GLOBAL SECTIONS OF LINE BUNDLES ON A WONDERFUL COMPACTIFICATION OF THE GENERAL LINEAR GROUP

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Contents

1. Introduction

Let k be a field of characteristic zero and let E and F be two n-dimensional vector spaces. In [K1] we have introduced a certain compactification $\mathrm{KGl}(E,F)$ of the variety $\mathrm{Isom}(E,F) \cong \mathrm{Gl}_n$ of linear isomorphisms from E to F which in many respects is analogous to De Concinis and Procesis so called *wonderful compactification* of adjoint semi-simple algebraic groups (cf. [CP]).

In particular, there is a natural action of the group $\mathbf{G} := \mathrm{Gl}(E) \times \mathrm{Gl}(F)$ on $\mathrm{KGl}(E,F)$ extending the one arising from right and left multiplication on $\mathrm{Isom}(E,F)$. Furthermore $\mathrm{KGl}(E,F)$ is smooth, the boundary, i.e. the complement of $\mathrm{Isom}(E,F)$ in $\mathrm{KGl}(E,F)$, is a divisor with normal crossings and the closures of the orbits of the $\mathbf{G}$-action are precisely the nonempty intersections of the irreducible components of the boundary.

We will see in 3.1 below that the Picard group of $\mathrm{KGl}(E,F)$ is generated by (the ideal sheaves of) the boundary components $Z_0,\ldots,Z_{n-1}$ and $Y_0,\ldots,Y_{n-1}$. Every line bundle expressed in terms of these generators is equipped with a canonical linearization of the $\mathbf{G}$-action and thus the space of global sections of its restriction to some orbit closure is naturally a finite dimensional $\mathbf{G}$-module.

In this paper we show how such a space of global sections decomposes into a direct sum of simple $\mathbf{G}$-modules. More precisely, we prove the following

Theorem: *(Cf. Theorem 4.3 for the exact formulation). Let L be a $\mathbf{G}$-linearized line bundle of the form $L = \mathcal{O}(\sum(m_iZ_i + l_iY_i))$ on $KGl(E,F)$ and let $I, J \subseteq [0, n-1]$ be subsets such that the intersection $\overline{\mathbf{O}}_{IJ} = (\cap_{i\in I} Z_i) \cap (\cap_{j\in J} Y_j)$ is nonempty. Then the decomposition of the*

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$\mathbf{G}$-module $H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^*L)$ into simple submodules is given by a canonical isomorphism

$$H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^*L) \xrightarrow{\sim} \bigoplus_{(a,b)\in A_{IJ}(L)} H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b)) \quad ,$$

where $A_{IJ}(L) \subset \mathbb{Z}^n \times \mathbb{Z}^n$ is a finite set defined explicitly in terms of I, J and L, where $\mathbf{Fl}$ is the product of the two complete flag manifolds associated to the vector spaces E and F respectively and where $\mathcal{O}_{\mathbf{Fl}}(a,b)$ is the product specified by (a,b) of successive quotients of tautological vector bundles on $\mathbf{Fl}$.

In [K2] we have shown the relevance of $\mathrm{KGl}(E,F)$ for the Gieseker type degeneration of moduli stacks of vector bundles on curves: The normalization of the moduli stack of Gieseker vector bundles on an irreducible nodal curve with one singularity is isomorphic to $\mathrm{KGl}(\mathcal{E},\mathcal{F})$, where $\mathcal{E}$ and $\mathcal{F}$ are certain vector bundles on the moduli stack of vector bundles on the normalization of the curve. In a forthcoming paper we will apply the results of the present paper to obtain a canonical decomposition of generalized theta functions on the moduli stack of Gieseker vector bundles (cf. [K3]).

Our proof of Theorem 4.3 is inspired by [CP] §8, where the cohomology of line bundles on complete symmetric varieties is computed. At one notable point however we have to argue differently, since to show that certain simple submodules occur in the space of global sections De Concini and Procesi make use of the fact that certain line bundles are ample (cf. [CP], Proposition 8.4), and it turns out (cf. 6.1) that the corresponding statement is false in the case of $\mathrm{KGl}(E,F)$. Instead, we produce (in 5.3) explicit sections which generate the simple submodules in question.

After finishing this paper I have learned that A. Tchoudjem has studied the cohomology of line bundles on compactifications of arbitrary reductive groups [T]. Part of our result can probably be deduced from his, but certainly not all, since he does not deal with cohomology of the strata and does not obtain a *canonical* decomposition.

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2. Notation

For an $(n\times n)$-matrix x with coefficients x_{ij} in some ring R and nonempty subsets $A, B \subset [1,n]$ of the same cardinality m we denote by $\det_{AB}(x) \in R$ the determinant of the $(m\times m)$-sub-matrix $(x_{ij})_{i\in A, j\in B}$ of x.

3. Preliminary results

Let k be a field of characteristic zero. We fix two k-vector spaces E and F of rank n. Let $\mathrm{KGl}(E,F)$ be the compactification of the scheme $\mathrm{Isom}(E,F)$ introduced in [K1]. We quickly recall the definition of $\mathrm{KGl}(E,F)$. Consider the closed subschemes

$$\begin{array}{ccccccccc}
Y_0^{(0)} & \hookrightarrow & Y_1^{(0)} & \hookrightarrow & \cdots & \hookrightarrow & Y_{n-1}^{(0)} & & \\
 & & \uparrow & & & & \uparrow & & \\
 & & Z_{n-1}^{(0)} & \hookrightarrow & \cdots & \hookrightarrow & Z_1^{(0)} & \hookrightarrow & Z_0^{(0)}
\end{array}$$

of $X^{(0)} := \mathbb{P}(\mathrm{Hom}(E,F)^\vee \oplus k)$ where after the choice of a basis for E and for F the subscheme $Y_i^{(0)}$ is defined by the homogeneous ideal in $\mathrm{Sym}\,(\mathrm{Hom}(E,F)^\vee \oplus \mathcal{O}_S) \cong k[x_{00}, x_{ij}(i,j \in [1,n])]$ generated by the $(i+1)\times(i+1)$-sub-minors of the matrix $(x_{ij})_{i,j\in[1,n]}$, the subscheme $Z_0^{(0)}$ is the hyper-plane at infinity and $Z_{n-i}^{(0)}$ is the intersection of $Y_i^{(0)}$ with $Z_0^{(0)}$. By definition, $\mathrm{KGl}(E,F)$ is the result of successively blowing up the scheme $X^{(0)}$ as follows:

$$X^{(0)} \longleftarrow X^{(1)} \longleftarrow X^{(2)} \longleftarrow \cdots\cdots \longleftarrow X^{(n-1)} = \mathrm{KGl}(E,F)$$

where $X^{(i)}$ arises from $X^{(i-1)}$ by blowing up the (disjoint) union of the subschemes $Y_{i-1}^{(i-1)}$ and $Z_{n-i}^{(i-1)}$ and where we define $Y_{j-1}^{(i)}, Z_{n-j}^{(i)} \subset X^{(i)}$ to be the proper transforms of $Y_{j-1}^{(i-1)}$ and $Z_{n-j}^{(i-1)}$ respectively if $j \neq i$ and to be the exceptional divisors lying above $Y_{i-1}^{(i-1)}$ and $Z_{n-i}^{(i-1)}$ respectively if $j = i$.

We have shown in [K1] that $\mathrm{KGl}(E,F)$ is a smooth projective variety over k and that the complement of $\mathrm{Isom}(E,F)$ in $\mathrm{KGl}(E,F)$ is a divisor with normal crossings whose irreducible components are $Z_1,\ldots,Z_{n-1}$ and $Y_1,\ldots,Y_{n-1}$, where $Z_i := Z_i^{(n-1)}$ and $Y_i := Y_i^{(n-1)}$.

The main result in [K1] says that the scheme $\mathrm{KGl}(E,F)$ parametrizes what we call generalized isomorphisms from E to F. We refer to [K1] §5 for the definition of generalized isomorphisms. In particular, there is a universal generalized isomorphism

$$E\otimes\mathcal{O} \underset{(M_0,\mu_0)}{\overset{0}{\longleftarrow}} E_1 \underset{(M_1,\mu_1)}{\overset{1}{\longleftarrow}} E_2 \quad \cdots \quad E_{n-1} \underset{(M_{n-1},\mu_{n-1})}{\overset{n-1}{\longleftarrow}} E_n \xrightarrow{\sim} F_n \underset{(L_{n-1},\lambda_{n-1})}{\overset{n-1}{\longrightarrow}} F_{n-1} \quad \cdots \quad F_2 \underset{(L_1,\lambda_1)}{\overset{1}{\longrightarrow}} F_1 \underset{(L_0,\lambda_0)}{\overset{0}{\longrightarrow}} F\otimes\mathcal{O}$$

on $\mathrm{KGl}(E,F)$.

The M_i and L_i are line bundles on $\mathrm{KGl}(E,F)$ and the μ_i and λ_i are global sections of M_i and L_i, which vanish precisely along the boundary components Z_i and Y_i respectively. In particular, we have canonical isomorphisms $M_i = \mathcal{O}(Z_i)$ and $L_i = \mathcal{O}(Y_i)$ which identify μ_i and λ_i with the canonical 1-sections in $\mathcal{O}(Z_i)$ and $\mathcal{O}(Y_i)$ respectively.

Let $\mathbf{G}$ be the product of the two algebraic k-groups $\mathrm{Gl}(E)$ and $\mathrm{Gl}(F)$. There is a natural action of $\mathbf{G}$ on $\mathrm{KGl}(E,F)$. In terms of R-valued points (R a k-algebra) it is given by $g.\Phi = g_2\Phi g_1^{-1}$, where $g = (g_1,g_2) \in \mathbf{G}(R) = (\mathrm{Gl}(E)\times\mathrm{Gl}(F))(R)$ and Φ is a generalized isomorphism from $E\otimes_k R$ to $F\otimes_k R$ (cf. [K1], pp 590-591). From this description it is clear that the line bundles M_i and L_i are canonically $\mathbf{G}$-linearized. Notice that also the trivial line bundles $\det(E)\otimes_k\mathcal{O}$ and $\det(F)\otimes_k\mathcal{O}$ carry canonical nontrivial $\mathbf{G}$-linearization.

Lemma 3.1. *There is a canonical isomorphism of* $\mathbf{G}$*-linearized line bundles on KGl*(E,F)*:*

$$(\det E)^{-1}\otimes_k \bigotimes_{i=0}^{n-1} M_i^{n-i} = (\det F)^{-1}\otimes_k \bigotimes_{i=0}^{n-1} L_i^{n-i} \quad . \qquad (*)$$

The Picard group of the variety KGl(E,F) *is generated by the isomorphism classes of the line bundles* M_i *and* L_i *(*$i\in[0,n-1]$*) and the only relations come from the isomorphism* $(*)$*.*

Proof. Let Φ denote the universal generalized isomorphism on $\mathrm{KGl}(E,F)$. The canonical isomorphism stated in the lemma is an immediate consequence of the fact that the canonical morphism

$$\det_{[1,n][1,n]}\Phi : \det(E)\otimes_k \mathcal{O} \to \bigotimes_{i=1}^{n}\left(\bigotimes_{j=0}^{i-1} L_j^{-1}\otimes\bigotimes_{j=0}^{n-i} M_j\right)\otimes_k \det(F)$$

is nowhere vanishing (cf. [K1], 6.5).

From the proof of [K1] 4.1 it is clear that for $i\in[1,n-1]$ the two subschemes $Y_{i-1}^{(i-1)}$ and $Z_{n-i}^{(i-1)}$ of $X^{(i-1)}$ are disjoint and each of them is irreducible and smooth of codimension ≥ 2. Therefore the divisor class group of $X^{(i)}$ is the direct sum of the divisor class group of $X^{(i-1)}$ and the free abelian group generated by the two divisors $Y_{i-1}^{(i)}$ and $Z_{n-i}^{(i)}$. Now the divisor class group of $X^{(0)}$ is generated by the hyper-plane $Z_0^{(0)}$, therefore by induction it follows that the classes of $Y_0,\ldots,Y_{n-2}$, $Z_0,\ldots,Z_{n-1}$ freely generate the divisor class group of $\mathrm{KGl}(E,F)$. □

For each pair of subsets $I,J\subseteq[0,n-1]$ with $\min(I)+\min(J)\geq n$ we have defined in [K1] §9 the closed subscheme $\overline{\mathbf{O}}_{IJ}=\overline{\mathbf{O}}_{IJ}(E,F)$ in $\mathrm{KGl}(E,F)$ as the intersection of the components Z_i $(i\in I)$ and Y_j $(j\in J)$. It is clear from [K1] §9 that the subschemes $\overline{\mathbf{O}}_{IJ}$ are precisely the closures of the orbits of $\mathbf{G}$ acting on $\mathrm{KGl}(E,F)$.

If $I\supseteq I'$ and $J\supseteq J'$ then we have $\overline{\mathbf{O}}_{IJ}\subseteq\overline{\mathbf{O}}_{I'J'}$. In particular, we have $\overline{\mathbf{O}}_{\emptyset\emptyset}=\mathrm{KGl}(E,F)$ and the smallest of the closed subschemes $\overline{\mathbf{O}}_{IJ}$ are of the form

$$\overline{\mathbf{O}}_{r,s}:=\overline{\mathbf{O}}_{[s,n-1],[r,n-1]}$$

for $r,s\in[0,n]$, $r+s=n$. (The set $[s,n-1]$ contains r elements while the set $[r,n-1]$ contains s elements, that's why we write $\overline{\mathbf{O}}_{r,s}$ instead of $\overline{\mathbf{O}}_{s,r}$). Let

$$i_{IJ}:\overline{\mathbf{O}}_{IJ}\hookrightarrow\mathrm{KGl}(E,F)\qquad\text{and}\qquad i_{r,s}:\overline{\mathbf{O}}_{r,s}\hookrightarrow\mathrm{KGl}(E,F)$$

denote the inclusion morphisms.

Let $\mathrm{Fl}(E)$ and $\mathrm{Fl}(F)$ denote the full flag manifolds associated to the vector spaces E and F respectively and let $\mathbf{Fl}:=\mathrm{Fl}(E)\times\mathrm{Fl}(F)$. For $a=(a_1,\ldots,a_n)\in\mathbb{Z}^n$, $b=(b_1,\ldots,b_n)\in\mathbb{Z}^n$ we define the invertible $\mathcal{O}_{\mathbf{Fl}}$-module

$$\mathcal{O}_{\mathbf{Fl}}(a,b):=\bigotimes_{i=1}^{n}(\mathcal{E}_i/\mathcal{E}_{i-1})^{\otimes a_i}\otimes\bigotimes_{i=1}^{n}(\mathcal{F}_i/\mathcal{F}_{i-1})^{\otimes b_i}\quad,$$

where $0=\mathcal{E}_0\subset\mathcal{E}_1\subset\cdots\subset\mathcal{E}_n=E\otimes\mathcal{O}_{\mathbf{Fl}}$ and $0=\mathcal{F}_0\subset\mathcal{F}_1\subset\cdots\subset\mathcal{F}_n=F\otimes\mathcal{O}_{\mathbf{Fl}}$ are the two universal flags on $\mathbf{Fl}$. The variety $\mathbf{Fl}$ is endowed with a canonical $\mathbf{G}$-action and the line bundles $\mathcal{O}_{\mathbf{Fl}}(a,b)$ come with a canonical $\mathbf{G}$-linearization.

Lemma 3.2. *For each pair $r,s\in[0,n]$ with $r+s=n$ we have a canonical isomorphism $\overline{\mathbf{O}}_{r,s}\xrightarrow{\sim}\mathbf{Fl}$, which is compatible with the $\mathbf{G}$-action on the two varieties. Furthermore, we have a canonical isomorphism of $\mathbf{G}$-linearized line bundles on $\overline{\mathbf{O}}_{r,s}$:*

$$i_{r,s}^*\left(\bigotimes_{i=0}^{n-1}(M_i^{m_i}\otimes L_i^{l_i})\otimes(\det E)^e\otimes(\det F)^d\right)=\mathcal{O}_{\overline{\mathbf{O}}_{r,s}}(a,b)\quad,$$

where $\mathcal{O}_{\overline{\mathbf{O}}_{r,s}}(a,b)$ *is the line bundle corresponding to* $\mathcal{O}_{\mathbf{Fl}}(a,b)$ *via the isomorphism* $\overline{\mathbf{O}}_{r,s} \xrightarrow{\sim} \mathbf{Fl}$ *and where* $(a,b) \in \mathbb{Z}^n \times \mathbb{Z}^n$ *is defined by*

$$a_i - e = -b_{n-i+1} + d = \begin{cases} l_{n-i+1} - l_{n-i} & \text{if } i \in [1,s] \\ m_{i-1} - m_i & \text{if } i \in [s+1,n] \end{cases}$$

(It is understood that $m_n = l_n = 0$*).*

Proof. This follows easily from [K1] Theorem 9.3 and its proof. □

Proposition 3.3. *Let* $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_n)$ *be two elements in* $\mathbb{Z}^n$*. Then* $H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b)) \neq 0$ *if and only if* a*,* b *are increasing, i.e. if* $a_1 \leq a_2 \leq \cdots \leq a_n$ *and* $b_1 \leq b_2 \leq \cdots \leq b_n$*. The association*

$$(a,b) \mapsto H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b))$$

establishes a bijection between the set of all increasing $a, b \in \mathbb{Z}^n$ *and the set of simple* $\mathbf{G}$*-modules. Furthermore,* $H^p(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b)) = 0$ *for all* $p \geq 2$ *and all increasing* $a, b \in \mathbb{Z}^n$*.*

Proof. This is a special case of the Borel-Bott-Weil theorem (cf. e.g. [J] II. 5.5). □

4. Statement of the theorem

We keep the notations introduced in section 3.

Definition 4.1. Let L be a line bundle on KGl(E,F) of the form

$$L = \bigotimes_{i=0}^{n-1} (M_i^{m_i} \otimes L_i^{l_i}) \otimes (\det E)^e \otimes (\det F)^d \quad .$$

Let $I, J \subseteq [0, n-1]$ and let $i_1 := \min(I)$, $j_1 := \min(J)$ where it is understood that $\min(\emptyset) = n$. Assume $i_1 + j_1 \geq n$. We denote by $A_{IJ}(L)$ the set of all elements $(a,b) \in \mathbb{Z}^n \times \mathbb{Z}^n$, which have the following properties:

(1) $a_1 \leq a_2 \leq \cdots \leq a_n$
(2) $\sum_{j=i+1}^{n} (a_j - e) \leq m_i$ for all $i \in [n - j_1, n-1]$ and equality holds for $i \in I$.
(3) $\sum_{j=1}^{n-i} (a_j - e) \geq -l_i$ for all $i \in [n - i_1, n-1]$ and equality holds for $i \in J$.
(4) For all $i \in [1, n]$ the equality $a_i - e = -b_{n-i+1} + d$ holds.

For abbreviation we denote by $A(L)$ the set $A_{\emptyset,\emptyset}(L)$.

Remark 4.2. Notice that for $r, s \in [0, n]$ with $r + s = n$ the set $A_{[s,n-1],[r,n-1]}(L)$ contains at most the single element (a,b) defined in 3.2.

Theorem 4.3. *Let* L *be a line bundle on KGl*(E,F) *of the form*

$$L = \bigotimes_{i=0}^{n-1} (M_i^{m_i} \otimes L_i^{l_i}) \otimes (\det E)^e \otimes (\det F)^d$$

and let $I, J \subseteq [0, n-1]$ *be subsets with* $\min(I) + \min(J) \geq n$*. Then the following holds:*

1. *The* $\mathbf{G}$*-module* $H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^* L)$ *comes with a canonical decomposition as follows:*

$$H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^* L) = \bigoplus_{(a,b) \in A_{IJ}(L)} H^0(\mathbf{Fl}, \mathcal{O}(a,b)) \quad .$$

2. This decomposition is compatible with restriction in the sense that the following is a commutative diagram of $\mathbf{G}$*-modules:*

$$\begin{array}{ccccc} H^0(KGl, L) & & \xrightarrow{\quad Res \quad} & & H^0(\overline{\mathbf{O}}_{I,J}, i_{I,J}^* L) \\ \| & & & & \| \\ \bigoplus_{(a,b)\in A(L)} H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b)) & \twoheadrightarrow & \bigoplus_{(a,b)\in A(L)\cap A_{I,J}(L)} H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b)) & \hookrightarrow & \bigoplus_{(a,b)\in A_{I,J}(L)} H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b)) \end{array}$$

where the lower arrows are the canonical projection and inclusion morphisms induced by the inclusions $A(L) \cap A_{I,J}(L) \subseteq A(L)$ *and* $A(L) \cap A_{I,J}(L) \subseteq A_{I,J}(L)$ *respectively.*

3. Let

$$L' = \bigotimes_{i=0}^{n-1} (M_i^{m'_i} \otimes L_i^{l'_i}) \otimes f^*(\det E)^e \otimes f^*(\det F)^d \quad ,$$

where $m'_i \le m_i$ *and* $l'_j \le l_j$ *and equality holds, if* $i \in I$ *and* $j \in J$ *respectively. Then we have a commutative diagram of* $\mathbf{G}$*-modules as follows:*

$$\begin{array}{ccc} H^0(\overline{\mathbf{O}}_{I,J}, i_{I,J}^* L') & \xhookrightarrow{\otimes \mu^{m-m'} \otimes \lambda^{l-l'}} & H^0(\overline{\mathbf{O}}_{I,J}, i_{I,J}^* L) \\ \| & & \| \\ \bigoplus_{(a,b)\in A_{I,J}(L')} H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b)) & \hookrightarrow & \bigoplus_{(a,b)\in A_{I,J}(L)} H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b)) \end{array}$$

where the upper horizontal arrow is induced by the section

$$\left(\mu_0^{m_0 - m'_0} \otimes \ldots \otimes \mu_{n-1}^{m_{n-1} - m'_{n-1}} \otimes \lambda_0^{l_0 - l'_0} \otimes \ldots \otimes \lambda_{n-1}^{l_{n-1} - l'_{n-1}} \right) \Big|_{\overline{\mathbf{O}}_{I,J}}$$

of $i_{I,J}^*(L \otimes (L')^{-1})$ *and the lower horizontal arrow is induced by the inclusion* $A_{I,J}(L') \subseteq A_{I,J}(L)$.

5. Proof of the theorem

We fix a basis $(v_1, \ldots, v_n)$ for E and $(w_1, \ldots, w_n)$ for F. Let $B_1 \subseteq \mathrm{Gl}(E)$ and $B_2 \subseteq \mathrm{Gl}(F)$ be the Borel subgroups consisting of linear automorphisms fixing the flags

$$\{0\} \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset E \qquad \text{and} \qquad \{0\} \subset \langle w_n \rangle \subset \langle w_n, w_{n-1} \rangle \subset \cdots \subset F$$

respectively. Let $U_1 \subset B_1$ and $U_2 \subset B_2$ be the maximal unipotent subgroups of B_1 and B_2 respectively. Then $\mathbf{B} := B_1 \times B_2$ is a Borel subgroup of $\mathbf{G}$ and $\mathbf{U} := U_1 \times U_2$ is its maximal unipotent subgroup.

Let $V := \mathbf{U} \times \mathbb{A}^n$ and let $\xi_1, \ldots, \xi_n \in H^0(V, \mathcal{O}_V)$ be the pull back of the coordinate functions on $\mathbb{A}^n$. Let $V^o \subset V$ be the maximal open subset where all the ξ_i are invertible. For every pair $r, s \in [0, n]$ with $r + s = n$ we have a morphism $j_{r,s}^o : V^o \to \mathrm{Isom}(E, F)$, which on R-valued points (R a k-algebra) is defined by

$$j_{r,s}^o(x, y, z) = y \circ \zeta_r(z) \circ x$$

for $x \in U_1(R)$, $y \in U_2(R)$, $z = (z_1, \dots, z_n) \in (R^\times)^n$, where $\zeta_r(z) : E \otimes_k R \xrightarrow{\sim} F \otimes_k R$ is the isomorphism defined with respect to the given basis of E and F by the diagonal matrix $\mathrm{diag}(\zeta_{r,1}(z), \dots, \zeta_{r,n}(z))$ whose entries are

$$\zeta_{r,i}(z) = \begin{cases} z_i^{-1} \dots z_r^{-1} & \text{for } i \in [1,r] \\ z_{r+1} \dots z_i & \text{for } i \in [r+1,n] \end{cases}$$

Recall from [K1], 4.1 that KGl(E,F) is covered by open affine subschemes $X(\alpha,\beta,\ell)$, where α and β run through the set of permutations of $[1,n]$ and ℓ runs through the set $[0,n]$. It is clear from the definition of the $X(\alpha,\beta,\ell)$ that the morphism $j^o_{r,s}$ extends to an open immersion

$$j^{(r,s)} : V \longrightarrow \mathrm{KGl}(E,F)$$

whose image is the open affine subscheme $X(\mathrm{id},\mathrm{id},r)$.

Recall from [K1], pp 590-591 that the action of $\mathbf{G}$ on KGl(E,F) may be described in terms of R-valued points by

$$g.\Phi = g_2 \circ \Phi \circ g_1^{-1} \quad ,$$

where $g = (g_1, g_2) \in \mathbf{G}(R) = \mathrm{Gl}(E)(R) \times \mathrm{Gl}(F)(R)$ and Φ is a generalized isomorphism from $E \otimes_k R$ to $F \otimes_k R$. Recall also from [K1], 7.4 and 4.3 that in terms of R-valued points the open subscheme $X(\mathrm{id},\mathrm{id},r) \subset \mathrm{KGl}(E,F)$ consists of those generalized isomorphisms Φ from $E \otimes R$ to $F \otimes R$ which have the property that $\det_{[1,p],[1,p]}(\Phi)$ is invertible for $p \in [1,n]$ and that furthermore the sections $\mu_0, \dots, \mu_{s-1}$ and $\lambda_0, \dots, \lambda_{r-1}$ occurring in Φ are invertible. From these observations it follows easily that the subschemes $X(\mathrm{id},\mathrm{id},r)$ are $\mathbf{B}$-invariant.

Thus for each $r, s \in [0,n]$ with $r + s = n$ the immersion $j^{(r,s)}$ induces an action of $\mathbf{B}$ on V. Explicitly, on R-valued points this action is given by

$$b \cdot_r (x,y,z) := (\rho x \rho^{-1} u_1^{-1}, u_2 \tau y \tau^{-1}, z') \quad ,$$

where $b = (u_1\rho, u_2\tau) \in \mathbf{B}(R)$, $u_i \in U_i(R)$, $\rho = \mathrm{diag}(\rho_1, \dots, \rho_n)$ and $\tau = \mathrm{diag}(\tau_1, \dots, \tau_n)$ are R-valued points of the maximal torus of B_1 and B_2 respectively and $z' = (z'_1, \dots, z'_n) \in R^n$ is defined by

$$z'_i := \begin{cases} \rho_{i+1}^{-1}\rho_i\tau_{i+1}\tau_i^{-1} z_i & \text{if } i \in [1, r-1] \\ \rho_r \tau_r^{-1} z_r & \text{if } i = r \\ \rho_{r+1}^{-1}\tau_{r+1} z_{r+1} & \text{if } i = r+1 \\ \rho_i^{-1}\rho_{i-1}\tau_i\tau_{i-1}^{-1} z_i & \text{if } i \in [r+2, n] \end{cases}$$

Let I, J be two subsets of $[0, n-1]$ and assume $\min(I) + \min(J) \geq n$. By this assumption there exist $r, s \in [0,n]$ with $r + s = n$ and $I \subseteq [s, n-1]$, $J \subseteq [r, n-1]$. It is clear from the definitions and [K1] 4.1 that the closed subscheme V_{IJ} of V defined by the cartesian diagram

$$\begin{array}{ccc} V_{IJ} & \hookrightarrow & V \\ {\scriptstyle j^{(r,s)}_{IJ}} \downarrow & & \downarrow {\scriptstyle j^{(r,s)}} \\ \overline{\mathbf{O}}_{IJ} & \underset{i_{IJ}}{\hookrightarrow} & \mathrm{KGl}(E,F) \end{array}$$

is cut out by the equations $\xi_{n-i} = 0$ for $i \in I$ and $\xi_{i+1} = 0$ for $i \in J$.

Proposition 5.1. *Let L be a $\mathbf{G}$-linearized line bundle on KGl(E,F) and let $I, J \subseteq [0, n-1]$ with $\min(I) + \min(J) \geq n$. Then for each simple $\mathbf{G}$-module W the $\mathbf{G}$-module $H^0(\overline{\mathbf{O}}_{IJ}, i^*_{IJ} L)$ contains W at most with multiplicity one as a submodule.*

Proof. (Analogous to the proof of Lemma 8.2 in [CP]). Let s_1 and s_2 be two global sections of i_{IJ}^*L which generate $\mathbf{B}$-invariant lines in $H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^*L)$ on which $\mathbf{B}$ operates by the same character. I claim that $\overline{\mathbf{O}}_{IJ}$ contains a dense open $\mathbf{B}$-orbit Ω. Indeed, let $z = (z_1, \ldots, z_n) \in \mathbb{A}^n$, where $z_i = 0$ if $n-i \in I$ or $i-1 \in J$, and $z_i = 1$ else. Then by the preceding discussion it follows easily that choosing $r, s \in [0, n]$ such that $r + s = n$, $I \subseteq [s, n-1]$, $J \subseteq [r, n-1]$, the image of the point $(1, z) \in V_{IJ} \subseteq \mathbf{U} \times \mathbb{A}^n$ by the morphism $j_{IJ}^{(r,s)}$ is contained in a dense open $\mathbf{B}$-orbit in $\overline{\mathbf{O}}_{IJ}$. Therefore $\mathsf{s}_1/\mathsf{s}_2$ is a rational function on $\overline{\mathbf{O}}_{IJ}$, which is necessarily constant, since its restriction on Ω is constant. □

Proposition 5.2. *Let* $L = \bigotimes_{i=0}^{n-1}(M_i^{m_i} \otimes L_i^{l_i})$ *and let* $I, J \subseteq [0, n-1]$ *with* $\min(I) + \min(J) \geq n$. *If the* $\mathbf{G}$*-module* $H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^*L)$ *contains an irreducible* $\mathbf{G}$*-module* W *as a submodule, then there exists an element* $(a, b) \in A_{IJ}(L)$ *such that* W *is isomorphic to the* $\mathbf{G}$*-module* $H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a, b))$.

Proof. (Analogous to the proof of Proposition 8.2 in [CP]). Let $r, s \in [0, n]$ with $r + s = n$ and $I \subseteq [s, n-1]$, $J \subseteq [r, n-1]$. Since V_{IJ} is isomorphic to an affine space $\mathbb{A}^N$ for some N, there exists a nowhere vanishing section s_0 of the line bundle $L_{IJ} := (j_{IJ}^{(r,s)})^* i_{IJ}^* L$ on V_{IJ}. The group $\mathbf{B}$ acts on $H^0(V_{IJ}, L_{IJ})$ and for any $b \in \mathbf{B}$ the section $b \cdot \mathsf{s}_0$ is again nowhere vanishing and thus a scalar multiple of s_0, since invertible functions on V_{IJ} are constant. Therefore $\mathbf{B}$ acts by a character on the line generated by s_0.

Now let $\mathsf{s} \in W$ be a highest weight vector with respect to $\mathbf{B}$. Thus s is a global section of i_{IJ}^*L which generates a $\mathbf{B}$-invariant line inside the space $H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^*L)$. Let s_1 be its pull back by $j_{IJ}^{(r,s)} : V_{IJ} \to \overline{\mathbf{O}}_{IJ}$. Then we have $\mathsf{s}_1 = f\mathsf{s}_0$ for some regular function f on V_{IJ}. Clearly f generates a $\mathbf{B}$-invariant line in $H^0(V_{IJ}, \mathcal{O})$, therefore f is left unchanged by the action of the maximal unipotent subgroup $\mathbf{U}$ and it follows that f must be a polynomial in the ξ_i, where $i \in [1, n]$, $n - i \notin I$, $i - 1 \notin J$. In fact f must be a monomial in these ξ_i, since otherwise the $\mathbf{B}$-translates of f would generate a subspace of dimension ≥ 2 of $H^0(V_{IJ}, \mathcal{O})$.

It follows from the above that there is a divisor

$$D = \sum_{i \in [0, n-1]} \beta_i Z_i \quad + \sum_{i \in [0, n-1]} \alpha_i Y_i$$

on $\mathrm{KGl}(E, F)$, where $\beta_i \geq 0$, $\alpha_i \geq 0$ for all i and $\beta_i = 0$ if $i \in I$, $\alpha_i = 0$ if $i \in J$, such that the pull back of D to $\overline{\mathbf{O}}_{IJ} \cap X(\mathrm{id}, \mathrm{id}, r)$ coincides with the restriction of the vanishing divisor of s to this open subscheme of $\overline{\mathbf{O}}_{IJ}$. Therefore there is a global section s' of $i_{IJ}^*L(-D)$ whose image under the canonical map $i_{IJ}^*L(-D) \to i_{IJ}^*L$ is s and whose restriction to $\overline{\mathbf{O}}_{IJ} \cap X(\mathrm{id}, \mathrm{id}, r)$ is nowhere vanishing. Since the intersection $\overline{\mathbf{O}}_{r,s} \cap X(\mathrm{id}, \mathrm{id}, r)$ is nonempty, it follows that the restriction of s' to the closed subscheme $\overline{\mathbf{O}}_{r,s} \subseteq \overline{\mathbf{O}}_{IJ}$ is a nonzero section of $i_{r,s}^*L(-D)$.

Let $m_i' := m_i - \beta_i$ and $l_i' := l_i - \alpha_i$ and let $(a, b) \in \mathbb{Z}^n \times \mathbb{Z}^n$ be defined by

$$a_i = -b_{n-i+1} = \begin{cases} l'_{n-i+1} - l'_{n-i} & \text{if } i \in [1, s] \\ m'_{i-1} - m'_i & \text{if } i \in [s+1, n] \end{cases}$$

with the convention that $m_n' = l_n' = 0$. By 3.2 we have $i_{r,s}^*L(-D) = \mathcal{O}_{\overline{\mathbf{O}}_{r,s}}(a, b)$.

Consider the following diagram of $\mathbf{G}$-modules:

$$H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^*L) \longleftarrow H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^*L(-D)) \longrightarrow H^0(\overline{\mathbf{O}}_{r,s}, \mathcal{O}_{\overline{\mathbf{O}}_{r,s}}(a, b)) \quad .$$

The left arrow is injective and maps s' to s. The right arrow maps s' to a non-zero element and by 3.2 and 3.3 the object on the right is a simple $\mathbf{G}$-module. Therefore $H^0(\overline{\mathbf{O}}_{r,s}, \mathcal{O}_{\overline{\mathbf{O}}_{r,s}}(a,b))$ is isomorphic to W as $\mathbf{G}$-module. Let us gather what we know about (a,b):

(1) Since, as we have seen above, the line bundle $\mathcal{O}_{\overline{\mathbf{O}}_{r,s}}(a,b)$ has a non-vanishing global section, it follows from 3.3 that $a_1 \leq \cdots \leq a_n$.
(2) We have $\sum_{j=i+1}^n a_j = m_i' \leq m_i$ for $i \in [s, n-1]$ and equality holds if $i \in I$.
(3) We have $\sum_{j=1}^{n-i} a_j = -l_i' \geq -l_i$ for $i \in [r, n-1]$ and equality holds if $i \in J$.
(4) By definition, $a_i = -b_{n-i+1}$.

Let $i_1 := \min(I)$ and $j_1 := \min(J)$. In the above argument we can choose any r, s with $r \in [n-i_1, j_1]$ and *a priori* (a,b) depends on r, s but by 3.3 the fact that $H^0(\overline{\mathbf{O}}_{r,s}, \mathcal{O}_{\overline{\mathbf{O}}_{r,s}}(a,b))$ and W are isomorphic as $\mathbf{G}$-modules determines (a,b) which is therefore independent of r, s. It follows that the inequality in 2. holds for all $i \in [n-j_1, n-1]$ and the inequality in 3. holds for all $i \in [n-i_1, n-1]$, i.e. we have $(a,b) \in A_{IJ}(L)$. □

Let x_{ij}/x_{00} ($i,j \in [1,n]$) denote the coordinate functions on $\mathrm{Gl}_n = \mathrm{Isom}(E,F)$ interpreted as rational functions on $\mathrm{KGl}(E,F)$. For each integer $p \in [1,n]$ we define the rational function d_p on KGl_n as the determinant of the $p \times p$ sub-matrix of $(x_{ij}/x_{00})_{i,j\in[1,n]}$ with indices in $[1,p] \times [1,p]$, i.e. we set $d_p := \det_{[1,p][1,p]}(x_{ij}/x_{00})$. For convenience we define $d_0 := 1$.

Lemma 5.3. *Let $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. Then we have the following equality of divisors on KGl(E,F):*

$$\mathit{div} \prod_{i=1}^n \left(\frac{d_i}{d_{i-1}}\right)^{a_{n-i+1}} = \sum_{i=0}^{n-1}\left(-\sum_{j=i+1}^n a_j\right) Z_i + \sum_{i=0}^{n-1}\left(\sum_{j=1}^{n-i} a_j\right) Y_i + \sum_{i=1}^{n-1}(a_{n-i+1} - a_{n-i})\Delta_i \quad ,$$

where Δ_i denotes the closure in KGl(E,F) of the subscheme $\{d_i = 0\} \subset Gl_n$. Furthermore, for every $I, J \subseteq [0, n-1]$ with $\min(I) + \min(J) \geq n$ the closed subscheme $\overline{\mathbf{O}}_{IJ}$ is not contained in any of the Δ_i.

Proof. Observe that in the notation of [K1] we have $d_i/d_{i-1} = t_i(\mathrm{id},\mathrm{id})/t_0$. The formula is now an easy consequence of [K1], 4.1. Let Φ be the universal generalized isomorphism on $\mathrm{KGl}(E,F)$. Then Δ_i is the locus of vanishing of the global section $\det_{[1,i][1,i]}\Phi$ of the line bundle $\bigotimes_{\nu=1}^i(\bigotimes_{j=0}^{n-\nu} M_j \otimes \bigotimes_{j=0}^{\nu-1} L_j^{-1})$ and the complement of the union of all Δ_i is precisely the union of the open sets $X(\mathrm{id},\mathrm{id},\ell)$ where ℓ runs through $[0,n]$ (cf. [K1], 7.4). If we choose ℓ such that $I \subseteq [n-\ell, n-1]$ and $J \subseteq [\ell, n-1]$ (which is always possible) then the intersection of $\overline{\mathbf{O}}_{IJ}$ with $X(\mathrm{id},\mathrm{id},\ell)$ is clearly nonempty. □

We now come to the proof of Theorem 4.3.

Proof of the first statement: Let L and I, J be as in the theorem and assume first that $e = d = 0$. If $A_{IJ}(L)$ is empty, then $H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^* L) = (0)$ by 5.2 and therefore the statement 1 of the theorem trivially holds in this case.

Assume $A_{IJ}(L)$ is non-empty, let $(a,b) \in A_{IJ}(L)$ and let $r, s \in [0,n]$ with $r + s = n$ and $I \subseteq [s, n-1]$, $J \subseteq [r, n-1]$. Let $L' := \bigotimes_{i=0}^{n-1}(M_i^{m_i'} \otimes L_i^{l_i'})$, where $m_i' := \sum_{j=i+1}^n a_j$ and $l_i' := -\sum_{j=1}^{n-i} a_j$. From lemma 5.3 it follows that there exists a global section of L' whose restriction to $\overline{\mathbf{O}}_{r,s}$ is nonzero. By 3.2 we have $i_{r,s}^*L' = \mathcal{O}_{\overline{\mathbf{O}}_{r,s}}(a,b)$. Therefore we have a non-vanishing restriction morphism

$$H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^*L') \to H^0(\overline{\mathbf{O}}_{r,s}, \mathcal{O}_{\overline{\mathbf{O}}_{r,s}}(a,b)) \quad .$$

Together with 3.3 and 5.1 it follows that the $\mathbf{G}$-module $H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^*L')$ contains an irreducible submodule $W \cong H^0(\overline{\mathbf{O}}_{r,s}, \mathcal{O}_{\overline{\mathbf{O}}_{r,s}}(a,b))$ exactly with multiplicity one. In particular, by the above restriction morphism the submodule $W \subseteq H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^*L')$ is canonically identified with $H^0(\overline{\mathbf{O}}_{r,s}, \mathcal{O}_{\overline{\mathbf{O}}_{r,s}}(a,b))$ which in turn is canonically isomorphic to $H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b))$.

I claim that the identification of W with $H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b))$ is independent of the choice of the numbers r, s. For this it is clearly sufficient to show that the composite morphism

$$H^0(\mathrm{KGl}, L') \to H^0(\overline{\mathbf{O}}_{r,s}, \mathcal{O}_{\overline{\mathbf{O}}_{r,s}}(a,b)) \xrightarrow{\sim} H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b)) \tag{$*$}$$

does not depend on r, s. This will be shown below. For the moment we assume this fact.

It is clear from the definition of $A_{IJ}(L)$ that

$$\left.\left(\mu_0^{m_0-m_0'} \otimes \ldots \otimes \mu_{n-1}^{m_{n-1}-m_{n-1}'} \otimes \lambda_0^{l_0-l_0'} \otimes \ldots \otimes \lambda_{n-1}^{l_{n-1}-l_{n-1}'}\right)\right|_{\overline{\mathbf{O}}_{IJ}}$$

is a non-vanishing global section of $i_{IJ}^*(L \otimes (L')^{-1})$, and therefore defines a canonical injective morphism

$$H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^*L') \to H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^*L) \quad .$$

It follows that $H^0(\overline{\mathbf{O}}_{IJ}, i_{IJ}^*L)$ contains an irreducible $\mathbf{G}$-submodule (the image of W), which is canonically isomorphic to $H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b))$. This together with 5.1 and 5.2 clearly implies statement 1 of the theorem in the case $e = d = 0$.

For arbitrary e, d the result is easily deduced from that special case, the key observation being that we have a canonical isomorphism

$$\mathcal{O}_{\mathbf{Fl}}(a,b) \otimes (\det E)^e \otimes (\det F)^d = \mathcal{O}_{\mathbf{Fl}}(a + (e,\ldots,e), b + (d,\ldots,d))$$

of $\mathbf{G}$-linearized line bundles on $\mathbf{Fl}$.

Independence of (r,s): It remains to be shown that the morphism $(*)$ does not does not depend on r, s. In fact, since $H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b))$ is a simple $\mathbf{G}$-module, it suffices to produce a point z in $\mathbf{Fl}$ and a global section s of L', whose image in $H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b))$ evaluates to a nonzero element in in the fiber $\mathcal{O}_{\mathbf{Fl}}(a,b)[z]$ of $\mathcal{O}_{\mathbf{Fl}}(a,b)$ at z, which is independent of r, s.

By 5.3, the rational function $\prod_{i=1}^n (d_i/d_{i-1})^{a_{n-i+1}}$ gives rise to a global section of $\mathcal{O}_{\mathrm{KGl}}(D)$, where

$$D := \sum_{i=0}^{n-1} (m_i' Z_i + l_i' Y_i)$$

Let $\mathsf{s} \in H^0(\mathrm{KGl}, L')$ be the element, which corresponds to this section via the canonical isomorphism $\mathcal{O}_{\mathrm{KGl}}(D) \xrightarrow{\sim} L'$ and let $z \in \mathbf{Fl}$ be the point given by the pair of flags

$$\{0\} \subset \langle v_n \rangle \subset \langle v_n, v_{n-1} \rangle \subset \cdots \subset E \qquad \text{and} \qquad \{0\} \subset \langle w_1 \rangle \subset \langle w_1, w_2 \rangle \subset \cdots \subset F.$$

Then the image of s by the morphism

$$H^0(\mathrm{KGl}, L') \to H^0(\overline{\mathbf{O}}_{r,s}, i^*_{r,s}L') \xrightarrow{\sim} H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b)) \to \mathcal{O}_{\mathbf{Fl}}(a,b)[z] =$$
$$= \bigotimes_{i=1}^n \left(\frac{\langle v_n,\dots,v_{n-i+1}\rangle}{\langle v_n,\dots,v_{n-i}\rangle}\right)^{\otimes a_i} \otimes \bigotimes_{i=1}^n \left(\frac{\langle w_1,\dots,w_i\rangle}{\langle w_1,\dots,w_{i-1}\rangle}\right)^{\otimes b_i} = \langle \bigotimes_{i=1}^n (v_i \otimes w_i^{-1})^{a_{n-i+1}} \rangle$$

is precisely the generator $\otimes_{i=1}^n (v_i \otimes w_i^{-1})^{a_{n-i+1}}$ of the fiber of $\mathcal{O}_{\mathbf{Fl}}(a,b)$ at z. This is not difficult to check, if one observes that the point $(1,0)$ of $V = \mathbf{U} \times \mathbb{A}^n$ and the point z correspond as follows:

$$\begin{array}{ccccccccc} \mathbf{U} \times \mathbb{A}^n & = & V & \xrightarrow{j^{(r,s)}} & \mathrm{KGl}(E,F) & \hookleftarrow & \overline{\mathbf{O}}_{r,s} & \xleftarrow{\cong} & \mathbf{Fl} \\ (1,0) & & \longmapsto & & \Phi_0 & & \longleftarrow & & z \end{array}$$

for some generalized isomorphism Φ_0 from E to F.

Thus the image of s in $\mathcal{O}_{\mathbf{Fl}}(a,b)[z]$ does not depend on r, s as was to be shown.

Proof of the second statement: Let $(a,b) \in A(L)$ and let $(a',b') \in A_{I,J}(L)$. Consider the composite morphism

$$H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b)) \longrightarrow H^0(\mathrm{KGl}, L) \longrightarrow H^0(\overline{\mathbf{O}}_{I,J}, i^*_{i,j}L) \longrightarrow H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a',b')) \qquad (\dagger)$$

If $(a,b) \neq (a',b')$, then this morphism is clearly 0, since then domain and target are non-isomorphic simple $\mathbf{G}$-modules. If $(a,b) = (a',b')$, let $L' := \bigotimes_{i=0}^{n-1}(M_i^{m'_i} \otimes L_i^{l'_i}) \otimes f^*(\det E)^e \otimes f^*(\det F)^d$, where $m'_i := \sum_{j=i+1}^n a_j$ and $l'_i := -\sum_{j=1}^{n-i} a_j$, let $r, s \in [0,n]$ with $I \subseteq [s, n-1]$ and $J \in [r, n-1]$ and consider the commutative diagram

$$\begin{array}{ccccc} H^0(\mathrm{KGl}, L) & & \longrightarrow & & H^0(\overline{\mathbf{O}}_{I,J}, i^*_{I,J}L) \\ \uparrow & & & & \uparrow \\ H^0(\mathrm{KGl}, L') & & \longrightarrow & & H^0(\overline{\mathbf{O}}_{I,J}, i^*_{I,J}L') \\ \downarrow & & & & \downarrow \\ H^0(\overline{\mathbf{O}}_{r,s}, i^*_{r,s}L') & = & H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b)) & = & H^0(\overline{\mathbf{O}}_{r,s}, i^*_{r,s}L') \end{array}$$

where the horizontal arrows are the restriction morphisms and the vertical arrows are defined as in the proof of the first statement of the theorem. Since all $\mathbf{G}$-modules in this diagram contain the simple submodule $H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b))$ with multiplicity one, it follows that the morphism $(\dagger)$ has to be the identity in this case.

Proof of the third statement: Let $(a',b') \in A_{I,J}(L')$ and let $(a,b) \in A_{I,J}(L)$. Consider the composite morphism

$$H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a',b')) \longrightarrow H^0(\overline{\mathbf{O}}_{I,J}, i^*_{I,J}L') \longrightarrow H^0(\overline{\mathbf{O}}_{I,J}, i^*_{i,j}L) \longrightarrow H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b)) \qquad (\dagger\dagger)$$

If $(a',b') \neq (a,b)$, this morphism vanishes by the same argument as above. If $(a',b') = (a,b)$, then the assertion that $(\dagger\dagger)$ is the identity morphism follows similarly as above from the commutative diagram

$$\begin{array}{ccccc}
H^0(\overline{\mathbf{O}}_{I,J}, i^*_{I,J}L') & & \xhookrightarrow{\otimes\mu^{m-m'}\otimes\lambda^{l-l'}} & & H^0(\overline{\mathbf{O}}_{I,J}, i^*_{I,J}L) \\
\Big\uparrow{\scriptstyle \otimes\mu^{m'-m''}\otimes\lambda^{l'-l''}} & & & & {\scriptstyle \otimes\mu^{m-m''}\otimes\lambda^{l-l''}}\Big\uparrow \\
H^0(\overline{\mathbf{O}}_{I,J}, i^*_{I,J}L'') & & = & & H^0(\overline{\mathbf{O}}_{I,J}, i^*_{I,J}L'') \\
\Big\downarrow & & & & \Big\downarrow \\
H^0(\overline{\mathbf{O}}_{r,s}, i^*_{r,s}L'') & = & H^0(\mathbf{Fl}, \mathcal{O}_{\mathbf{Fl}}(a,b)) & = & H^0(\overline{\mathbf{O}}_{r,s}, i^*_{r,s}L'')
\end{array}$$

where $r, s \in [0, n]$ with $I \subseteq [s, n-1]$ and $J \in [r, n-1]$, where $m''_i := \sum_{j=i+1}^{n} a_j$, $l''_i := -\sum_{j=1}^{n-i} a_j$ and where $L'' := \bigotimes_{i=0}^{n-1}(M_i^{m''_i} \otimes L_i^{l''_i}) \otimes f^*(\det E)^e \otimes f^*(\det F)^d$.

6. Non-ampleness

Since large parts of our proof of Theorem 4.3 are analogous to the proof of Theorem 8.3 in [CP], it is natural to ask whether also the analogue of the ampleness result stated in Proposition 8.4 in [CP] holds. We will see below that the answer is negative.

Adopting the notation of [CP] let $\overline{X}$ be the complete symmetric variety associated to the data (G, σ), where G is a semi-simple simply connected algebraic group over the complex numbers and σ is a nontrivial involution on G. Let $S_1, \ldots, S_\ell$ be the closures of the 1-codimensional orbits of the natural action of G on $\overline{X}$. By [CP], 8.1 the unique closed orbit $Y = \cap_{i=1}^{\ell} S_i$ is isomorphic to G/P for some parabolic $P \subset G$ and the restriction of line bundles defines an injective homomorphism $i^* : \mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(Y)$. Proposition 8.4 in [CP] can be formulated as follows:

Let L be a line bundle on Pic($\overline{X}$) such that $H^0(Y, i^*L) \neq 0$ and let $\omega_{\overline{X}}$ denote the dualizing line bundle on $\overline{X}$. Then the line bundle $\omega_{\overline{X}}^{-1}(-S_1 - \cdots - S_\ell) \otimes L$ is ample.

The following result shows that the analogue of this Proposition in our context is false.

Proposition 6.1. *Let $\omega_{KGl} := \det \Omega^1_{KGl(E,F)}$ denote the dualising line bundle on KGl(E, F). Let $a_1 \leq \cdots \leq a_n$, $a_i \in \mathbb{Z}$ and let $L = \bigotimes_{i=0}^{n-1}(M_i^{m_i} \otimes L_i^{l_i})$, where $m_i := \sum_{j=i+1}^{n} a_j$ and $l_i := -\sum_{j=1}^{n-i} a_j$. Then $H^0(\overline{\mathbf{O}}_{r,s}, i^*_{r,s}L)$ is nonzero for every $r, s \in [0, n]$ with $r + s = n$, but neither the line bundle $\omega_{KGl}^{-1}(-\sum_{i=0}^{n-1}(Z_i + Y_i)) \otimes L$ nor the line bundle L itself is ample.*

Proof. Using the inductive blowing up procedure which defines KGl(E, F) it is easy to see that $\omega_{\mathrm{KGl}} \cong \bigotimes_{i=1}^{n-1}(M_i \otimes L_i)^{i(i-n)-1}$.

Let $e_1, \ldots, e_n$ be the canonical basis of $N := \mathbb{Z}^n$. Let Δ be the smooth complete fan in $N_{\mathbb{Q}} = \mathbb{Q}^n$ whose one-dimensional cones are generated by the vectors $\pm\sum_{i\in I} e_i$, where I runs through the nonempty subsets of $[1, n]$ and whose n-dimensional cones are the sets $\sigma(\alpha, \ell) := \{x \in \mathbb{Q}^n \mid x_{\alpha(1)} \leq \cdots \leq x_{\alpha(\ell)} \leq 0 \leq x_{\alpha(\ell+1)} \leq \cdots \leq x_{\alpha(n)}\}$ where α runs through the set of permutations of $[1, n]$ and ℓ runs through the set $[0, n]$. Let KT be the smooth complete torus embedding associated to Δ.

Let $\tilde{T}$ be the torus embedding defined in [K1] p 563, and let $Z_{i,\tilde{T}}$, $Y_{i,\tilde{T}}$ be the divisors on $\tilde{T}$ defined there. The variety $\tilde{T}$ can be identified with an open subscheme of KT and KT can be identified with the closure in $\mathrm{KGl}(E,F)$ of a maximal torus in $\mathrm{Gl}_n \cong \mathrm{Isom}(E,F)$ such that the restriction of the line bundles M_i and L_i to $\tilde{T}$ are $\mathcal{O}_{\tilde{T}}(Z_{i,\tilde{T}})$ and $\mathcal{O}_{\tilde{T}}(Y_{i,\tilde{T}})$ respectively.

From 3.2 and 3.3 it is immediate that for any $r,s \in [0,n]$ with $r+s=n$ the restriction of L to $\overline{\mathbf{O}}_{r,s}$ has non-vanishing global sections. On the other hand, with the help of criterion [C], 3.1 it is easy to see that neither the restriction of L nor that of $\omega_{\mathrm{KGl}}^{-1}(-\sum_{i=0}^{n-1}(Z_i+Y_i)) \otimes L$ to the closed subscheme KT are ample. □

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