{A.35})$$

The associated Picard-Fuchs operators look:

$$\begin{aligned} \mathcal{D}_1 &= (\theta_1 - 2\theta_2)(\theta_1 - \theta_0) - z_1(2\theta_1 - \theta_0)(1 + 2\theta_1 - \theta_0) \\ \mathcal{D}_2 &= -(z_2(-1 + \theta_1 - 2\theta_2)(\theta_1 - 2\theta_2)) + \theta_2^2 \\ \mathcal{D}_0 &= z_0(\theta_1 - \theta_0)\theta_0 - (2\theta_1 - \theta_0)\theta_0. \end{aligned} \quad (\text{A.36})$$

The corrections to the mirror map are $S_0 = A$, $S_1 = B - 2A$, $S_2 = -2B$, with

$$\begin{aligned} A &= - \sum_{n_1 > 0, 2n_2 \leq n_1} \frac{(2n_1 - 1)!}{(n_1 - 2n_2)!n_2!^2 n_1!} z_1^{n_1} z_2^{n_2}, \\ B &= - \sum_{n_2 > 0} \frac{(2n_2 - 1)!}{n_2!^2} z_2^{n_2}. \end{aligned} \quad (\text{A.37})$$

The superpotential (2.23) takes the form:

$$W(z_a, z_0) = \sum_{\substack{n_a \geq 0, n_0 > 0 \\ n_0 > n_1}} \frac{(-1)^{n_1} (n_0 - n_1 - 1)!}{(n_0 - 2n_1)! (n_1 - 2n_2)! n_2!^2 n_0} z_0^{n_0} z_1^{n_1} z_2^{n_2}. \quad (\text{A.38})$$

The first few generating functions for disc instanton numbers $N_{n_1, n_2}(x) = \sum_{n_0} N_{n_1, n_2, n_0} x^{n_0}$ take the form:

²⁰ We could not reproduce the instanton numbers given in Table 9 of [10] for the phase III sketched in their Fig. 19. In fact this table is identical to Table 8(II) up to a minus sign and trivial relabelling and should simply describe the D-brane on the edge linearly equivalent to that of phase II.

	0	1	2
0	0	0	0
1	x	x	0
2	0	$\frac{1}{x-1}x(-3+x+x^2)$	0
3	0	$\frac{1}{x-1}x(-5+x+x^2+x^3)$	$\frac{1}{x-1}x(-5+x+x^2+x^3)$

(A.39)

Some explicit disc instanton numbers N_{n_1, n_2, n_0} are (in the same conventions as in Table A.1):

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Table A.3: Disc instanton numbers for the D-brane ending on the outer edge $|x_1|^2 = |x_3|^2 = 0$ of F_2 .

Closed formulae for some disc instanton numbers N_{n_1, n_2, n_0} are:

$$\begin{aligned}
N_{n_1, n_2, n_0} &= 0, \quad n_2 \geq n_1, \quad (n_1, n_2) \neq (1, 1), \\
N_{1, 0, n_0} &= N_{1, 1, n_0} = 0, \quad n_0 \neq 1 \\
N_{1, 0, 1} &= N_{1, 1, 1} = 1 \\
N_{n_1, 1, n_0} &= \begin{cases} n_1 - 1, & n_0 > n_1 \\ 2n_1 - n_0, & 1 \leq n_0 \leq n_1 \\ 0, & n_0 = 0 \end{cases} \\
N_{4, 2, n_0} &= \frac{1}{4}(65 - 6n_0 + n_0^2) - \frac{1}{4}\epsilon(2, n_0) .
\end{aligned} \tag{A.40}$$

The flop to a phase where the brane has moved to an inner edge of the toric diagram is described by the charge vectors

$$\begin{aligned}
l^{(1)} &= \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 1 & -1 \end{pmatrix} \\
l^{(2)} &= \begin{pmatrix} 0 & 0 & -2 & 0 & 1 & 0 & 0 \end{pmatrix} . \\
l^{(0)} &= \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}
\end{aligned} \tag{A.41}$$

The Picard-Fuchs operators look:

$$\begin{aligned}
\mathcal{D}_1 &= (\theta_1 - 2\theta_2)(\theta_1 - \theta_0) - z_1(\theta_1 - \theta_0)(\theta_1 + \theta_0) \\
\mathcal{D}_2 &= \theta_2^2 - z_2(\theta_1 - 2\theta_2 - 1)(\theta_1 - 2\theta_2) \\
\mathcal{D}_0 &= \theta_0(\theta_0 - \theta_1) + z_0(\theta_1 - \theta_0)(\theta_1 + \theta_0) .
\end{aligned} \tag{A.42}$$

The corrections to the mirror map become $S_0 = A$, $S_1 = A + B$, $S_2 = -2B$, with

$$\begin{aligned}
A(z_a) &= - \sum_{n > 0, n_2 \geq 0} \frac{(2n - 1)!}{(n - 2n_2)! n_2!^2 n!} (z_0 z_1)^n z_2^{n_2} , \\
B(z_a) &= - \sum_{n_2 > 0} \frac{(2n_2 - 1)!}{n_2!^2} z_2^{n_2} .
\end{aligned} \tag{A.43}$$

The superpotential (3.9) takes the form:

$$W(z_a, z_0) = \sum_{n_a \geq 0, n_0 \neq n_1} \frac{(n_0 + n_1 - 1)!}{n_0! (n_1 - 2n_2)! n_2!^2 (n_0 - n_1)} z_0^{n_0} z_1^{n_1} z_2^{n_2} . \tag{A.44}$$

The first generating functions $N_{n_1, n_2}(x) = \sum_{n_0} N_{n_1, n_2, n_0} x^{n_0}$ (2.27) for disc instanton numbers are:

	0	1	2
0	0	0	0
1	-1	-1	0
2	0	$\frac{1}{x-1}(1+x-2x^2-x^3)$	0
3	0	$\frac{1}{x-1}(1+x+x^2-3x^3-2x^4)$	$\frac{1}{x-1}(1+x+x^2-3x^3-2x^4)$
4	0	$\frac{1}{x-1}(1+x+x^2+x^3-4x^4-3x^5)$	$\frac{1}{(x-1)^3(x-1)}(2+2x-2x^2+4x^3-30x^4-4x^5+46x^6-3x^7-16x^8)$

(A.45)

Some explicit instanton numbers N_{n_1, n_2, n_0} are:

0	0	1	2	3	4	5	1	0	1	2	3	4	5	2	0	1	2	3	4	5
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	-1	-1	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0
2	0	-1	0	0	0	0	2	0	-2	0	0	0	0	2	0	0	0	0	0	0
3	0	-1	-1	0	0	0	3	0	-2	-2	0	0	0	3	0	-3	-3	0	0	0
4	0	-1	-2	-1	0	0	4	0	-2	-6	-2	0	0	4	0	-3	-10	-3	0	0
5	0	-1	-4	-4	-1	0	5	0	-2	-12	-12	-2	0	5	0	-3	-23	-23	-3	0
6	0	-1	-6	-11	-6	-1	6	0	-2	-20	-40	-20	-2	6	0	-3	-40	-88	-40	-3
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4	0	-4	-20	-4	0	0	4	0	0	0	0	0	0	4	0	3	18	3	0	
5	0	-4	-40	-40	-4	0	5	0	-5	-75	-75	-5	0	5	0	0	0	0	0	
6	0	-4	-70	-164	-70	-4	6	0	-5	-118	-295	-118	-5	6	0	-6	-210	-600	-210	

Table A.4: Disc instanton numbers for the D-brane ending on the inner edge $|x_0|^2 = |x_3|^2 = 0$ of F_2 .

Closed formulae for some disc instanton numbers N_{n_1, n_2, n_0} are:

$$\begin{aligned}
N_{n_1, n_2, n_0} &= 0, \quad n_2 \geq n_1, \quad (n_1, n_2) \neq (1, 1), \\
N_{1, 0, n_0} &= N_{1, 1, n_0} = 0, \quad n_0 > 0, \\
N_{1, 0, 0} &= N_{1, 1, 0} = -1 \\
N_{n_1, 1, n_0} &= \begin{cases} n_1 - 1, & n_0 > n_1 \\ 0, & n_0 = n_1 \\ -n_0 - 1, & 0 \leq n_0 < n_1 \end{cases} \\
N_{4, 2, n_0} &= \frac{1}{4}(57 - 2n_0 + n_0^2) - \frac{1}{4}\epsilon(2, n_0), \quad n_0 > 4.
\end{aligned}
\tag{A.46}$$

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